Annals of Mathematics **190** (2019), 113–143
https://doi.org/10.4007/annals.2019.190.1.2

Cichoń's maximum

By Martin Goldstern, Jakob Kellner, and Saharon Shelah

Abstract

Assuming four strongly compact cardinals, it is consistent that all entries in Cichoń's diagram (apart from $\mathrm{add}(\mathcal{M})$ and $\mathrm{cof}(\mathcal{M})$, whose values are determined by the others) are pairwise different; more specifically, $\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}$.

Introduction

Independence. How many Lebesgue null sets are required to cover the real line? Obviously countably many are not enough, as the countable union of null sets is null; and obviously continuum many are enough, as $\bigcup_{r\in\mathbb{R}}\{r\} = \mathbb{R}$.

The answer to our question is a cardinal number called the covering number of the null ideal, or $\mathrm{cov}(\mathcal{N})$. As we have just seen,

$$\aleph_0 = |\mathbb{N}| < \mathrm{cov}(\mathcal{N}) \leq |\mathbb{R}| = 2^{\aleph_0}.$$

In particular, if the Continuum Hypothesis (CH) holds (i.e., if there are no cardinalities strictly between $|\mathbb{N}|$ and $|\mathbb{R}|$, or equivalently, if $\aleph_1 = 2^{\aleph_0}$), then $\mathrm{cov}(\mathcal{N}) = 2^{\aleph_0}$; but without CH, the answer could also be some cardinal less than $2^{\aleph_0}$. According to Cohen's famous result [Coh63], CH is independent of the usual axiomatization of mathematics, the Zermelo Fraenkel axioms of set theory including the Axiom of Choice, abbreviated ZFC. That is, we can prove that the ZFC axioms neither imply CH nor imply $\neg$CH. For this result, Cohen introduced the method of forcing, which has been continuously expanded and refined ever since. Forcing also proves that the value of $\mathrm{cov}(\mathcal{N})$ is independent. For example, $\mathrm{cov}(\mathcal{N}) = \aleph_1 < 2^{\aleph_0}$ is consistent, as is $\aleph_1 < \mathrm{cov}(\mathcal{N}) = 2^{\aleph_0}$.

Keywords: set theory of the reals, Cichoń's diagram, forcing, compact cardinals
AMS Classification: Primary: 03E17.
The first author was supported by Austrian Science Fund (FWF): P29575 & I3081 and National Science Foundation NSF DMS-1362974. The second author was supported by Austrian Science Fund (FWF): P26737 & P30666. The third author was supported by European Research Council grant ERC-2013-ADG 338821. This is publication number 1122 of the third author.

Cichoń's diagram. The covering number $\mathrm{cov}(\mathcal{N})$ is a so-called cardinal characteristic of the continuum. Other well-studied characteristics include the following:

- $\mathrm{add}(\mathcal{N})$ is the smallest number of Lebesgue null sets whose union is not null.
- $\mathrm{non}(\mathcal{N})$ is the smallest cardinality of a non-null set.
- $\mathrm{cof}(\mathcal{N})$ is the smallest size of a cofinal family of null sets, i.e., a family that contains for each null set N a superset of N.
- Replacing "null" with "meager," we can analogously define the characteristics $\mathrm{add}(\mathcal{M})$, $\mathrm{non}(\mathcal{M})$, $\mathrm{cov}(\mathcal{M})$, and $\mathrm{cof}(\mathcal{M})$.
- In addition, we define $\mathfrak{b}$ as the smallest size of an unbounded family, i.e., a family $\mathcal{H}$ of functions from $\mathbb{N}$ to $\mathbb{N}$ such that for every $f : \mathbb{N} \to \mathbb{N}$, there is some $h \in \mathcal{H}$ that is not almost everywhere bounded by f.

  Equivalently, $\mathfrak{b} = \mathrm{add}(\mathcal{K}) = \mathrm{non}(\mathcal{K})$, where $\mathcal{K}$ is the σ-ideal generated by the compact subsets of the irrationals.
- $\mathfrak{d}$ is the smallest size of a dominating family, i.e., a family $\mathcal{H}$ such that for every $f : \mathbb{N} \to \mathbb{N}$, there is some $h \in \mathcal{H}$ such that $(\exists n \in \mathbb{N})\, (\forall m > n)\, h(m) > f(m)$.

  Equivalently, $\mathfrak{d} = \mathrm{cov}(\mathcal{K}) = \mathrm{cof}(\mathcal{K})$.
- For the ideal ctbl of countable sets, we trivially get $\mathrm{add}(\mathrm{ctbl}) = \mathrm{non}(\mathrm{ctbl}) = \aleph_1$ and $\mathrm{cov}(\mathrm{ctbl}) = \mathrm{cof}(\mathrm{ctbl}) = 2^{\aleph_0}$.

The characteristics we have mentioned so far,[1] and the basic relations between them, can be summarized in Cichoń's diagram:

$$\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0}. \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

An arrow from $\mathfrak{x}$ to $\mathfrak{y}$ indicates that ZFC proves $\mathfrak{x} \leq \mathfrak{y}$. Moreover, $\mathrm{cof}(\mathcal{M}) = \max(\mathfrak{d}, \mathrm{non}(\mathcal{M}))$ and $\mathrm{add}(\mathcal{M}) = \min(\mathfrak{b}, \mathrm{cov}(\mathcal{M}))$. A (by now) classical series of theorems [Bar84], [BJS93], [CKP85], [JS90], [Kam89], [Mil81], [Mil84], [RS83] and [RS85] proves these (in)equalities in ZFC and shows that they are the only ones provable. More precisely, all assignments of the values $\aleph_1$ and $\aleph_2$ to the characteristics in Cichoń's Diagram are consistent with ZFC, provided they do not contradict the above (in)equalities. (A complete proof can be found in [BJ95, Ch. 7].)

[1] There are many other cardinal characteristics (see, for example, [Bla10]), but the ones in Cichoń's diagram seem to be considered to be the most important ones.

Note that Cichoń's diagram shows a fundamental asymmetry between the ideals of Lebesgue null sets and of meager sets. (We will mention another one in the context of large cardinals.) Any such asymmetry is hidden if we assume CH, as under CH not only all the characteristics are $\aleph_1$, but even the Erdős-Sierpiński Duality Theorem holds [Oxt80, Ch. 19]: There is an involution $f:\mathbb{R}\to\mathbb{R}$ (i.e., a bijection such that $f\circ f=\mathrm{Id}$) such that $A\subseteq\mathbb{R}$ is meager if and only if $f''A$ is null.

So it is settled which assignments of $\aleph_1$ and $\aleph_2$ to Cichoń's diagram are consistent. It is more challenging to show that the diagram can contain more than two different cardinal values. For recent progress in this direction, see, e.g., [Mej13], [GMS16], [FGKS17], [KTT18].

The result of this paper is in some respect the strongest possible, as we show that consistently *all* the entries are pairwise different (apart from the two ZFC-provable equalities mentioned above). Of course one can ask more; see the questions in Section 4.[2] In particular, we use large cardinals in the proof.

Large cardinals. As mentioned, ZFC is an axiom system for the whole of mathematics. A much "weaker" axiom system (for the natural numbers) is PA (Peano arithmetic).

Gödel's Incompleteness Theorem shows that a theory such as PA or ZFC can never prove its own consistency. On the other hand, it is trivial to show in ZFC that PA is consistent. (As in ZFC we can construct $\mathbb{N}$ and prove that it satisfies PA.) We can say that ZFC has a higher consistency strength than PA.

One axiom of ZFC is INF, the statement "there is an infinite cardinal." If we remove INF from ZFC, we end up with a theory ZFC^0 that can still describe concrete hereditarily finite objects and can be interpreted (admittedly in a not very natural way) as a weak version of PA that has the same consistency strength as PA.[3] So we can say that adding an infinite cardinal to ZFC^0 increases the consistency strength.

There are notions of cardinal numbers much "stronger" than just "infinite." Often, such large cardinal assumptions (abbreviated LC in the following) have the following form:

> There is a cardinal $\kappa>\aleph_0$ that behaves towards the smaller cardinals in a similar way as $\aleph_0$ behaves to finite numbers.

A forcing proof shows, e.g.,

> If ZFC is consistent, then ZFC+$\neg$CH is consistent,

[2] Section 4 also contains information on some progress made since the paper was submitted.

[3] More concretely, $\mathrm{ZF}_{\mathrm{fin}} := \mathrm{ZFC}^0 + \neg\mathrm{INF}$ can be seen to be "equivalent" to PA (i.e., mutually interpretable). This goes back to Ackermann [Ack37]; see the survey [KW07].

and this implication can be proved in a very weak system such as PA. However, we cannot prove (not even in ZFC) for any large cardinal that

"if ZFC is consistent, then ZFC+LC is consistent"

because in ZFC+LC we can prove the consistency of ZFC. We say that LC has a higher consistency strength than ZFC.

An instance of a large cardinal (in fact a very weak one, a so-called inaccessible cardinal) appears in another striking example of the asymmetry between measure and category. The following statement is equiconsistent with an inaccessible cardinal [Sol70], [She84]:

All projective[4] sets of reals are Lebesgue measurable.

In contrast, according to [She84] no large cardinal assumption is required to show the consistency of

All projective sets of reals have the property of Baire.

So we can assume "for free" that all (reasonable) sets have the Baire property, whereas we have to provide additional consistency strength for Lebesgue measurability.

In the case of our paper, we require (the consistency of) the existence of four compact cardinals to prove our main result. It seems unlikely that any large cardinals are actually required; but a proof without them would probably be considerably more complicated. It is not unheard of that ZFC results first have (simpler) proofs using large cardinal assumptions; an example can be found in [She04].

Annotated Contents. From now on, we assume that the reader is familiar with some basic properties of the characteristics defined above, as well as with the associated forcing notions Cohen, amoeba, random, Hechler and eventually different, all of which can be found, e.g., in [BJ95].

This paper consists of three parts. In Section 1, we present a finite support ccc (countable chain condition) iteration $\mathbb{P}^5$ forcing that $\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}$. This result is not new: Such a forcing was introduced in [GMS16], and we follow this construction quite closely. However, we need the Generalized Continuum Hypothesis (GCH) in the ground model, whereas [GMS16] requires $2^\chi \gg \lambda$ for some $\chi < \lambda$. Also, we describe how the inequalities are "strongly witnessed" (see Definitions 1.8 and 1.15).

[4]This is the smallest family containing the Borel sets and closed under continuous images, complements, and countable unions. In practice, all sets used in mathematics that are defined without using AC are projective. Alternatively we could use the statement: "ZF (without the Axiom of Choice) holds, and all sets of reals are Lebesgue measurable."

In Section 2, we show how to construct (under GCH) for κ strongly compact and $\theta > \kappa$ regular a "BUP-embedding" from κ to θ, i.e., an elementary embedding $j : V \to M$ with critical point κ and $\mathrm{cf}(j(\kappa)) = |j(\kappa)| = \theta$ such that M is transitive and $<\kappa$-closed and such that $j''S$ is cofinal in $j(S)$ for every $\leq\kappa$-directed partial order S. For a ccc forcing P, we investigate $j(P)$ and show that $j(P)$ forces the same values to some characteristics in Cichoń's diagram as P and different values to others, in a very controlled way — assuming that there were "strong witnesses" for P forcing the initial values, as described in Section 1.

Section 3 contains the main result of this paper: Assuming four strongly compact cardinals, we let k be the composition of four such BUP-embeddings, mapping $\mathbb{P}^5$ to a ccc forcing $\mathbb{P}^9$. We then show that $\mathbb{P}^9$ forces

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathrm{non}(\mathcal{M}) \\ < \mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0};$$

i.e., we get for increasing cardinals λ_i the constellation of Figure 1.

$$\begin{tikzcd}
 & \lambda_2 \arrow[r] & \lambda_4 \arrow[r] & \boxed{} \arrow[r] & \lambda_8 \arrow[r] & \lambda_9. \\
 & & \lambda_3 \arrow[u] \arrow[r] & \lambda_6 \arrow[u] & & \\
\aleph_1 \arrow[r] & \lambda_1 \arrow[uu] \arrow[r] & \boxed{} \arrow[u] \arrow[r] & \lambda_5 \arrow[u] \arrow[r] & \lambda_7 \arrow[uu] &
\end{tikzcd}$$

Figure 1. Our cardinal configuration. (The λ_i are increasing.)

Boolean ultrapowers as used in this paper were investigated by Mansfield [Man71] and recently applied, e.g., by the third author with Malliaris [MS16] and with Raghavan [RS], where Boolean ultrapowers of forcing notions are used to force specific values to certain cardinal characteristics. Recently the third author developed a method of using Boolean ultrapowers to control characteristics in Cichoń's diagram. A first (and simpler) application of these methods is given in [KTT18].

We mention some open questions in Section 4.

Acknowledgments. We thank three anonymous referees for pointing out several unclarities and typos, and we thank Moti Gitik and Diego Mejía for suggestions to improve the presentation.

1. **The initial forcing**

1.1. *Good iterations and the* LCU *property.* We want to show that some forcing $\mathbb{P}^5$ results in $\mathfrak{x} = \lambda_i$ for certain characteristics $\mathfrak{x}$. So we have to show two "directions," $\mathfrak{x} \leq \lambda_i$ and $\mathfrak{x} \geq \lambda_i$. For most of the characteristics, one direction will use the fact that $\mathbb{P}^5$ is "good" — a notion introduced by Judah and the third author [JS90] and Brendle [Bre91]. We now recall the basic facts of good iterations and specify the instances of the relations we use.

Assumption 1.1. *We will consider binary relations* R *on* $X = \omega^\omega$ (*or on* $X = 2^\omega$) *that satisfy the following*: *There are relations* R^k *such that* $\mathrm{R} = \bigcup_{k\in\omega} \mathrm{R}^k$, *each* R^k *is a closed subset* (*and in fact absolutely defined*) *of* $X \times X$, *and for* $g \in X$ *and* $k \in \omega$, *the set* $\{f \in X : f\,\mathrm{R}^k\,g\}$ *is nowhere dense* (*and of course closed*). *Also, for all* $g \in X$, *there is some* $f \in X$ *with* $f\,\mathrm{R}\,g$.

