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COMPACT CARDINALS AND EIGHT VALUES IN CICHOŃ'S DIAGRAM

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Abstract. Assuming three strongly compact cardinals, it is consistent that

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

Under the same assumption, it is consistent that

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

Introduction. We assume the reader is familiar with the definitions and some basic properties (which can all be found, e.g., in [2]) of the cardinal characteristics in Cichoń's diagram:

$$\begin{array}{ccccccc}
\mathrm{cov}(\mathcal{N}) & \to & \mathrm{non}(\mathcal{M}) & \to & \mathrm{cof}(\mathcal{M}) & \to & \mathrm{cof}(\mathcal{N}) \to 2^{\aleph_0} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & \mathfrak{b} & \longrightarrow & \mathfrak{d} & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\aleph_1 \to \mathrm{add}(\mathcal{N}) & \to & \mathrm{add}(\mathcal{M}) & \to & \mathrm{cov}(\mathcal{M}) & \to & \mathrm{non}(\mathcal{N})
\end{array}$$

An arrow between $\mathfrak{x}$ and $\mathfrak{y}$ indicates that ZFC proves $\mathfrak{x} \leq \mathfrak{y}$. Moreover, $\max(\mathfrak{d}, \mathrm{non}(\mathcal{M})) = \mathrm{cof}(\mathcal{M})$ and $\min(\mathfrak{b}, \mathrm{cov}(\mathcal{M})) = \mathrm{add}(\mathcal{M})$. These are the only "simple" restrictions in the following sense: every assignment of $\aleph_1$ and $\aleph_2$ to the entries of Cichoń's diagram that honors these restrictions can be shown to be consistent. It is more challenging to get more than two simultaneously different values, for recent progress in this direction see, e.g., [4, 6, 11].

This article consists of two parts: In the first one, we present a finite support ccc iteration P^4 forcing that $\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathfrak{d} = 2^{\aleph_0}$ (and actually something stronger, cf. Lemmas 1.18 and 1.20). This is nothing new: The forcing and all required properties were presented in [11]. We recall all the facts that are required for our result, in a form convenient for our purposes.

In the second part, we investigate the (iterated) Boolean ultrapower P^7 of P^4. Assuming three strongly compact cardinals, this ultrapower (again a finite support ccc iteration) forces

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0},$$

i.e., we get the following values in the diagram (for some increasing cardinals λ_i):

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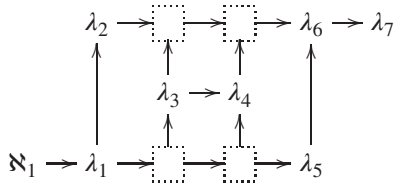

It seems unlikely that the large cardinals assumption is actually needed, but we would expect a proof without it to be considerably more complicated.

The kind of Boolean ultrapower that we use was investigated in [10], and recently applied, e.g., in [13] and [14] (where a Boolean ultrapower of a forcing notion is applied to cardinal characteristics of the reals). Recently Shelah developed a method of using Boolean ultrapowers to control characteristics in Cichoń's diagram. The current article is a relatively simple application of these methods. A more complicated one, in an upcoming article [5] by Goldstern, Shelah and the first author, shows that all entries in Cichońs diagram can be pairwise different.

§1. The initial forcing.

1.1. Good iterations. The forcing P^4 we are about to define has many pleasant properties because it is "good", a notion first explored in [7] and [3]. We now recall the basic facts of good iterations, and specify the instances of the relations we use.

ASSUMPTION 1.1. *We will consider binary relations* R *on* $X = \omega^\omega$ *(or on* $X = 2^\omega$*) that satisfy the following*: *There are relations* R^n *such that* $\mathrm{R} = \bigcup_{n\in\omega} \mathrm{R}^n$*, each* R^n *is a closed subset* (*and in fact absolutely defined*) *of* $X \times X$*, and for* $g \in X$ *and* $n \in \omega$*, the set* $\{f \in X : f\,\mathrm{R}^n\,g\}$ *is nowhere dense. Also, for all* $g \in X$ *there is some* $f \in X$ *with* $f\,\mathrm{R}\,g$.

We will actually use another space as well, the space C of strictly positive rational sequences $(q_n)_{n\in\omega}$ such that $\sum_{n\in\omega} q_n \leq 1$. It is easy to see that C is homeomorphic to ω^ω, when we equip the rationals with the discrete topology and use the product topology.

We use the following instances of relations R on X; it is easy to see that they all satisfy the assumption (in case of $X = C$ we use the homeomorphism mentioned above):

DEFINITION 1.2.
1. $X = C$: $f\,\mathrm{R}_1\,g$ if $(\forall^* n \in \omega)\, f(n) \leq g(n)$.
   (We use "$\forall^* n \in \omega$" for "$(\exists n_0 \in \omega)(\forall n > n_0)$".)
2. $X = 2^\omega$: $f\,\mathrm{R}_2\,g$ if $(\forall^* n \in \omega)\, f \restriction I_n \neq g \restriction I_n$,
   where $(I_n)_{n\in\omega}$ is the increasing interval partition of ω with $|I_n| = 2^{n+1}$.
3. $X = \omega^\omega$: $f\,\mathrm{R}_3\,g$ if $(\forall^* n \in \omega)\, f(n) \leq g(n)$.