We will actually use another space as well, the space $\mathcal{C}$ of strictly positive rational sequences $(q_n)_{n\in\omega}$ such that $\sum_{n\in\omega} q_n \leq 1$. It is easy to see that $\mathcal{C}$ is homeomorphic to ω^ω, when we equip the rationals with the discrete topology and use the product topology. Let us fix one such (absolutely defined) homeomorphism.

We use the following instances of relations R on X; it is easy to see that they all satisfy the assumption. (For $X_1 = \mathcal{C}$, we use the homeomorphism mentioned above.)

Definition 1.2.

(1) $X_1 = \mathcal{C}$: $f\,\mathrm{R}_1\,g$ if $(\forall^* n \in \omega)\, f(n) \leq g(n)$.
(We use $\forall^* n$ as abbreviation for $(\exists n_0)\,(\forall n > n_0)$.)
(2) Fix a partition $(I_n)_{n\in\omega}$ of ω with $|I_n| = 2^{n+1}$.
$X_2 = 2^\omega$: $f\,\mathrm{R}_2\,g$ if $(\forall^* n \in \omega)\, f \restriction I_n \neq g \restriction I_n$.
(3) $X_3 = \omega^\omega$: $f\,\mathrm{R}_3\,g$ if $(\forall^* n \in \omega)\, f(n) \leq g(n)$.
(4) $X_4 = \omega^\omega$: $f\,\mathrm{R}_4\,g$ if $(\forall^* n \in \omega)\, f(n) \neq g(n)$.

Note that Assumption 1.1 is satisfied, witnessed by the relations R_i^k defined by replacing $(\forall^* n \in \omega)$ with $(\forall n \geq k)$.

We say "f is bounded by g" if $f\,\mathrm{R}\,g$; and, for $\mathcal{Y} \subseteq \omega^\omega$, "$f$ is bounded by $\mathcal{Y}$" if $(\exists y \in \mathcal{Y})\, f\,\mathrm{R}\,y$. We say "unbounded" for "not bounded." (That is, f is unbounded by $\mathcal{Y}$ if $(\forall y \in \mathcal{Y})\, \neg f\,\mathrm{R}\,y$.) We call $\mathcal{X}$ an R-unbounded family if $\neg(\exists g)\,(\forall x \in \mathcal{X}) x\,\mathrm{R}\,g$, and an R-dominating family if $(\forall f)\,(\exists x \in \mathcal{X})\, f\,\mathrm{R}\,x$.

- Let $\mathfrak{b}_i$ be the minimal size of an R_i-unbounded family,
- and let $\mathfrak{d}_i$ be the minimal size of an R_i-dominating family.

We only need the following connections between R_i and the cardinal characteristics:

Lemma 1.3.

(1) $\mathrm{add}(\mathcal{N}) = \mathfrak{b}_1$ *and* $\mathrm{cof}(\mathcal{N}) = \mathfrak{d}_1$.
(2) $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}_2$ *and* $\mathrm{non}(\mathcal{N}) \geq \mathfrak{d}_2$.
(3) $\mathfrak{b} = \mathfrak{b}_3$ *and* $\mathfrak{d} = \mathfrak{d}_3$.
(4) $\mathrm{non}(\mathcal{M}) = \mathfrak{b}_4$ *and* $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}_4$.

Proof. (3) holds by definition. (1) can be found in [BJ95, 6.5.B]. (4) is a result of [Mil82], [Bar87]; cf. [BJ95, 2.4.1 and 2.4.7].

To prove (2), note that for fixed $f \in 2^\omega$, the set $\{g \in 2^\omega : \neg f \mathrel{\mathrm{R}_2} g\}$ is a null set — call it N_f. Let $\mathcal{F}$ be an R_2-unbounded family. Then $\{N_f : f \in \mathcal{F}\}$ covers 2^ω: Fix $g \in 2^\omega$. As g does not bound $\mathcal{F}$, there is some $f \in \mathcal{F}$ unbounded by g, i.e., $g \in N_f$. Let X be a non-null set. Then X is R_2-dominating: For any $f \in 2^\omega$, there is some $x \in X \setminus N_f$, i.e., $f \mathrel{\mathrm{R}_2} x$. □

We will also use

Lemma 1.4 ([BJ95]). *Amoeba forcing $\mathbb{A}$ adds a dominating element $\bar{b}$ of $\mathcal{C}$; i.e., $\mathbb{A} \Vdash \bar{q} \mathrel{\mathrm{R}_1} \bar{b}$ for all $\bar{q} \in \mathcal{C} \cap V$.*

Proof. Let us define a slalom $\mathcal{S}$ to be a function $\mathcal{S} : \omega \to [\omega]^{<\omega}$ such that $|\mathcal{S}(n)| > 0$ and $\sum_{n=1}^\infty \frac{|\mathcal{S}(n)|}{n^2} < \infty$.

Amoeba forcing will add a null set covering all old null sets, and therefore (according to [BJ95, 2.3.3]) a slalom $\mathcal{S}$ covering all old slaloms. Set $a_n := \frac{|\mathcal{S}(n)|}{n^2}$, $M := \sum_{n=1}^\infty a_n$, set M' the smallest natural number $\geq M$, and set $b_n := \frac{a_{n+1}}{M'}$. Then it is easy to see that $(b_n)_{n\in\omega} \in \mathcal{C}$ dominates every old sequence $(q_n)_{n\in\omega}$ in $\mathcal{C}$. □

Definition 1.5 ([JS90]). Let P be a ccc forcing, λ an uncountable regular cardinal, and R as above. P is (R, λ)-good if for each P-name $r \in \omega^\omega$, there is (in V) a nonempty set $\mathcal{Y} \subseteq \omega^\omega$ of size $<\lambda$ such that every f (in V) that is R-unbounded by $\mathcal{Y}$ is forced to be R-unbounded by r as well.

Note that λ-good trivially implies μ-good if $\mu \geq \lambda$ are regular.

How do we get good forcings? Let us just note the following results:

Lemma 1.6. *A finite support (henceforth abbreviated FS) iteration of Cohen forcing is good for any (R, λ), and the composition of two (R, λ)-good forcings is (R, λ)-good.*

Assume that $(P_\alpha, Q_\alpha)_{\alpha<\delta}$ is an FS ccc iteration. Then P_δ is (R, λ)-good if each Q_α is forced to satisfy the following:

(1) *For $\mathrm{R} = \mathrm{R}_1$, $|Q_\alpha| < \lambda$, or Q_α is σ-centered, or Q_α is a sub-Boolean-algebra of the random algebra.*
(2) *For $\mathrm{R} = \mathrm{R}_2$, $|Q_\alpha| < \lambda$, or Q_α is σ-centered.*
(4) *For $\mathrm{R} = \mathrm{R}_4$, $|Q_\alpha| < \lambda$.*

(Remark: For R_3, the same holds as for R_4 which, however, is of no use for our construction.)

Proof. (R, λ)-goodness is preserved by FS ccc iterations (in particular, compositions), as proved in [JS90]; cf. [BJ95, 6.4.11–12]. Also, ccc forcings of size $<\lambda$ are (R, λ)-good [BJ95, 6.4.7], which takes care of the case $|Q_\alpha| < \lambda$

(and, in particular, of Cohen forcing). So it remains to show that (for $i = 1, 2$) the "large" iterands in the list are (R_i, λ)-good.

For R_1, this follows from [JS90] and [Kam89]; cf. [BJ95, 6.5.17–18]. For R_2, this is proven in [Bre91], and as the proof is very short, we give it here: Write Q_α as union $\bigcup_{k\in\omega} Q^k$ of centered sets. Given the Q_α-name r, pick a countable elementary submodel N containing r and Q_α, and set $\mathcal{Y} = N \cap 2^\omega$. Assume towards a contradiction that f is unbounded by $\mathcal{Y}$, but is forced by p_0 to be bounded by r; i.e., p_0 forces $(\forall n > n_0) f \restriction I_n \neq r \restriction I_n$. Now p_0 may not be in N, but there is some $k_0 \in \omega$ such that $p_0 \in Q^{k_0}$. In N, we can pick for each $n \in \omega$ some $s_n \in 2^{I_n}$ such that no $q \in Q^{k_0}$ forces $r \restriction I_n \neq s_n$. (There are only finitely many $s \in 2^{I_n}$; if each s is forbidden by some q, then the common stronger element would prevent all possibilities for $r \restriction I_n$.) So in N, we get some $g \in 2^\omega$ such that $g \restriction I_n = s_n$. As f is unbounded by $\mathcal{Y}$ (or equivalently, by N), there is some $n > n_0$ such that $f \restriction I_n = g \restriction I_n = s_n$, which implies that p_0 (as an element of Q^{k_0}) does not force $r \restriction I_n \neq f \restriction I_n$, a contradiction. □

LEMMA 1.7. *Let $\lambda \leq \kappa \leq \mu$ be uncountable regular cardinals. After forcing with μ many Cohen reals $(c_\alpha)_{\alpha\in\mu}$, followed by an (R, λ)-good forcing, we get that for every real r in the final extension, the set $\{\alpha \in \kappa : c_\alpha$ is unbounded by $r\}$ is cobounded in κ. That is, $(\exists \alpha \in \kappa)\, (\forall \beta \in \kappa \setminus \alpha)\, \neg c_\beta \,\mathrm{R}\, r$.*

(The Cohen real c_β can be interpreted both as Cohen generic element of 2^ω and as Cohen generic element of ω^ω; we use the interpretation suitable for the relation R.)

Proof. Work in the intermediate extension after κ many Cohen reals; let us call it V_κ. The remaining forcing (i.e., $\mu \setminus \kappa$ many Cohens composed with the good forcing) is good; so applying the definition, we get (in V_κ) a set $\mathcal{Y}$ of size $<\lambda$.

As the initial Cohen extension is ccc, and $\kappa \geq \lambda$ is regular, we get some $\alpha \in \kappa$ such that each element y of $\mathcal{Y}$ already exists in the extension by the first α many Cohens, call it V_α. The set of reals M_y bounded by y is meager (and absolute). Any c_β for $\beta \in \kappa \setminus \alpha$ is Cohen over V_α, and therefore not in M_y, i.e., not bounded by y, i.e., not by $\mathcal{Y}$. So according to the definition of good, each such c_β is unbounded by r as well for the given r. □

In light of this result, let us revisit Lemma 1.3 with some new notation, the "linearly cofinally unbounded" property LCU:

Definition 1.8. For $i = 1, 2, 3, 4$, γ a limit ordinal, and P a ccc forcing notion, let $\mathsf{LCU}_i(P, \gamma)$ stand for the following:

> There is a sequence $(x_\alpha)_{\alpha\in\gamma}$ of P-names of elements of X_i (the domain of the relation R_i) such that for every such P-name y,
> $$(\exists \alpha \in \gamma)\, (\forall \beta \in \gamma \setminus \alpha)\, P \Vdash \neg x_\beta \,\mathrm{R}_i\, y.$$

LEMMA 1.9.

- $\mathsf{LCU}_i(P,\delta)$ *is equivalent to* $\mathsf{LCU}_i(P,\mathrm{cf}(\delta))$.
- *If* λ *is regular, then* $\mathsf{LCU}_i(P,\lambda)$ *implies* $\mathfrak{b}_i \leq \lambda$ *and* $\mathfrak{d}_i \geq \lambda$.

In particular,

(1) $\mathsf{LCU}_1(P,\lambda)$ *implies* $P \Vdash (\,\mathrm{add}(\mathcal{N}) \leq \lambda \;\&\; \mathrm{cof}(\mathcal{N}) \geq \lambda\,)$.
(2) $\mathsf{LCU}_2(P,\lambda)$ *implies* $P \Vdash (\,\mathrm{cov}(\mathcal{N}) \leq \lambda \;\&\; \mathrm{non}(\mathcal{N}) \geq \lambda\,)$.
(3) $\mathsf{LCU}_3(P,\lambda)$ *implies* $P \Vdash (\,\mathfrak{b} \leq \lambda \;\&\; \mathfrak{d} \geq \lambda\,)$.
(4) $\mathsf{LCU}_4(P,\lambda)$ *implies* $P \Vdash (\,\mathrm{non}(\mathcal{M}) \leq \lambda \;\&\; \mathrm{cov}(\mathcal{M}) \geq \lambda\,)$.

Proof. Assume $(\alpha_\beta)_{\beta\in\mathrm{cf}(\delta)}$ is increasing continuous and cofinal in δ. If $(x_\alpha)_{\alpha\in\delta}$ witnesses $\mathsf{LCU}_i(P,\delta)$, then $(x_{\alpha_\beta})_{\beta\in\mathrm{cf}(\delta)}$ witnesses $\mathsf{LCU}_i(P,\mathrm{cf}(\delta))$. And if $(x_\beta)_{\beta\in\mathrm{cf}(\delta)}$ witnesses $\mathsf{LCU}_i(P,\mathrm{cf}(\delta))$, then $(y_\alpha)_{\alpha\in\delta}$ witnesses $\mathsf{LCU}_i(P,\delta)$, where $y_\alpha := x_\beta$ for $\alpha \in [\alpha_\beta, \alpha_{\beta+1})$.

The set $\{x_\alpha : \alpha \in \lambda\}$ is certainly forced to be R_i-unbounded; and given a set $Y = \{y_j : j < \theta\}$ of $\theta < \lambda$ many P-names, each has a bound $\alpha_j \in \lambda$ so that $(\forall \beta \in \lambda \setminus \alpha_j)\, P \Vdash \neg x_\beta \,\mathrm{R}_i\, y_j$, so for any $\beta \in \lambda$ above all α_j, we get $P \Vdash \neg x_\beta \,\mathrm{R}_i\, y_j$ for all j; i.e., Y cannot be dominating. □

Remark 1.10. Note that $\mathfrak{b}_i \leq \lambda$ is equivalent to the existence of a sequence $(x_\alpha : \alpha \in \lambda)$ with the property $(\forall y)\,(\exists \alpha)\,\neg\big(x_\alpha R_i y\big)$; such a sequence might be called a "'witness" for $\mathfrak{b}_i \leq \lambda$. In LCU we demand a stronger property; a sequence $(x_\alpha : \alpha < \lambda)$ with this stronger property could informally be called a "strong witness" for $\mathfrak{b}_i \leq \lambda$. Similarly, the next subsection introduces a different notion, COB, corresponding to "strong witnesses" for $\mathfrak{d}_i \leq \mu$.

1.2. *The initial forcing* $\mathbb{P}^5$: *Partial forcings and the* COB *property.* Assume we have a forcing iteration $(P_\beta, Q_\beta)_{\beta<\alpha}$ with limit P_α, where each Q_β is forced by P_β to be a set of reals such that the generic filter of Q_β is determined (in a Borel way)[5] from some generic real η_β. Fix some $w \subseteq \alpha$. We define the P_α-name Q_α to consist of all random forcing conditions that can be Borel-calculated from generics at w alone.

More explicitly,

Definition 1.11.

(a) q is in Q_α if there are in the ground model V a countable subset $u \subseteq w$ and a Borel function $B : \mathbb{R}^u \to \mathbb{R}$ such that $q = B(\,(\eta_\beta)_{\beta\in u}\,)$ is a random condition.

[5] More specifically, we require that the Borel function for Q_β is already fixed in the ground model. For example, assume Q_β is random forcing, defined as the set of all positive pruned trees T, i.e., trees $T \subseteq 2^{<\omega}$ without leaves such that $[T]$ has positive measure. Then the generic filter G for this forcing is determined by the generic real η (the random real), and G consists of those trees T such that $\eta \in [T]$, which is a Borel relation. See [KTT18, §1.2] for a formal definition and more details.

Being a random condition is a Borel property (if we fix some suitable representation of random forcing). Accordingly, we can restrict ourselves to the case that B is a Borel function whose image consists of random conditions only.

(b) We call a pair (B, u) as above "a w-groundmodel-code" or just "code." Note that this code is a ground model object. So Q_α consists exactly of the evaluations of such codes.

(c) We call a condition $(p, q) \in P_\alpha * Q_\alpha$ "determined at position α" if there is a code (B, u) such that p forces that (B, u) is a code for q. (Note that generally we only have a P_α-name for a code.) Given some (p, q), we can obviously find $p' \leq p$ such that (p', q) is determined at α.

(d) We will later also consider so-called "groundmodel-code-sequences" for elements of Q_α, that is, (in V) a sequence $(B_n, u_n)_{n\in\omega}$ of codes, where u_n is in w_α. Of course not every ω-sequence of Q_α-conditions in the P_α-extension is described by a ground model sequence. (In particular, there will only be few ground model sequences, but many new ω-sequences in the extension.)

Clearly, in the P_α extension, Q_α is a subforcing (not necessarily a complete one) of the full random forcing, and if p, q in Q_α are incompatible in Q_α, then they are incompatible in random forcing. (Two compatible conditions p, q have a canonical conjunction $p \wedge q$ (the intersection), and if p and q are both Borel-calculated from w, then so is the intersection.) In particular, Q_α is ccc.

We call this forcing "partial random forcing defined from w." Analogously, we define the "partial Hechler," "partial eventually different"[6] and "partial amoeba" forcings. The same argument shows that these forcings are also ccc.

Assume that λ is regular uncountable and that $\mu < \lambda$ implies $\mu^{\aleph_0} < \lambda$. Then $|w| < \lambda$ implies that the sizes of the partial forcings defined by w are $<\lambda$.

We will assume the following throughout the paper:

Assumption 1.12. *Let $\aleph_1 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \lambda_5$ be regular cardinals such that $\mu < \lambda_i$ implies $\mu^{\aleph_0} < \lambda_i$. Furthermore, let λ_3 be the successor of a regular cardinal χ with $\chi^{\aleph_0} = \chi$, and $\lambda_5^{<\lambda_4} = \lambda_5$.*

We set $\delta_5 = \lambda_5 + \lambda_5$, and we partition $\delta_5 \setminus \lambda_5$ into unbounded sets S^1, S^2, S^3 and S^4. Fix for each $\alpha \in \delta_5 \setminus \lambda_5$ some $w_\alpha \subseteq \alpha$ such that each $\{w_\alpha : \alpha \in S^i\}$ is cofinal[7] *in $[\delta_5]^{<\lambda_i}$.*

The reader can assume that $(\lambda_i)_{i=1,\dots,5}$, $(S^i)_{i=1,\dots,4}$ as well as $(w_\alpha)_{\alpha\in S^i}$ for $i = 1, 2, 3$ have been fixed once and for all (let us call them "fixed parameters"),

[6] See 1.22 for the definition.

[7] That is, if $\alpha \in S^i$, then $|w_\alpha| < \lambda_i$, and for all $u \subseteq \delta_5$, $|u| < \lambda_i$, there is some $\alpha \in S^i$ with $w_\alpha \supseteq u$.

whereas we will investigate various possibilities for $\bar{w} = (w_\alpha)_{\alpha\in S^4}$ in Sections 1.3 and 1.4. (We will call such a $\bar{w}$ that satisfies the assumption a "cofinal parameter.")

Definition 1.13. Let $\mathbb{P}^5 = (P_\alpha, Q_\alpha)_{\alpha\in\delta_5}$ be the FS iteration, where Q_α is Cohen forcing for $\alpha \in \lambda_5$ and

$$Q_\alpha \text{ is the partial} \left\{ \begin{array}{c} \text{amoeba} \\ \text{random} \\ \text{Hechler} \\ \text{eventually different} \end{array} \right\} \text{forcing defined from } w_\alpha \text{ if } \alpha \text{ is in} \left\{ \begin{array}{l} S^1 \\ S^2 \\ S^3 \\ S^4. \end{array} \right.$$

According to Lemma 1.6, $\mathbb{P}^5$ is $(\lambda_i, \mathrm{R}_i)$-good for $i = 1, 2, 4$, so Lemmas 1.7 and 1.9 give us

LEMMA 1.14. *$\mathsf{LCU}_i(\mathbb{P}^5, \kappa)$ holds for $i = 1, 2, 4$ and each regular cardinal κ in $[\lambda_i, \lambda_5]$.*

So, in particular, $\mathbb{P}^5$ forces $\mathrm{add}(\mathcal{N}) \le \lambda_1$, $\mathrm{cov}(\mathcal{N}) \le \lambda_2$, $\mathrm{non}(\mathcal{M}) \le \lambda_4$ and $\mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \lambda_5 = 2^{\aleph_0}$; i.e., the respective characteristics in the left half of Cichoń's diagram are small enough. It is easy to see that they are also large enough:

For example, the partial amoebas and the fact that $(w_\alpha)_{\alpha\in S^1}$ is cofinal ensure that $\mathbb{P}^5$ forces $\mathrm{add}(\mathcal{N}) \ge \lambda_1$. Let $(N_k)_{k\in\mu}$, $\aleph_1 \le \mu < \lambda_1$ be a family of $\mathbb{P}^5$-names of null sets. Each N_k is a Borel-code, i.e., a real, i.e., a sequence of natural numbers, each of which is decided by a maximal antichain (labeled with natural numbers). Each condition in such an antichain has finite support, hence it only uses finitely many coordinates in δ_5. So all in all we get a set w^* of size $\le\mu$ that already decides all N_k. (That is, for each $k \in \mu$, there are a Borel function B in V and a sequence $(\alpha_j)_{j\in\omega}$ in V of elements of w^* such that $N_k = B(\eta_{\alpha_0}, \eta_{\alpha_1}, \dots)$.) There is some $\beta \in S^1$ such that $w_\beta \supseteq w^*$, and the partial amoeba forcing at β sees all the null sets N_k and therefore covers their union.

We will reformulate this in a slightly cumbersome manner that can be conveniently used later on, using the "cone of bounds" property COB:

Definition 1.15. For a ccc forcing notion P, regular uncountable cardinals λ, μ and $i = 1, 3, 4$, let $\mathsf{COB}_i(P, \lambda, \mu)$ stand for the following:

There are a $<\lambda$-directed partial order $(S, \prec)$ of size μ and a sequence $(g_s)_{s\in S}$ of P-names for reals such that for each P-name f of a real, we have

$$(\exists s \in S)\,(\forall t \succ s)\, P \Vdash f\, \mathrm{R}_i\, g_t.$$

So s is the tip of a cone that consists of elements bounding f.

LEMMA 1.16. *For $i = 1, 3, 4$, $\mathsf{COB}_i(P, \lambda, \mu)$ implies*

$$P \Vdash (\, \mathfrak{b}_i \ge \lambda \,\&\, \mathfrak{d}_i \le \mu \,).$$

Proof. The set $(g_s)_{s\in S}$ is a dominating family of size μ, so $\mathfrak{d}_i \le \mu$. To show $\mathfrak{b}_i \ge \lambda$, assume $(f_\alpha)_{\alpha\in\theta}$ is a sequence of P-names of length $\theta < \lambda$. For each f_α, there is a cone of upper bounds with tip $s_\alpha \in S$, i.e., $(\forall t \succ s_\alpha)\, P \Vdash f_\alpha \,\mathrm{R}_i\, g_t$. As S is $<\lambda$-directed, there is some t above all tips s_α. Accordingly, $P \Vdash f_\alpha \,\mathrm{R}_i\, g_t$ for all α; i.e., $\{f_\alpha : \alpha \in \theta\}$ is not unbounded. $\square$

So, for example, $\mathsf{COB}_1(P,\lambda,\mu)$ implies $\lambda_1 \le \mathfrak{b}_1 = \mathrm{add}(\mathcal{N})$, etc. The definition and lemma would work for $i = 2$ as well, but this would not be useful[8] as we do not have $\mathfrak{b}_2 \le \mathrm{cov}(\mathcal{N})$. So instead, we define COB_2 separately:

Definition 1.17. For P, λ and μ as above, let $\mathsf{COB}_2(P,\lambda,\mu)$ stand for the following:

> There are a $<\lambda$-directed partial order $(S,\prec)$ of size μ and a sequence $(g_s)_{s\in S}$ of P-names for reals such that for each P-name f of a null set, we have $(\exists s \in S)(\forall t \succ s)\, P \Vdash g_t \notin f$.

Lemma 1.18.

(1) $\mathsf{COB}_1(P,\lambda,\mu)$ *implies* $P \Vdash (\,\mathrm{add}(\mathcal{N}) \ge \lambda \,\&\, \mathrm{cof}(\mathcal{N}) \le \mu\,)$.
(2) $\mathsf{COB}_2(P,\lambda,\mu)$ *implies* $P \Vdash (\,\mathrm{cov}(\mathcal{N}) \ge \lambda \,\&\, \mathrm{non}(\mathcal{N}) \le \mu\,)$.
(3) $\mathsf{COB}_3(P,\lambda,\mu)$ *implies* $P \Vdash (\,\mathfrak{b} \ge \lambda \,\&\, \mathfrak{d} \le \mu\,)$.
(4) $\mathsf{COB}_4(P,\lambda,\mu)$ *implies* $P \Vdash (\,\mathrm{non}(\mathcal{M}) \ge \lambda \,\&\, \mathrm{cov}(\mathcal{M}) \le \mu\,)$.

Proof. The cases $i \ne 2$ are direct consequences of Lemmas 1.3 and 1.16. The proof for $i = 2$ is analogous to the proof of Lemma 1.16. $\square$

Lemma 1.19. $\mathsf{COB}_i(\mathbb{P}^5,\lambda_i,\lambda_5)$ *holds* (*for* $i = 1,2,3,4$).

Proof. Set $S = S^i$ and $s \prec t$ if $w_s \subsetneq w_t$. As λ_i is regular, $(S,\prec)$ is $<\lambda_i$-directed. Let g_s be the generic added at s (e.g., the partial random real in case of $i = 2$, etc). A $\mathbb{P}^5$-name f depends (in a Borel way) on the subsequence of generics indexed by a countable set $w^* \subseteq \delta$. Fix some $s \in S^i$ such that $w_s \supseteq w^*$. Pick any $t \succ s$. Then $w_t \supseteq w_s$, so w_t contains all information to calculate f, so we can show that $P \Vdash f \,\mathrm{R}_i\, g_t$. Let us list the possible cases: $i = 2$: A partial random real g_t will avoid the null set f. $i = 3$: A partial Hechler real g_t will dominate f. $i = 4$: A partial eventually different real g_t will be eventually different from f. As for $i = 1$, we use[9] Lemma 1.4. $\square$

To summarize what we know so far about $\mathbb{P}^5$,

- COB_i holds for $i = 1,2,3,4$, so the left-hand characteristics are large.

---

[8]More specifically, this definition would give us the property $g_t \notin f$ only for the null sets of the specific form $f = \{h : \neg r \,\mathrm{R}_2\, h\} = N_r$ for some $r \in 2^\omega$, whereas we will define COB_2 to deal with all names f of null sets.

[9]Alternatively, we could use, instead of amoeba, some other Suslin ccc forcing that more directly adds an R_1-dominating element of $\mathcal{C}$.

- LCU_i holds for $i = 1, 2, 4$, so the left-hand characteristics other than $\mathfrak{b}$ are small.

However, LCU_3 (corresponding to "$\mathfrak{b}$ small") is missing, and we cannot get it by a simple "preservation of $(\mathrm{R}_3, \lambda_3)$-goodness" argument. Instead, we will argue in the following two subsections that it is possible to choose the parameter $(w_\alpha)_{\alpha \in S^4}$ in such a way that LCU_3 holds as well.

1.3. *Dealing with* $\mathfrak{b}$ *without GCH.* In this subsection, we follow (and slightly modify) the main construction of [GMS16]. In this subection (and this subsection only) we will assume the following (in addition to Assumption 1.12, i.e., in particular, to the assumption $\lambda_3 = \chi^+$):

ASSUMPTION 1.20 (This subsection only). $2^\chi = |\delta_5| = \lambda_5$.

Set $S^0 = \lambda_5 \cup S^1 \cup S^2 \cup S^3$. So $\delta_5 = S^0 \cup S^4$, and $\mathbb{P}^5$ is an FS ccc iteration along δ_5 such that $\alpha \in S^0$ implies $|Q_\alpha| < \lambda_3$, i.e., $|Q_\alpha| \leq \chi$. Let us fix P_α-names

$$i_\alpha : Q_\alpha \to \chi \text{ injective} \tag{1.21}$$

(for $\alpha \in S^0$). Note that we can strengthen each $p \in \mathbb{P}^5$ to some q such that $\alpha \in \mathrm{supp}(q) \cap S^0$ implies $q \restriction \alpha \Vdash i_\alpha(q(\alpha)) = \check{\jmath}$ for some $j \in \chi$.

For $\alpha \in S^4$, Q_α is a partial eventually different forcing. At this point, we should specify which variant of this forcing we actually use.[10]

Definition 1.22.

- Eventually different forcing $\mathbb{E}$ consists of all tuples (s, k, φ), where $s \in \omega^{<\omega}$, $k \in \omega$, and $\varphi : \omega \to [\omega]^{\leq k}$ satisfies $s(i) \notin \varphi(i)$ for all $i \in \mathrm{dom}(s)$.
- We define $(s', k', \varphi') \leq (s, k, \varphi)$ if $s \subseteq s'$, $k \leq k'$, and $\varphi(i) \subseteq \varphi'(i)$ for all i.
- The generic object $g^* = \bigcup_{(s,k,\varphi) \in G_{\mathbb{E}}} s$ is a function such that each condition (s, k, φ) forces that s is an initial segment of g^*, and $g^*(i) \notin \varphi(i)$ for all i.
- We call $s \in \omega^{<\omega}$ the "stem" of (s, k, φ) and $k \in \omega$ the "width."

A density argument shows that g^* will be eventually different from all functions $f : \omega \to \omega$ from V.

The following is easy to see:

- If $p, q \in \mathbb{E}$ are compatible, then they have a greatest lower bound.
- Any finite set of conditions with the same stem has a lower bound (again with the same stem). So $\mathbb{E}$ is σ-centered.
- If $q = (s', k', \varphi')$ and $p = (s, k, \varphi)$ and s' extends s, then p and q are compatible if and only if $s'(i) \notin \varphi(i)$ for all $i \in \mathrm{dom}(s')$.

[10] In the previous subsection it did not matter which variant we use.

- If a condition $q^* = (s^*, k^*, \varphi^*)$ is compatible with each condition in a finite set $B \subseteq \mathbb{E}$, and s^* extends s for each $(s, k, \varphi) \in B$, then the set $B \cup \{q^*\}$ has a lower bound. (Use s^* as stem, and take the pointwise union of all φ that occur in $B \cup \{q^*\}$.)

We will not force with $\mathbb{E}$, but with a partial version of $\mathbb{E}$. In the P_α-extension (for $\alpha \in S^4$), this partial forcing $Q_\alpha = \mathbb{E}'$ is a (generally not complete) subforcing of $\mathbb{E}$ that is easily seen to be closed under conjunctions (i.e., under the partial operation "greatest lower bound" of finite sets of conditions). Note that this implies that compatibility is absolute between $\mathbb{E}$ and $\mathbb{E}'$, and that the previous items also hold for $\mathbb{E}'$. For later reference, let us explicitly state the last item:

FACT 1.23. *Assume $\mathbb{E}' \subseteq \mathbb{E}$ is closed under conjunctions. If a condition $q^* = (s^*, k^*, \varphi^*)$ in $\mathbb{E}'$ is compatible with each condition in a finite set $B \subseteq \mathbb{E}'$, and s^* extends s for each $(s, k, \varphi) \in B$, then the set $B \cup \{q^*\}$ has a lower bound in $\mathbb{E}'$.*

Definition 1.24. Let D be a non-principal ultrafilter on ω, and let $\bar{p} = (p_n)_{n\in\omega} = (s, k, \varphi_n)_{n\in\omega}$ be a sequence of conditions in $\mathbb{E}$ with the same stem and the same width. We define $\lim_D \bar{p}$ to be (s, k, φ_∞), where for all i and all j we have $j \in \varphi_\infty(i) \Leftrightarrow \{n : j \in \varphi_n(i)\} \in D$.

The following is easy to see: $\lim_D \bar{p} \in \mathbb{E}$ and if $q \leq \lim_D \bar{p}$, then the set $B := \{n \in \omega : p_n \text{ compatible with } q\}$ is in D. (*Proof.* Note that $q = (s', k', \varphi') \leq \lim_D \bar{p} = (s, k, \varphi_\infty)$. So for each $i \in \mathrm{dom}(s')$, $s'(i) \notin \varphi_\infty(i)$, and by the definition of the limit, $A^i := \{n : s'(i) \notin \varphi_n(i)\} \in D$. If $n \in \bigcap_{i\in\mathrm{dom}(s')} A^i$, then p_n is compatible with q.)

As B is defined using only compatibility, the statement still holds for compatibility preserving subforcings. We state it for later reference in the following form:

FACT 1.25. *Assume that $\mathbb{E}'$ is a subforcing of $\mathbb{E}$ closed under conjunctions, let $\bar{p}$ be a sequence of $\mathbb{E}'$ conditions with the same stem and width, and assume that $\lim_D(\bar{p}) \in \mathbb{E}'$ and that $q \leq_{\mathbb{E}'} \lim_D(\bar{p})$. Then $B := \{n \in \omega : p_n \text{ compatible with } q\}$ is in D.*

Definition 1.26.

- A "partial guardrail" is a function h defined on a subset of δ_5 such that $h(\alpha) \in \chi$ for $\alpha \in S^0 \cap \mathrm{dom}(h)$, and $h(\alpha) \in \omega^{<\omega} \times \omega$ for $\alpha \in S^4 \cap \mathrm{dom}(h)$.
- A "countable guardrail" is a partial guardrail with countable domain. A "full guardrail" is a partial guardrail with domain δ_5.

We will use the following lemma, which is a consequence of the Engelking-Karłowicz theorem [EK65] on the density of box products (cf. [GMS16, 5.1]):

Lemma 1.27 (As $|\delta_5| \le 2^\chi$ and $\chi^{\aleph_0} = \chi$). *There is a family H^* of full guardrails with $|H^*| = \chi$, such that each countable guardrail is extended by some $h \in H^*$. We will fix such an H^* and enumerate it as $(h^*_\varepsilon)_{\varepsilon\in\chi}$.*

Note that the notion of guardrail (and the density property required in Lemma 1.27) only depends on χ, δ_5, S^0 and S^4, i.e., on fixed parameters. Thus we can fix an H^* that will work for all cofinal parameters $\bar{w} = (w_\alpha)_{\alpha\in S^4}$.

Once we have decided on $\bar{w}$, and thus have defined $\mathbb{P}^5$, we can define the following:

Definition 1.28. A condition $p \in \mathbb{P}^5$ follows the full guardrail h if

- for all $\alpha \in S^0 \cap \mathrm{dom}(p)$, the empty condition of P_α forces that $p(\alpha) \in Q_\alpha$ and $i_\alpha(p(\alpha)) = h(\alpha)$ (where i_α is defined in (1.21)), and
- for all $\alpha \in S^4 \cap \mathrm{dom}(p)$,
  - $p\restriction\alpha$ forces that the pair of stem and width of $p(\alpha)$ is equal to $h(\alpha)$ and, moreover,
  - p is determined at α.[11]

As we are dealing with an FS iteration, the set of conditions p determined at each position $\alpha \in \mathrm{dom}(p)$ is easily seen to be dense (by induction). So note that

- the set of conditions p such that there is *some* guardrail h such that p follows h, is dense; while
- for each *fixed* guardrail h, the set of all conditions p following h is *centered* (i.e., each finitely many such p are compatible).

Definition 1.29. • A "Δ-system with root ∇ following the full guardrail h" is a family $\bar{p} = (p_i)_{i\in I}$ of conditions all following h, where $(\mathrm{dom}(p_i) : i \in I)$ is a Δ-system with root ∇ in the usual sense (so $\nabla \subseteq \delta_5$ is finite).

- We will be particularly interested in countable Δ-systems. Let $(p_n : n \in \omega)$ be such a Δ-system with root ∇ following h, and assume that $\bar{D} = (D_\alpha : \alpha \in u)$ is a sequence such that $u \supseteq \nabla \cap S^4$ and each D_α is a P_α-name of an ultrafilter on ω. Then we define the $\lim_{\bar{D}} \bar{p}$ to be the following function with domain ∇:
  - If $\beta \in \nabla \cap S^0$, then $\lim_{\bar{D}} \bar{p}(\beta)$ is the common value of all $p_n(\beta)$. (Recall that this value is already determined by the guardrail h.)
  - If $\alpha \in \nabla \cap S^4$, then $\lim_{\bar{D}} \bar{p}(\alpha)$ is (forced by $\mathbb{P}^5_\alpha$ to be) $\lim_{D_\alpha}(p_n(\alpha))_{n\in\omega}$.

Note that in general, $\lim_{\bar{D}} \bar{p}$ will not be a condition in $\mathbb{P}^5$: For $\alpha \in S^4 \cap \nabla$, the object $\lim_{\bar{D}} \bar{p}(\alpha)$ will be forced to be in the eventually different forcing $\mathbb{E}$, but not necessarily in the *partial* eventually different forcing $Q_\alpha \subseteq \mathbb{E}$.

[11] This was defined in 1.11(c); we already know in V a code (B, u) that evaluates to $p(\alpha)$.

Also note the following: If $\bar p$ is a countable Δ-system, and $\alpha \in \nabla \cap S^4$, then $(p_n(\alpha))_{n\in\omega}$ is a ground-model-code-sequence (see Definition 1.11(d)). This follows trivially from the definition of "p_n follows h" and the fact that $\bar p$ is in V.

Recall that we assume all of the parameters defining $\mathbb{P}^5 = (P_\alpha, Q_\alpha)_{\alpha\in\delta_5}$ to be fixed, apart from $(w_\alpha)_{\alpha\in S^4}$. Once we fix w_α for $\alpha \in S^4 \cap \beta$, we know P_β.

Lemma/Construction 1.30. *We can construct by induction on $\alpha \in \delta_5$ the sequences $(D^\varepsilon_\alpha)_{\varepsilon\in\chi}$ and, if $\alpha \in S^4$, also w_α, such that*

(a) *Each D^ε_α is a P_α-name of a nonprincipal ultrafilter extending $\bigcup_{\beta<\alpha} D^\varepsilon_\beta$.*
(b) *For each countable Δ-system $\bar p$ in P_α that follows the guardrail $h^*_\varepsilon \in H^*$, $\lim_{(D^\varepsilon_\beta)_{\beta<\alpha}} \bar p$ is in P_α $\cdots$*
(c) *$\cdots$ and forces that $A_{\bar p} := \{n \in \omega : p_n \in G_\alpha\}$ is in D^ε_α.*
(d) *(If $\alpha \in S^4$) $w_\alpha \subseteq \alpha$, $|w_\alpha| < \lambda_4$, and for all ground-model-code-sequences*[12] *for elements of Q_α, the D^ε_α-limit is forced to be in Q_α as well (for all $\varepsilon \in \chi$).*

(Actually, the set of w_α satisfying this is an ω_1-club set in $[\alpha]^{<\lambda_4}$.[13]*)*

Proof. (b) *for α limit*: The root of a Δ-system is finite and therefore below some $\beta < \alpha$, so the limit exists (by induction) already in P_β.

(a) *and* (c) *for α limit*: It is enough to show, for each $\varepsilon \in \chi$, that P_α forces that the following generates a proper filter (i.e., any finite intersection of elements of this set is nonempty):

$$\bigcup_{\beta<\alpha} D^\varepsilon_\beta \ \cup\ \{A_{\bar p} : \bar p \text{ is a countable } \Delta\text{-system following } h^*_\varepsilon \text{ and } \lim_{(D^\varepsilon_\beta)_{\beta<\alpha}} \bar p \in G_\alpha\}.$$

(Then we let D^ε_α be any ultrafilter extending this set.)

So assume towards a contradiction that $q \in P_\alpha$ forces that $A \cap A_{\bar p^0} \cap \cdots \cap A_{\bar p^{n-1}} = \emptyset$, where $A \in D^\varepsilon_{\beta_0}$ for some $\beta_0 < \alpha$ (we can assume β_0 is already decided in V) and $\bar p^i$ as above with $q \leq \lim_{(D^\varepsilon_\beta)_{\beta<\alpha}} \bar p^i$ for $i < n$. Let $\beta_1 < \alpha$ be the maximum of the union of the roots of the $\bar p^i$, and set $\beta_2 := \max(\mathrm{supp}(q))$ and $\gamma := \max(\beta_0, \beta_1, \beta_2)+1$. By the induction hypothesis, q forces $A' := A \cap \bigcap_{i<n} A_{\bar p^i \restriction \gamma} \in D^\varepsilon_\gamma$ (as $\lim_{(D^\varepsilon_\beta)_{\beta<\gamma}} \bar p^i \restriction \gamma = \lim_{(D^\varepsilon_\beta)_{\beta<\alpha}} \bar p^i$, since the root lies below γ). As A' is a P_γ-name, we can find $q' \leq q$ in P_γ and $\ell \in \omega$ such that $q' \Vdash \ell \in A'$. We now find $q'' \leq q'$ in P_α by defining $q''(\beta)$ for each element β of the finite set $\bigcup_{i<n} \mathrm{supp}(p^i_\ell) \setminus \gamma$. For such β in S^0, the guardrail gives a specific value $h^*_\varepsilon(\beta) \in Q_\beta$, which we use for $q''(\beta)$ as well. For

[12] See Definition 1.11(d).

[13] That is, for each $w^* \in [\alpha]^{<\lambda_4}$, there is a $w_\alpha \supseteq w^*$ satisfying (d), and if $(w^i)_{i\in\omega_1}$ is an increasing sequence of sets satisfying (d), then the limit $w_\alpha := \bigcup_{i\in\omega_1} w^i$ satisfies (d) as well.

$\beta \in S^4$, all conditions $p^i_\ell(\beta)$ (where defined) have the same stem and width $h^*_\varepsilon(\beta)$; hence there is a common extension $q''(\beta)$.

Clearly q'' forces that ℓ is in the allegedly empty set, the desired contradiction.

(b) *for* $\alpha = \gamma + 1$ *successor*: Assume the nontrivial case, $\gamma \in S^4$. Write the Δ-system as $(p_i, q_i)_{i\in\omega}$ with $(p_i, q_i) \in P_\gamma * Q_\gamma$. As noted above, $(q_n)_{n\in\omega}$ is a ground-model-code-sequence, and by induction (d) holds for w_γ. So it is forced that the D^ε_γ-limit q^* of the q_n is in Q_γ. Again by induction, the limit p^* of the p_n exists as well, and (p^*, q^*) is the required limit.

(a) *and* (c) *for* $\alpha = \gamma+1$ *successor*: We again have to show that P_α forces that the following is a filter base for each $\varepsilon \in \chi$:

$$D^\varepsilon_\gamma \cup \{A_{\bar p} : \bar p \text{ is a countable } \Delta\text{-system following } h^*_\varepsilon \text{ and } \lim_{(D^\varepsilon_\beta)_{\beta<\alpha}} \bar p \in G_\alpha\}.$$

As above, assume that q forces $A \cap A_{\bar p^0} \cap \cdots \cap A_{\bar p^{n-1}} = \emptyset$.

We can assume that $q \restriction \gamma$ forces that $q(\gamma)$ is stronger than the limit of all $\bar p^i(\gamma)$ (for $i < n$). Thus, by Fact 1.25, each $B_i := \{\ell \in \omega : q(\gamma) \text{ compatible with } p^i_\ell(\gamma)\}$ is forced to be in D^ε_γ.

By induction, $q \restriction \gamma$ forces that $A' := A \cap \bigcap_{i<n} A_{\bar p^i \restriction \gamma} \in D^\varepsilon_\gamma$, and therefore it also forces that $B' = A' \cap \bigcap_{i<n} B_i$ is in the ultrafilter and, in particular, nonempty. Work in the P_γ-extension by some generic filter containing $q \restriction \gamma$. Fix some $\ell \in B'$. By the definition of B_i, $q(\gamma)$ is compatible with each $p^i_\ell(\gamma)$ for $i < n$. According to Fact 1.23 there is a common lower bound q''.

Note that $q \restriction \gamma \Vdash_{P_\gamma} q'' \Vdash_{Q_\gamma} \ell \in A_{\bar p^i}$. That is, $q \restriction \gamma * q'' \leq q$ forces that ℓ is an element of the allegedly empty set.

(d) For any $w \subseteq \alpha$, let Q^w be the (P_α-name for) the partial eventually different forcing defined using w. Start with some $w^0 \subseteq \alpha$ of size $<\lambda_4$. There are $|w^0|^{\aleph_0}$ many ground-model sequences in Q^{w^0}. For any ε and any such sequence, the D^ε_α-limit is a real; so we can extend w^0 by a countable set to some w' such that $Q^{w'}$ contains the limit. We can do that for all $\varepsilon \in \chi$ and all sequences, resulting in some $w^1 \supseteq w^0$ still of size $<\lambda_4$. We iterate this construction and get w^i for $i \leq \omega_1$, taking the unions at limits. Then $w_\alpha := w^{\omega_1}$ is as required, as $Q_\alpha := Q^{w_\alpha} = \bigcup_{i<\omega_1} Q^{w_i}$.

So this proof actually shows that the set of w_α with the desired property is an ω_1-club. □

After carrying out the construction of this lemma, we get a forcing notion $\mathbb{P}^5$ satisfying the following:

LEMMA 1.31. $\mathsf{LCU}_3(\mathbb{P}^5, \kappa)$ *for* $\kappa \in [\lambda_3, \lambda_5]$, *witnessed by the sequence* $(c_\alpha)_{\alpha<\kappa}$ *of the first* κ *many Cohen reals.*

Proof. We want to show that for every $\mathbb{P}^5$-name y, there are coboundedly many $\alpha \in \kappa$ such that $\mathbb{P}^5 \Vdash \neg c_\alpha \leq^* y$.

Assume that p^* forces that there are unboundedly many $\alpha \in \kappa$ with $c_\alpha \leq^* y$, and enumerate them as $(\alpha_i)_{i\in\kappa}$ in increasing order (so, in particular, $\alpha_i \geq i$). Pick $p_i \leq p^*$ deciding α_i to be some β_i, and also deciding n_i such that $(\forall m \geq n_i)\, c_{\alpha_i}(m) \leq y(m)$. We can assume that $\beta_i \in \mathrm{dom}(p_i)$. Note that β_i is a Cohen position (as $\beta_i < \kappa \leq \lambda_5$), and we can assume that $p_i(\beta_i)$ is a Cohen condition in V (and not just a P_{β_i}-name for such a condition). By thinning out, we may assume

- all n_i are equal to some n^*;
- $(p_i)_{i\in\kappa}$ forms a Δ-system with root ∇;
- $\beta_i \notin \nabla$, hence all β_i are distinct.

(For any $\beta \in \kappa$, at most $|\beta|$ many p_i can force $\alpha_i = \beta$, as p_i forces that $\alpha_i \geq i$ for all i.)

- $p_i(\beta_i)$ is always the same Cohen condition s, without loss of generality of length $n^{**} \geq n^*$; otherwise extend s.

Pick the first ω many elements $(p_i)_{i\in\omega}$ of this Δ-system. Now extend each p_i to p'_i by extending the Cohen condition $p_i(\beta_i) = s$ to $s^\frown i$ (i.e., forcing $c_{\alpha_i}(n^{**}) = i$). Note that $(p'_i)_{i\in\omega}$ is still a countable Δ-system, following some new countable guardrail and therefore some full guardrail $h^*_\varepsilon \in H^*$.

Accordingly, the limit $\lim_{(D^\varepsilon_\alpha)_{\alpha\in\delta_5}} \bar{p}'$ forces that infinitely many of the p'_i are in the generic filter. But each such p'_i forces that $c_{\alpha_i}(n^{**}) = i \leq y(n^{**})$, a contradiction. □

1.4. *Recovering GCH.* For the rest of the paper we will assume the following for the ground model V (in addition to Assumption 1.12):

ASSUMPTION 1.32. *GCH holds.*

(Note that this is incompatible with Assumption 1.20.)

Recall that all parameters used to define $\mathbb{P}^5$ are fixed, apart from $\bar{w} = (w_\alpha)_{\alpha\in S^4}$.

LEMMA 1.33. *We can choose $\bar{w}$ such that $\mathsf{LCU}_3(\mathbb{P}^5, \kappa)$ holds for all regular $\kappa \in [\lambda_3, \lambda_5]$.*

For the proof, we will use the following easy observation:

LEMMA 1.34. *Assume χ is a cardinal and B a set and $X^0 \in [B]^\chi$, $\mathbb{R}$ is a χ^+-cc forcing notion, and C is an $\mathbb{R}$-name such that the empty condition forces that C is an ω_1-club subset of $[B]^\chi$. Then there is a set $X \supseteq X^0$ (in the ground model) such that the empty condition forces $X \in C$.*

Proof. By induction, choose (in the ground model) sequences $X^\alpha, \tilde{X}^\alpha$ for $\alpha < \omega_1$ such that X^α is in $[B]^\chi$, the sequence of the X^α is increasing with α, $\tilde{X}^\alpha$ is an R-name, and the empty condition forces the following: "$\tilde{X}^\alpha$ is in C and is a superset of X^α, and the sequence of the $\tilde{X}^\alpha$ is increasing (not necessarily continuous)." Moreover, the empty condition forces $\tilde{X}^\alpha \subseteq X^{\alpha+1}$. (In a limit step γ, we set $X^\gamma = \bigcup_{\alpha<\gamma} X^\alpha$, and in a successor step $\alpha+1$, we use χ^+-cc to cover the name $\tilde{X}^\alpha$.) Then $X = \bigcup_{\alpha\in\omega_1} X^\alpha$ is as required. $\square$

Proof of Lemma 1.33. Let $\mathbb{R}$ be a $<\chi$-closed χ^+-cc partial order that forces $2^\chi = \lambda_5$. In the $\mathbb{R}$-extension V^*, Assumption 1.20 holds, and Assumption 1.12 still holds for the fixed parameters.[14]

So in V^*, we can perform the inductive Construction 1.30, where now "ground model" refers to V^*, not V (e.g., when we talk about determined positions, or ground-model-code-sequences, etc.). Actually, we can construct in V the following, by induction on $\alpha \in \delta_5$, and starting with some cofinal $\bar{w}^{\text{initial}} = (w_\alpha^{\text{initial}})_{\alpha\in S^4}$ in V,

- An $\mathbb{R}$-name $(D_\alpha^\varepsilon)_{\varepsilon\in\chi}$ (forced to be constructed) according to 1.30(a,b,c).
- If $\alpha \in S^4$, some $w_\alpha \supseteq w_\alpha^{\text{initial}}$ in V such that $\mathbb{R}$ forces w_α satisfies 1.30(d). (We can do this by Lemma 1.34, as the set of potential w_α's is an ω_1-clubset of $[\alpha]^{<\lambda_4}$.)

So we get in V a cofinal parameter $\bar{w}$ satisfying the following: In the $\mathbb{R}$-extension V^*, the same parameters define a forcing (call it $\mathbb{P}^{*,5}$) satisfying $\mathsf{LCU}_3(\mathbb{P}^{*,5}, \kappa)$ in V^*.

$\mathbb{P}^{*,5}$ is basically the same as $\mathbb{P}^5$. More formally,

> In the $\mathbb{R}$-extension V^*, $\mathbb{P}^5 = (P_\alpha, Q_\alpha)_{\alpha<\delta_5}$ (the iteration constructed in V) is canonically densely embedded into $\mathbb{P}^{*,5} = (P_\alpha^*, Q_\alpha^*)_{\alpha<\delta_5}$ (the iteration constructed in V^* using the same parameters).

Proof. By induction, we show (in the $\mathbb{R}$-extension) that P_α^* forces that Q_α^* (evaluated by the P_α^*-generic) is equal to Q_α (evaluated by the induced P_α-generic, as per induction hypothesis). Every element of Q_α^* is a Borel function (which already exists in V) applied to the generics at a countable sequence of indices in w_α (which also already exists in V).

This implies

> In V, $\mathsf{LCU}_3(\mathbb{P}^5, \kappa)$ holds for all $\kappa \in [\lambda_3, \lambda_5]$, witnessed by the first κ many Cohen reals.

[14] In particular, $(w_\alpha)_{\alpha\in S^i}$ is still cofinal in $[\delta_5]^{<\lambda_i}$: For $i = 1, 2$, the forcing $\mathbb{R}$ does not add any new elements of $[\delta_5]^{<\lambda_i}$ as $\mathbb{R}$ is λ_i-closed; for $i = 3$, any new subset of δ_5 of size $\theta < \lambda_3$ is contained in a ground model set of size at most $\theta \times \chi < \lambda_3$, as $\mathbb{R}$ is χ^+-cc.

Proof. Let y be a $\mathbb{P}^5$-name of a real. In V^*, we can interpret y as $\mathbb{P}^{*,5}$-name, and as $\mathsf{LCU}_3(\mathbb{P}^{*,5},\kappa)$ holds, we get $(\exists\alpha\in\kappa)\,(\forall\beta\in\kappa\setminus\alpha)\mathbb{P}^{*,5}\Vdash c_\beta\not\leq^* y$, where c_β is the Cohen added at β. As $\chi<\kappa$, there is in V an upper bound $\alpha^*<\kappa$ for the possible values of α. For any $\beta\in\kappa\setminus\alpha^*$, we have (in V) $\mathbb{P}^5\Vdash c_\beta\not\leq^* y$ (by absoluteness). □

To summarize,

THEOREM 1.35. *Assuming GCH and given λ_i as in Assumption 1.12, we can find parameters*[15] *such that the FS ccc iteration $\mathbb{P}^5$ as defined in 1.13 satisfies, for $i=1,2,3,4$,*

- $\mathsf{LCU}_i(\mathbb{P}^5,\kappa)$ *holds for any regular cardinal κ in $[\lambda_i,\lambda_5]$;*
- $\mathsf{COB}_i(\mathbb{P}^5,\lambda_i,\lambda_5)$ *holds.*

So, in particular, $\mathbb{P}^5$ *forces* $\mathrm{add}(\mathcal{N})=\lambda_1$, $\mathrm{cov}(\mathcal{N})=\lambda_2$, $\mathfrak{b}=\lambda_3$, $\mathrm{non}(\mathcal{M})=\lambda_4$ *and* $\mathrm{cov}(\mathcal{M})=\mathfrak{d}=\mathrm{non}(\mathcal{N})=\mathrm{cof}(\mathcal{N})=\lambda_5=2^{\aleph_0}$.

For the rest of the paper, we fix these parameters and thus the forcing $\mathbb{P}^5$.

2. Boolean ultrapowers

In Sections 2.1 and 2.2 we describe how to get an elementary embedding (which we call a *BUP-embedding*) $j:V\to M$ with $\mathrm{cr}(j)=\kappa$ and $\mathrm{cf}(j(\kappa))=|j(\kappa)|=\theta$, assuming κ is strongly compact and $\theta>\kappa$ is a regular cardinal with $\theta^\kappa=\theta$.

In Sections 2.3 and 2.4 we show how to use such embeddings to transform a ccc forcing P to $j(P)$ while preserving some of the values forced to the entries of Cichoń's diagram (and changing others).

2.1. *Boolean ultrapowers.* Boolean ultrapowers generalize ordinary ultrapowers by using arbitrary Boolean algebras instead of the power set algebra.

We assume that κ is strongly compact and that B is a κ-distributive, κ^+-cc, atomless complete Boolean algebra. Then every κ-complete filter in B can be extended to a κ-complete ultrafilter U.[16] Also, there is a maximal antichain A_0 in B of size κ such that $A_0\cap U=\emptyset$ (i.e., U is not κ^+-complete).[17] For now, fix some κ-complete ultrafilter U.

The Boolean algebra B can be used as forcing notion. As usual, V (or the ground model) denotes the universe we "start with." In the following, we will not actually force with B (and in this subsection and the following subsection,

[15] That is, we set $\delta_5=\lambda_5+\lambda_5$, and we find $(S^i)_{i=1,\dots,4}$ and $\bar w=(w_\alpha)_{\alpha\in\delta_5}$.

[16] For this, neither κ^+-cc nor atomless is required, and κ-complete is sufficient. The proof is straightforward; the first proof that we are aware of has been published in [KT64].

[17] *Proof.* Let A be a maximal antichain in the open dense set $B\setminus U$, by κ^+-cc $|A|\leq\kappa$. Also, A cannot have size $<\kappa$, as otherwise it would meet the κ-complete U.

we will not force with anything, rather we always remain in V), but we still use forcing notation. In particular, we call the usual B-names "forcing names."

A *BUP-name* (or labeled antichain) x is a function $A \to V$ whose domain is a maximal antichain of B. We may write $A(x)$ to denote A. Each BUP-name corresponds to a forcing name[18] for an element of V. We will identify the BUP-name and the corresponding forcing name. In turn, every forcing name τ for an element of V has a forcing-equivalent BUP-name. In particular, there is a *standard BUP-name* $\check{v}$ for each $v \in V$.

We can calculate, for two BUP-names x and y, the Boolean value $[\![x = y]\!]$. We call x and y *equivalent* if $[\![x = y]\!] \in U$ (the κ-complete ultrafilter fixed above).

For example, any two standard BUP-names for the same $v \in V$ trivially are equivalent (as $\mathbb{1}_B \in U$). So we can speak (modulo equivalence) of *the* standard BUP-name for v.

The *Boolean ultrapower* M^- consists of the equivalence classes $[x]$ of BUP-names x; and we define $[x] \in^- [y]$ by $[\![x \in y]\!] \in U$. We are interested in the $\in$-structure $(M^-, \in^-)$. We let $j^- : V \to M^-$ map v to $[\check{v}]$.

Given BUP-names $x_1, \dots, x_n$ and an $\in$-formula φ, there is a well-defined truth value $[\![\varphi^V(x_1, \dots, x_n)]\!]$. (It is the weakest element of B forcing that in the ground model $\varphi(x_1, \dots, x_n)$ holds, which makes sense as $x_1, \dots, x_n$ are guaranteed to be in the ground model.)

A straightforward induction (which can be found in [KTT18, §2]) shows

- Łoś's theorem: $(M^-, \in^-) \models \varphi([x_1], \dots, [x_n])$ if and only if $[\![\varphi^V(x_1, \dots, x_n)]\!] \in U$;
- $j^- : (V, \in) \to (M^-, \in^-)$ is an elementary embedding;
- in particular, $(M^-, \in^-)$ is a ZFC model.

As U is σ-complete, $(M^-, \in^-)$ is well-founded. So we let M be the transitive collapse of $(M^-, \in^-)$, and let $j : V \to M$ be the composition of j^- with the collapse. We denote the collapse of $[x]$ by x^U. So, in particular, $\check{v}^U = j(v)$.

FACTS 2.1.

- *$M \models \varphi(x_1^U, \dots, x_n^U)$ if and only if $[\![\varphi^V(x_1, \dots, x_n)]\!] \in U$. In particular, $j : V \to M$ is an elementary embedding.*
- *If $|Y| < \kappa$, then $j(Y) = j''Y$. In particular, j restricted to κ is the identity. M is closed under $<\kappa$-sequences.*
- *$j(\kappa) \neq \kappa$, i.e., $\kappa = \mathrm{cr}(j)$.*

As we have already mentioned, an arbitrary forcing name for an element of V has a forcing-equivalent BUP-name, i.e., a maximal antichain labeled with

[18] More specifically, to the forcing name $\{(\widecheck{x(a)}, a) : a \in A(x)\}$.

elements of V. If τ is a forcing name for an element of Y ($Y \in V$), then without loss of generality τ corresponds to a maximal antichain labeled with elements of Y. We call such an object y a "BUP-name for an element of $j(Y)$" (and not "for an element of Y," for the obvious reason: unlike in the case of a forcing extension, y^U is generally not in Y, but, by definition of $\in^-$, it is in $j(Y)$).

LEMMA 2.2. *If the partial order* $(S, \leq)$ *is* $\leq\kappa$*-directed, then* $j''S$ *is cofinal in* $j(S)$.

Proof. Let x^U be some element of $j(S)$; without loss of generality we can assume that x is a labeled antichain that only uses elements of S as labels. The size of the antichain is at most κ, so all labels have some common upper bound s_0. Then $[\![x \leq s_0]\!]$ is $\mathbb{1}_B$, and thus in U; so $(M^-, \in^-) \models [x] \leq \check{s}_0$, i.e., $j(s_0) \geq x^U$ as required. $\square$

For later reference, let us summarize what we know about j in the form of a definition.

Definition 2.3. A BUP-embedding is an elementary embedding $j : V \to M$ (M transitive) with critical point κ, such that M is $<\kappa$-closed and such that $j''S$ is cofinal in $j(S)$ for every $\leq\kappa$-directed partial order S.

So the embedding j defined as above for a κ-distributive, κ^+-cc atomless complete Boolean algebra and a κ-complete ultrafilter U is a BUP-embedding.

LEMMA 2.4. *Let* j *be a BUP-embedding with* $\mathrm{cr}(j) = \kappa$.

- *If* $|A| < \kappa$, *then* $j''A = j(A)$.
- *If* S *is a* $<\lambda$*-directed partial order for some regular* $\lambda < \kappa$, *then* $j(S)$ *is* $<\lambda$*-directed.*
- *If* $\mathrm{cf}(\alpha) \neq \kappa$, *then* $j''\alpha$ *is cofinal in* $j(\alpha)$ *and so, in particular,* $\mathrm{cf}(j(\alpha)) = \mathrm{cf}(\alpha)$.

Proof. For the second item, use that M believes that $j(S)$ is $<\lambda$-directed and that M is $<\kappa$-closed. For the last item, assume $\mathrm{cf}(\alpha) = \lambda \neq \kappa$, witnessed by some strictly increasing cofinal function $f : \lambda \to \alpha$. If $\lambda < \kappa$, then M thinks that $j(f)$ is strictly increasing cofinal from $j(\lambda) = \lambda$ to $j(\alpha)$, which is absolute. If $\lambda > \kappa$, then α is a $\leq\kappa$-directed (linear) order, so $j''\alpha$ is cofinal in $j(\alpha)$. So $j''f$, i.e., $(j(\zeta), j(f(\zeta)))_{\zeta\in\lambda}$, witnesses that $\mathrm{cf}(j''\lambda) = \mathrm{cf}(j''\alpha) = \mathrm{cf}(j(\alpha))$, and $\mathrm{cf}(j''\lambda) = \mathrm{cf}(\lambda) = \lambda$ (as these orders are isomorphic). $\square$

2.2. *The algebra and the filter.* For a strongly compact cardinal, we can get large $\mathrm{cf}(j(\kappa))$ as follows:

LEMMA 2.5. *Let* κ *be strongly compact,* $\theta > \kappa$ *and* $\mathrm{cf}(\theta) > \kappa$. *Then there is a BUP-embedding* j *with* $\mathrm{cr}(j) = \kappa$ *such that*

(1) $\mathrm{cf}(j(\kappa)) = \mathrm{cf}(\theta)$ *and* $j(\kappa) \geq \theta$;

(2) $|j(\mu)| \leq \max(\mu, \theta)^\kappa$ *for any* μ;
(3) *in particular, if* $\theta^\kappa = \theta$ *and* $\kappa \leq \mu \leq \theta$, *then* $|j(\mu)| = \theta$.

We will use this in the following form:

Definition 2.6. A "BUP-embedding from κ to θ" is a BUP-embedding j with critical point κ such that $\mathrm{cf}(j(\kappa)) = |j(\kappa)| = \theta$. (In particular, κ and θ are regular.)

The lemma immediately implies

Corollary 2.7. *Assume* κ *is strongly compact and* $\theta > \kappa$ *is a regular cardinal such that* $\theta^\kappa = \theta$. *Then there is a BUP-embedding* j *from* κ *to* θ. (*In addition,* $|j(\mu)| = \theta$ *whenever* $\kappa \leq \mu \leq \theta$.)

Proof of Lemma 2.5. Let B be the complete Boolean algebra generated by the forcing notion $P_{\kappa,\theta}$ consisting of partial functions from θ to κ with domain of size $<\kappa$, ordered by extension. Clearly B is $<\kappa$-distributive (as $P_{\kappa,\theta}$ is even $<\kappa$-closed) and κ^+-cc.

The forcing adds a canonical generic function $f^* : \theta \to \kappa$. So for each $\delta \in \theta$, $f^*(\delta)$ is a forcing name for an element of κ, and thus a BUP-name for an element of $j(\kappa)$.

Let x be some other BUP-name for an element of $j(\kappa)$, i.e., an antichain A of size κ labeled with elements of κ. As $P_{\kappa,\theta}$ is dense in $B \setminus \{\mathbb{0}_B\}$, we can assume that $A \subseteq P_{\kappa,\theta}$. Let $\delta \in \theta$ be bigger than the supremum of the domain of a for each $a \in A$. We call such a pair (x, δ) "suitable" and set $b_{x,\delta} := [\![f^*(\delta) > x]\!]$. We claim that these elements generate a κ-complete filter. To see this, fix suitable pairs (x_i, δ_i) for $i < \mu < \kappa$; we have to show that $\bigwedge_{i\in\mu} b_{x_i,\delta_i} \neq \mathbb{0}$. Enumerate $\{\delta_i : i \in \mu\}$ increasing (and without repetitions) as δ^ℓ for $\ell \in \gamma \leq \mu$. Set $A_\ell = \{i : \delta_i = \delta^\ell\}$. Given q_ℓ, define $q_{\ell+1} \in P_{\kappa,\theta}$ as follows: $q_{\ell+1} \leq q_\ell$; $\delta^\ell \in \mathrm{supp}(q_{\ell+1}) \subseteq \delta^\ell \cup \{\delta^\ell\}$; and $q_{\ell+1} \restriction \delta^\ell$ decides for all $i \in A_\ell$ the values of x_i to be some α_i; and $q_{\ell+1}(\delta^\ell) = \sup_{i\in A_\ell}(\alpha_i) + 1$. This ensures that $q_{\ell+1}$ is stronger than b_{x_i,δ_i} for $i \in A_\ell$. For any limit ordinal $\ell \leq \gamma$, let q_ℓ be the union of $\{q_k : k < \ell\}$. Then q_γ is stronger than each b_{x_i,δ_i}.

As κ is strongly compact, we can extend the κ-complete filter generated by all b_{x_i,δ_i} to a κ-complete ultrafilter U. Then the sequence $f^*(\delta)^U_{\delta\in\theta}$ is strictly increasing (as $(f^*(\delta), \delta')$ is suitable for all $\delta < \delta'$) and cofinal in $j(\kappa)$ (as we have just seen); so $\mathrm{cf}(j(\kappa)) = \mathrm{cf}(\theta)$ and $j(\kappa) \geq \theta$.

To get an upper bound for $j(\mu)$ for any cardinal μ, we count all possible BUP-names for elements of $j(\mu)$. As we can assume that the antichains are subsets of $P_{\kappa,\theta}$, which has size $\theta^{<\kappa}$, we get the upper bound $|j(\mu)| \leq [\theta^{<\kappa}]^\kappa \times \mu^\kappa = \max(\theta, \mu)^\kappa$. □

2.3. *The ultrapower of a forcing notion.* We now investigate the relation of a forcing notion $P \in V$ and its image $j(P) \in M$, which we use as forcing notion over V. (Think of P as being one of the forcings of Section 1; it has no relation with the Boolean algebra B used to construct j.)

Note that as $j(P) \in M$ and M is transitive, every $j(P)$-generic filter G over V is trivially generic over M as well, and we will use absoluteness between $M[G]$ and $V[G]$ to prove various properties of $j(P)$.

LEMMA 2.8. *Let $j : V \to M$ be elementary, M transitive and $<\kappa$-closed with $\mathrm{cr}(j) = \kappa$. Assume that P is ν-cc for some $\nu < \kappa$.*

(1) *$j(P)$ is ν-cc.*
(2) *If τ is (in V) a $j(P)$-name for an element of $M[G]$, then there is a $j(P)$-name σ in M such that the empty condition forces $\sigma = \tau$.*
(3) *In particular, every $j(P)$-name for a real, a Borel-code, a countable sequence of reals, etc., is in M (more formally: has an equivalent name in M).*
(4) *$M[G]$ is $<\kappa$-closed in $V[G]$.*
(5) *If $\xi < \kappa$ and P forces $2^\xi = \lambda$, then $j(P)$ forces $2^\xi = |j(\lambda)|$.*
(6) *$j''P$, which is isomorphic to P via j, is a complete subforcing of $j(P)$.*

Proof. (1) If $A \subseteq j(P)$ has size ν, then $A \in M$, and by elementarity M thinks that A is not an antichain, which is absolute.

(2) τ corresponds to (A, f) where $A \subseteq j(P)$ is a maximal antichain and $f : A \to M$ maps a to a $j(P)$-name in M. As $j(P)$ is ν-cc and M $<\kappa$-closed, (A, f) is in M and we can interpret in M (A, f) as a $j(P)$-name σ.

This immediately implies (3) and (4). Given a $j(P)$-name τ for a ζ-sequence of elements of $M[G]$, $\zeta < \kappa$, we can interpret τ as a ζ-sequence of names $(\tau_i)_{i<\zeta}$, and find for each τ_i an equivalent $j(P)$-name σ_i in M. As M is $<\kappa$-closed, the sequence $(\sigma_i)_{i<\zeta}$ is in M and defines a $j(P)$-name in M equivalent to τ.

(Furthermore, if τ is a $j(P)$-name for a $<\kappa$-sequence in $M[G]$, we can use the fact that κ is regular and that $j(P)$ is κ-cc to get a bound $\zeta < \kappa$ for the length of τ.)

(5) $M[G]$ thinks that $|2^\xi| = j(\lambda)$, and $2^\xi \cap V[G] = 2^\xi \cap M[G]$.

(6) It is clear that $j''P$ is an incompatibility-preserving subforcing of $j(P)$: $j(p) \leq j(q)$ in $j''P$ if and only if $p \leq q$ in P (by definition) if and only if M thinks that $j(p) \leq j(q)$ in $j(P)$ (by elementarity) if and only if this holds in V (by absoluteness). The same argument works for compatibility instead of $\leq$. Similarly, assume $A \subseteq j''P$ is a maximal antichain. By definition, $B := j^{-1}(A) \subseteq P$ is one as well and, in particular, of size $<\nu$. Therefore $j(B) = B$, and by elementarity M thinks that $B \subseteq j(P)$ is maximal, which holds in V by absoluteness. $\square$

To round off the picture, let us mention the following fact (which is, however, not required for the rest of the paper):

Lemma 2.9. *If $P = (P_\alpha, Q_\alpha)_{\alpha<\delta}$ is a finite support (FS) ccc iteration of length δ, then $j(P)$ is an FS ccc iteration of length $j(\delta)$. (More formally, it is canonically equivalent to one.)*

Proof. M certainly thinks that $j(P) = (P^*_\alpha, Q^*_\alpha)_{\alpha<j(\delta)}$ is an FS iteration of length $j(\delta)$. By induction on α, we define the FS ccc iteration $(\tilde{P}_\alpha, \tilde{Q}_\alpha)_{\alpha<j(\delta)}$ and show that P^*_α is a dense subforcing of $\tilde{P}_\alpha$. Assume this is already the case for P^*_α. Then M thinks that Q^*_α is a P^*_α-name, so we can interpret it as $\tilde{P}_\alpha$-name and use it as $\tilde{Q}_\alpha$. Assume that (p, q) is an element (in V) of $\tilde{P}_\alpha * \tilde{Q}_\alpha$. So p forces that q is a name in M; we can strengthen p to some p' that decides q to be the name $q' \in M$. By induction we can further strengthen p' to $p'' \in P^*_\alpha$, and then $(p,'' q') \in P^*_{\alpha+1}$ is stronger than (p, q). (At limits there is nothing to do, as we use FS iterations.)

According to Lemma 2.8(1), $j(P)$ is ccc. □

2.4. *Preservation of values of characteristics.* Recall Definition 1.8 of LCU_i and Definitions 1.15 and 1.17 of COB_i.

Lemma 2.10. *Assume*[19] *that P is ccc and that j is a BUP-embedding with critical point κ. Then*

(1) $\mathsf{LCU}_i(P, \delta)$ *implies* $\mathsf{LCU}_i(j(P), j(\delta))$. *Thus if $\lambda \neq \kappa$ regular, then* $\mathsf{LCU}_i(P, \lambda)$ *implies* $\mathsf{LCU}_i(j(P), \lambda)$.

(2) *Assume* $\mathsf{COB}_i(P, \lambda, \mu)$. *If $\kappa > \lambda$, then* $\mathsf{COB}_i(j(P), \lambda, |j(\mu)|)$; *if $\kappa < \lambda$, then* $\mathsf{COB}_i(j(P), \lambda, \mu)$.

Proof. (1) Let $\bar{x} = (x_\alpha)_{\alpha<\delta}$ be the sequence of P-names that witnesses $\mathsf{LCU}_i(P, \delta)$. So M thinks the following: For every $j(P)$-name y of a real, we have

$$(\exists \alpha \in j(\delta))\, (\forall \beta \in j(\delta) \setminus \alpha)\, \neg \big((j(\bar{x}))_\beta \, \mathrm{R}_i \, y\big).$$

This is absolute, so $j(\bar{x})$ witnesses $\mathsf{LCU}_i(j(P), j(\delta))$.

The second claim follows from the fact that $\mathsf{LCU}_i(j(P), j(\delta))$ is equivalent to $\mathsf{LCU}_i(j(P), \mathrm{cf}(j(\delta)))$ and that $\mathrm{cf}(j(\lambda)) = \lambda$ for regular $\lambda \neq \kappa$.

(2) Let $(S, \prec)$ and $\bar{g}$ witness $\mathsf{COB}_i(P, \lambda, \mu)$. M thinks that

(∗)

for each $j(P)$-name f, $(\exists s \in j(S))\, (\forall t \in j(S))\, (\, t \succ s \to j(P) \Vdash f \, \mathrm{R}_i \, j(\bar{g})_t \,)$

(or, in the case $i = 2$, $j(P) \Vdash j(\bar{g})_t \notin f$, where f is the name of a null set). This is true in V as well: If f is a $j(P)$-name for a real, then we can assume $f \in M$, and so we can find $s \in j(S)$ such that for all $t \succ s$, $M[G] \models f \, \mathrm{R}_i \, j(\bar{g})_t$, which holds in $V[G]$ as well, as R_i is absolute.

[19]For most of the lemma, the requirements of Lemma 2.8 are sufficient. We use ccc only to simplify notation as we do not have to indicate where we calculate cofinalities (in V or the $j(P)$ extensions $V[G]$). We need BUP-embedding for the last part of (2) only.

If $\lambda < \kappa$, then $j(\lambda) = \lambda$, and $j(S)$ is λ-directed in M and therefore in V as well, so we get $\mathsf{COB}_i(j(P), \lambda, |j(\mu)|)$.

So assume $\lambda > \kappa$. We claim that $j''(S)$ and $j''\bar{g}$ witness $\mathsf{COB}_i(j(P), \lambda, \mu)$. Since $j''S$ is isomorphic to S, directedness is trivial. Given a $j(P)$-name f of a real, without loss of generality in M, there is in M a cone with tip $s \in j(S)$ as in $(*)$. As $j''S$ is cofinal in $j(S)$, there is some $s' \in S$ such that $j(s') \succ s$. Then for all $t \succ s'$, i.e., $j(t) \succ j(s')$, we get $j(P) \Vdash f \, \mathrm{R}_i \, j(g_t)$ (or, in case $i = 2$, $j(P) \Vdash j(g_t) \notin f$). □

We list the specific cases that we will use:

Corollary 2.11. *Let j be a BUP-embedding from κ to θ.*
(a) $\mathsf{LCU}_i(P, \lambda)$ *for a regular* $\lambda \neq \kappa$ *implies* $\mathsf{LCU}_i(j(P), \lambda)$.
(b) $\mathsf{LCU}_i(P, \kappa)$ *implies* $\mathsf{LCU}_i(j(P), \theta)$.
(c) $\mathsf{COB}_i(P, \lambda, \mu)$ *for* $\kappa > \lambda$ *and* $\kappa \leq \mu \leq \theta$ *implies* $\mathsf{COB}_i(j(P), \lambda, \theta)$.
(d) $\mathsf{COB}_i(P, \lambda, \mu)$ *for* $\kappa < \lambda$ *implies* $\mathsf{COB}_i(j(P), \lambda, \mu)$.

3. **A finite iteration of BUP-embeddings**

We now have everything required for the main result.

Theorem 3.1. *Assume GCH and that $\aleph_1 < \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 < \kappa_6 < \lambda_4 < \lambda_5 < \lambda_6 < \lambda_7 < \lambda_8 < \lambda_9$ are regular, λ_3 is a successor of a regular cardinal, λ_i is not successor of a cardinal with countable cofinality for $i = 1, 2, 4, 5$, and κ_i strongly compact for $i = 6, 7, 8, 9$. Then there is a ccc forcing notion $\mathbb{P}^9$ resulting in*

$$\begin{aligned} \mathrm{add}(\mathcal{N}) = \lambda_1 < \mathrm{cov}(\mathcal{N}) = \lambda_2 < \mathfrak{b} = \lambda_3 < \mathrm{non}(\mathcal{M}) = \lambda_4 < \mathrm{cov}(\mathcal{M}) \\ = \lambda_5 < \mathfrak{d} = \lambda_6 < \mathrm{non}(\mathcal{N}) = \lambda_7 < \mathrm{cof}(\mathcal{N}) = \lambda_8 < 2^{\aleph_0} = \lambda_9. \end{aligned}$$

Proof. For $i = 6, \dots, 9$, let j_i be a BUP-embedding from κ_i to λ_i, i.e., $\mathrm{cf}(j_i(\kappa_i)) = |j_i(\lambda_i)| = \lambda_i$. (Such an embedding exists according to Corollary 2.7.)

We use $\mathbb{P}^5$ of Theorem 1.35 and set $\mathbb{P}^{i+1} := j_{i+1}(\mathbb{P}^i)$ for $i = 5, 6, 7, 8$. In particular, $\mathbb{P}^9 = j_9(j_8(j_7(j_6(\mathbb{P}^5))))$.

We enumerate the relevant characteristics of Cichoń's diagram as $\mathfrak{x}_1, \dots, \mathfrak{x}_8$ in the desired increasing order as displayed in Figure 1. For $i = 1, \dots, 4$ (i.e., $\mathfrak{x}_i$ in the left half), we set $i^* := 9 - i$ (so $\mathfrak{x}_{i^*}$ is the dual of $\mathfrak{x}_i$ in the right half).

Recall that according to Lemmas 1.9 and 1.18, $\mathsf{LCU}_i(\lambda)$ implies $\mathfrak{x}_i \leq \lambda$ and $\mathfrak{x}_{i^*} \geq \lambda$. Furthermore, $\mathsf{COB}_i(\lambda, \mu)$ implies $\mathfrak{x}_i \geq \lambda$ and $\mathfrak{x}_{i^*} \leq \mu$.

Claim. $\mathbb{P}^9$ *forces* $2^{\aleph_0} = \lambda_9$.

Proof. By induction on $i = 5, \dots, 8$, each $\mathbb{P}^{i+1}$ forces $2^{\aleph_0} = j_{i+1}(\lambda_i) = \lambda_{i+1}$ (according to Lemma 2.8(5) and Corollary 2.7).

Claim. $\mathsf{LCU}_i(\mathbb{P}^9, \lambda_i)$ *holds for $i = 1, \dots, 4$ as well as* $\mathsf{LCU}_4(\mathbb{P}^9, \lambda_5)$.

Proof. The statements hold for $\mathbb{P}^5$ by Theorem 1.35 and are preserved by Corollary 2.11(a). This implies $\mathfrak{x}_i \leq \lambda_i$ for $i = 1, \dots, 4$, as well as $\mathfrak{x}_5 = \mathrm{cov}(\mathcal{M}) \geq \lambda_5$.

Claim. $\mathsf{LCU}_i(\mathbb{P}^9, \lambda_{i^*})$ *holds for* $i = 1, 2, 3$.

Proof. Note that $\kappa_{i^*+1} < \lambda_i < \kappa_{i*} < \lambda_5$. So $\mathsf{LCU}_i(\mathbb{P}^5, \kappa_{i^*})$ holds (Theorem 1.35). This implies $\mathsf{LCU}_i(\mathbb{P}^\ell, \kappa_{i^*})$ for $\ell = 5, \dots, i^* - 1$ (Corollary 2.11(a)), then $\mathsf{LCU}_i(\mathbb{P}^\ell, \lambda_{i^*})$ for $\ell = i^*$ (Corollary 2.11(b)), and then again $\mathsf{LCU}_i(\mathbb{P}^\ell, \lambda_{i^*})$ for $\ell = i^* + 1, \dots, 9$ (again Corollary 2.11(a)). This implies $\mathfrak{x}_\ell \geq \lambda_\ell$ for $\ell = 6, 7, 8$.

Claim. $\mathsf{COB}_i(\mathbb{P}^9, \lambda_i, \lambda_{i^*})$ *holds for* $i = 1, 2, 3, 4$.

Proof. $\mathsf{COB}_i(\mathbb{P}^5, \lambda_i, \lambda_5)$ holds by Theorem 1.35. It implies $\mathsf{COB}_i(\mathbb{P}^\ell, \lambda_i, \lambda_\ell)$ for $\ell = 5, \dots, i^*$ (while $\kappa_\ell > \lambda_i$) (Corollary 2.11(c)), then $\mathsf{COB}_i(\mathbb{P}^\ell, \lambda_i, \lambda_{i^*})$ for $\ell = i^* + 1, \dots, 9$ (Corollary 2.11(d)). This implies $\mathfrak{x}_i \geq \lambda_i$ for $i = 1, \dots, 4$ as well as $\mathfrak{x}_\ell \leq \lambda_\ell$ for $\ell = 5, \dots, 8$. $\square$

4. **Questions**

The result poses some obvious questions. (Since the initial submission of the paper, some of the questions found (*partial*) *answers*, which we mention in the following.)

(a) Can we prove the result without using large cardinals?

It would be quite surprising if compact cardinals are needed, but a proof without them will probably be a lot more complicated.

Answers.

- Gitik [Git19] points out that certain extender embeddings are BUP-embeddings, and that a variation of superstrongs is sufficient to construct the BUP-embeddings required in our construction.
- [BCM18] (building on [Mej19a]) gives a construction that requires only three (instead of four) strongly compact cardinals.
- Finally, in [GKMS19a] it is shown that we can indeed get the result without large cardinals.

(b) Does the result still hold for other specific values of λ_i, such as $\lambda_i = \aleph_{i+1}$?

In our construction, the regular cardinals λ_i for $i = 4, \dots, 9$ can be chosen quite arbitrarily (above the compact κ_6, that is). However, $\aleph_1$, λ_1, λ_2 and λ_3 each have to be separated by a compact cardinal (and furthermore λ_3 has to be a successor of a regular cardinal).

Answer. In [GKMS19a] it is shown that any choice of regular cardinals is possible (in particular, $\lambda_i = \aleph_{i+1}$). We also show that we can replace any number of instances of $<$ by $=$.

(c) Are other linear orders between the characteristics of Cichoń's diagram consistent?

Note that in this paper, we use an FS ccc iteration of length δ with uncountable cofinality, (cf. 2.9), which always results in $\mathrm{non}(\mathcal{M}) \leq \mathrm{cof}(\delta) \leq \mathrm{cov}(\mathcal{M})$. Under these restrictions, there are only four possible assignments. Of course there are a lot more[20] possibilities to assign $\lambda_1, \ldots, \lambda_8$ to Cichoń's diagram in a way that satisfies the known ZFC-provable (in)equalities. Figure 2(b) is an example. Such orders require entirely different methods. (Even to get just the five different values $\aleph_1 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 < \lambda_6 < \lambda_7 < \lambda_8 < \lambda_9$ in this figure turned out to be rather involved [FGKS17, §11].)

Partial answer. Another of the orders compatible with FS ccc iterations, the one of Figure 2(a), is consistent [KST19]. See also [Mej19b]. (A different initial forcing gives the modified ordering of the left hand side; then the same construction and proof as in this paper gives us the whole diagram.)

$$\begin{tikzcd}
& \lambda_3 \arrow[r] & \lambda_4 \arrow[r] & \square \arrow[r] & \lambda_8 \arrow[r] & \lambda_9 \\
& & \lambda_2 \arrow[u] \arrow[r] & \lambda_7 \arrow[u] & & \\
\aleph_1 \arrow[r] & \lambda_1 \arrow[uu] \arrow[r] & \square \arrow[u] \arrow[r] & \lambda_5 \arrow[u] \arrow[r] & \lambda_6 \arrow[uu] &
\end{tikzcd}$$

(a) An ordering compatible with FS ccc.

$$\begin{tikzcd}
& \lambda_2 \arrow[r] & \lambda_7 \arrow[r] & \square \arrow[r] & \lambda_8 \arrow[r] & \lambda_9 \\
& & \lambda_4 \arrow[u] \arrow[r] & \lambda_5 \arrow[u] & & \\
\aleph_1 \arrow[r] & \lambda_1 \arrow[uu] \arrow[r] & \square \arrow[u] \arrow[r] & \lambda_3 \arrow[u] \arrow[r] & \lambda_6 \arrow[uu] &
\end{tikzcd}$$

(b) Another one, incompatible with FS ccc.

Figure 2. Alternative orderings of the cardinal characteristics.

(d) Is it consistent that other cardinal characteristics that have been studied,[21] in addition to the ones in Cichoń's diagram, have pairwise different values as well?

Partial answer. In [GKMS19b], it is forced that additionally $\aleph_1 < \mathfrak{m} < \mathfrak{p} < \mathfrak{h} < \mathrm{add}(\mathcal{N})$ holds.

References

[Ack37] W. Ackermann, Die Widerspruchsfreiheit der allgemeinen Mengenlehre, *Math. Ann.* **114** no. 1 (1937), 305–315. MR 1513141. Zbl 0016.19501. https://doi.org/10.1007/BF01594179.

[20] In fact, we counted 57 in addition to the 4 that are compatible with FS ccc.

[21] The most important ones are described in [Bla10].

[Bar84] T. Bartoszyński, Additivity of measure implies additivity of category, *Trans. Amer. Math. Soc.* **281** no. 1 (1984), 209–213. MR 0719666. Zbl 0538.03042. https://doi.org/10.2307/1999530.

[Bar87] T. Bartoszyński, Combinatorial aspects of measure and category, *Fund. Math.* **127** no. 3 (1987), 225–239. MR 0917147. Zbl 0635.04001. https://doi.org/10.4064/fm-127-3-225-239.

[BJ95] T. Bartoszyński and H. Judah, *Set Theory. On the Structure of the Real Line*, A K Peters, Ltd., Wellesley, MA, 1995. MR 1350295. Zbl 0834.04001. https://doi.org/10.1201/9781439863466.

[BJS93] T. Bartoszyński, H. Judah, and S. Shelah, The Cichoń diagram, *J. Symbolic Logic* **58** no. 2 (1993), 401–423. MR 1233917. Zbl 0786.03034. https://doi.org/10.2307/2275212.

[Bla10] A. Blass, Combinatorial cardinal characteristics of the continuum, in *Handbook of Set Theory. Vols.* 1, 2, 3, Springer, Dordrecht, 2010, pp. 395–489. MR 2768685. Zbl 1198.03058. https://doi.org/10.1007/978-1-4020-5764-9_7.

[Bre91] J. Brendle, Larger cardinals in Cichoń's diagram, *J. Symbolic Logic* **56** no. 3 (1991), 795–810. MR 1129144. Zbl 0758.03021. https://doi.org/10.2307/2275049.

[BCM18] J. Brendle, M. A. Cardona, and D. A. Mejía, Filter-linkedness and its effect on preservation of cardinal characteristics, 2018. arXiv 1809.05004.

[CKP85] J. Cichoń, A. Kamburelis, and J. Pawlikowski, On dense subsets of the measure algebra, *Proc. Amer. Math. Soc.* **94** no. 1 (1985), 142–146. MR 0781072. Zbl 0593.28003. https://doi.org/10.2307/2044967.

[Coh63] P. Cohen, The independence of the continuum hypothesis, *Proc. Nat. Acad. Sci. U.S.A.* **50** (1963), 1143–1148. MR 0157890. Zbl 0192.04401. https://doi.org/10.1073/pnas.50.6.1143.

[EK65] R. Engelking and M. Karłowicz, Some theorems of set theory and their topological consequences, *Fund. Math.* **57** (1965), 275–285. MR 0196693. Zbl 0137.41904. https://doi.org/10.4064/fm-57-3-275-285.

[FGKS17] A. Fischer, M. Goldstern, J. Kellner, and S. Shelah, Creature forcing and five cardinal characteristics in Cichoń's diagram, *Arch. Math. Logic* **56** no. 7-8 (2017), 1045–1103. MR 3696076. Zbl 1404.03040. https://doi.org/10.1007/s00153-017-0553-8.

[Git19] M. Gitik, A remark on large cardinals in "Cichon maximum", 2019, unpublished note, April 18; available on author's webpage.

[GKMS19a] M. Goldstern, J. Kellner, D. Mejía, and S. Shelah, Cichoń's maximum without large cardinals, 2019. arXiv 1906.06608.

[GKMS19b] M. Goldstern, J. Kellner, D. Mejía, and S. Shelah, Controlling cardinal characteristics without adding reals, 2019. arXiv 1904.02617v1.

[GMS16] M. Goldstern, D. A. Mejía, and S. Shelah, The left side of Cichoń's diagram, *Proc. Amer. Math. Soc.* **144** no. 9 (2016), 4025–4042. MR 3513558. Zbl 06596163. https://doi.org/10.1090/proc/13161.

[JS90] H. Judah and S. Shelah, The Kunen-Miller chart (Lebesgue measure, the Baire property, Laver reals and preservation theorems for forcing), *J. Symbolic Logic* **55** no. 3 (1990), 909–927. MR 1071305. Zbl 0718.03037. https://doi.org/10.2307/2274464.

[Kam89] A. Kamburelis, Iterations of Boolean algebras with measure, *Arch. Math. Logic* **29** no. 1 (1989), 21–28. MR 1022984. Zbl 0687.03032. https://doi.org/10.1007/BF01630808.

[KW07] R. Kaye and T. L. Wong, On interpretations of arithmetic and set theory, *Notre Dame J. Formal Logic* **48** no. 4 (2007), 497–510. MR 2357524. Zbl 1137.03019. https://doi.org/10.1305/ndjfl/1193667707.

[KT64] H. J. Keisler and A. Tarski, From accessible to inaccessible cardinals. Results holding for all accessible cardinal numbers and the problem of their extension to inaccessible ones, *Fund. Math.* **53** (1963/1964), 225–308. MR 0166107. Zbl 0173.00802. https://doi.org/10.4064/fm-53-3-225-308.

[KST19] J. Kellner, S. Shelah, and A. Tănasie, Another ordering of the ten cardinal characteristics in Cichoń's diagram, *Comment. Math. Univ. Carolin.* **60** no. 1 (2019), 61–95. https://doi.org/10.14712/1213-7243.2015.273.

[KTT18] J. Kellner, A. R. Tănasie, and F. E. Tonti, Compact cardinals and eight values in Cichoń's diagram, *J. Symb. Log.* **83** no. 2 (2018), 790–803. MR 3835089. Zbl 06915725. https://doi.org/10.1017/jsl.2018.17.

[MS16] M. Malliaris and S. Shelah, Existence of optimal ultrafilters and the fundamental complexity of simple theories, *Adv. Math.* **290** (2016), 614–681. MR 3451934. Zbl 06538728. https://doi.org/10.1016/j.aim.2015.12.009.

[Man71] R. Mansfield, The theory of Boolean ultrapowers, *Ann. Math. Logic* **2** no. 3 (1970/71), 297–323. MR 0300887. Zbl 0216.29401. https://doi.org/10.1016/0003-4843(71)90017-9.

[Mej19a] D. A. Mejía, Matrix iterations with vertical support restrictions, in *Proceedings of the* 14*th and* 15*th Asian Logic Conferences*, World Sci. Publ., Hackensack, NJ, 2019, (Kim, Byunghan and Brendle, Jörg and Lee, Gyesik and Liu, Fenrong and Ramanujam, R and Srivastava, Shashi M and Tsuboi, Akito and Yu, Liang, eds.), pp. 213–248. MR 3890467. https://doi.org/10.1142/9789813237551_0012.

[Mej19b] D. A. Mejía, A note on "another ordering of the ten cardinal characteristics in Cichoń's Diagram" and further remarks, 2019, to appear in *Kyōto Daigaku Sūrikaiseki Kenkyūsho Kōkyūroku*. arXiv 1904.00165.

[Mej13] D. A. Mejía, Matrix iterations and Cichon's diagram, *Arch. Math. Logic* **52** no. 3-4 (2013), 261–278. MR 3047455. Zbl 1270.03087. https://doi.org/10.1007/s00153-012-0315-6.

[Mil81] A. W. Miller, Some properties of measure and category, *Trans. Amer. Math. Soc.* **266** no. 1 (1981), 93–114. MR 0613787. Zbl 0472.03040. https://doi.org/10.2307/1998389.

[Mil82] A. W. Miller, A characterization of the least cardinal for which the Baire category theorem fails, *Proc. Amer. Math. Soc.* **86** no. 3 (1982), 498–502. MR 0671224. Zbl 0506.03012. https://doi.org/10.2307/2044457.

[Mil84] A. W. Miller, Additivity of measure implies dominating reals, *Proc. Amer. Math. Soc.* **91** no. 1 (1984), 111–117. MR 0735576. Zbl 0586.03042. https://doi.org/10.2307/2045281.

[Oxt80] J. C. Oxtoby, *Measure and Category. A Survey of the Analogies between Topological and Measure Spaces*, second ed., *Graduate Texts in Math.* **2**, Springer-Verlag, New York, 1980. MR 0584443. Zbl 0435.28011. https://doi.org/10.1007/978-1-4684-9339-9_22.

[RS] D. Raghavan and S. Shelah, Boolean ultrapowers and iterated forcing, preprint.

[RS83] J. Raisonnier and J. Stern, Mesurabilité et propriété de Baire, *C. R. Acad. Sci. Paris Sér. I Math.* **296** no. 7 (1983), 323–326. MR 0697963. Zbl 0549.03038.

[RS85] J. Raisonnier and J. Stern, The strength of measurability hypotheses, *Israel J. Math.* **50** no. 4 (1985), 337–349. MR 0800191. Zbl 0602.03012. https://doi.org/10.1007/BF02759764.

[She84] S. Shelah, Can you take Solovay's inaccessible away?, *Israel J. Math.* **48** no. 1 (1984), 1–47. MR 0768264. Zbl 0596.03055. https://doi.org/10.1007/BF02760522.

[She04] S. Shelah, Two cardinal invariants of the continuum ($\mathfrak{d} < \mathfrak{a}$) and FS linearly ordered iterated forcing, *Acta Math.* **192** no. 2 (2004), 187–223. MR 2096454. Zbl 1106.03044. https://doi.org/10.1007/BF02392740.

[Sol70] R. M. Solovay, A model of set-theory in which every set of reals is Lebesgue measurable, *Ann. of Math.* (2) **92** (1970), 1–56. MR 0265151. Zbl 0207.00905. https://doi.org/10.2307/1970696.

(Received: June 30, 2017)
(Revised: May 20, 2019)

Institute of Discrete Mathematics and Geometry, TU Wien, Wien, Austria
E-mail: martin.goldstern@tuwien.ac.at
http://www.tuwien.ac.at/goldstern/

Institute of Discrete Mathematics and Geometry, TU Wien, Wien, Austria
E-mail: jakob.kellner@tuwien.ac.at
http://dmg.tuwien.ac.at/kellner/

Einstein Institute of Mathematics, Givat Ram, Jerusalem 91904, Israel and Rutgers University, Piscataway, NJ, USA
E-mail: shlhetal@mat.huji.ac.il
http://shelah.logic.at/