We say "f is bounded by g" if $f\,\mathrm{R}\,g$; and, for $\mathcal{Y} \subseteq \omega^\omega$, "$f$ is bounded by $\mathcal{Y}$" if $(\exists y \in \mathcal{Y})\, f\,\mathrm{R}\,y$. We say "unbounded" for "not bounded". (I.e., f is unbounded by $\mathcal{Y}$ if $(\forall y \in \mathcal{Y}) \neg f\,\mathrm{R}\,y$.) We call $\mathcal{X}$ an R-unbounded family, if $\neg(\exists g)(\forall x \in \mathcal{X})\, x\,\mathrm{R}\,g$, and an R-dominating family if $(\forall f)(\exists x \in \mathcal{X})\, f\,\mathrm{R}\,x$. Let $\mathfrak{b}_i$ be the minimal size of an R_i-unbounded family, and $\mathfrak{d}_i$ of an R_i-dominating family.

We only need the following connection between R_i and the cardinal characteristics:

LEMMA 1.3.
1. $\mathrm{add}(\mathcal{N}) = \mathfrak{b}_1$ *and* $\mathrm{cof}(\mathcal{N}) = \mathfrak{d}_1$.
2. $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}_2$ *and* $\mathrm{non}(\mathcal{N}) \geq \mathfrak{d}_2$.
3. $\mathfrak{b} = \mathfrak{b}_3$ *and* $\mathfrak{d} = \mathfrak{d}_3$.

PROOF. (3) holds by definition. (1) can be found in [2, 6.5.B]. To prove (2), note that for fixed $g \in 2^\omega$ the set $\{f \in 2^\omega : \neg g\, \mathrm{R}_2\, f\}$ is a null set, call it N_g. Let $\mathcal{G}$ be an R_2-unbounded family. Then $\{N_g : g \in \mathcal{G}\}$ covers 2^ω: Fix $f \in 2^\omega$. As f does not bound $\mathcal{G}$, there is some $g \in \mathcal{G}$ unbounded by f, i.e., $f \in N_g$. Let X be a non-null set. Then X is R_2-dominating: For any $g \in 2^\omega$ there is some $x \in X \setminus N_g$, i.e., $g\, \mathrm{R}_2\, x$. ⊣

DEFINITION 1.4 ([7]). Let P be a ccc forcing, λ an uncountable regular cardinal, and R as above. P is (R, λ)-*good*, if for each P-name $r \in \omega^\omega$ there is (in V) a nonempty set $\mathcal{Y} \subseteq \omega^\omega$ of size $<\lambda$ such that every f (in V) that is R-unbounded by $\mathcal{Y}$ is forced to be R-unbounded by r as well.

Note that λ-good trivially implies μ-good if $\mu \geq \lambda$ are regular.

How do we get good forcings? Let us just quote the following results:

LEMMA 1.5. *A FS iteration of Cohen forcing is good for any* (R, λ)*, and the composition of two* (R, λ)*-good forcings is* (R, λ)*-good.*
Assume that $(P_\alpha, Q_\alpha)_{\alpha<\delta}$ *is a FS ccc iteration. Then* P_δ *is* (R, λ)*-good, if each* Q_α *is forced to satisfy the following*:

1. *For* $\mathrm{R} = \mathrm{R}_1$: $|Q_\alpha| < \lambda$*, or* Q_α *is* σ*-centered, or* Q_α *is a sub-Boolean-algebra of the random algebra.*
2. *For* $\mathrm{R} = \mathrm{R}_2$: $|Q_\alpha| < \lambda$*, or* Q_α *is* σ*-centered.*
3. *For* $\mathrm{R} = \mathrm{R}_3$: $|Q_\alpha| < \lambda$.

PROOF. (R, λ)-goodness is preserved by FS ccc iterations (in particular compositions), as proved in [7], cf. [2, 6.4.11–12]. Also, ccc forcings of size $<\lambda$ are (R, λ)-good [2, 6.4.7], which takes care of the case of Cohens and of $|Q_\alpha| < \lambda$. So it remains to show that (for $i = 1, 2$) the "large" iterands in the list are (R_i, λ)-good. For R_1 this follows from [7] and [8], cf. [2, 6.5.17–18]. For R_2 this is proven in [3]. ⊣

LEMMA 1.6. *Let* $\lambda \leq \kappa \leq \mu$ *be uncountable regular cardinals. After forcing with* μ *many Cohen reals* $(c_\alpha)_{\alpha\in\mu}$*, followed by an* (R, λ)*-good forcing, we get: For every real* r *in the final extension, the set* $\{\beta \in \kappa : c_\beta$ *is unbounded by* $r\}$ *is cobounded in* κ*. I.e.,* $(\exists \alpha \in \kappa)\, (\forall \beta \in \kappa \setminus \alpha)\, \neg c_\beta\, \mathrm{R}\, r$.

(The Cohen real c_β can be interpreted both as Cohen generic element of 2^ω and as Cohen generic element of ω^ω; we use the interpretation suitable for the relation R.)

PROOF. Work in the intermediate extension after κ many Cohen reals, let us call it V_κ. The remaining forcing (i.e., $\mu \setminus \kappa$ many Cohens composed with the good forcing) is good; so applying Definition 1.4 we get (in V_κ) a set $\mathcal{Y}$ of size $<\lambda$.

As the initial Cohen extension is ccc, and $\kappa \geq \lambda$ is regular, we get some $\alpha \in \kappa$ such that each element y of $\mathcal{Y}$ already exists in the extension by the first α many Cohens, call it V_α. The set of reals M_y bounded by y is meager (and absolute). Any c_β for $\beta \in \kappa \setminus \alpha$ is Cohen over V_α, and therefore not in M_y, i.e., not bounded by y. As this holds for all y, c_β is unbounded by $\mathcal{Y}$, and thus, according to the definition of good, unbounded by r as well. ⊣

In the light of this result, let us revisit Lemma 1.3 with some new notation:

DEFINITION 1.7. For $i = 1, 2, 3$, $\lambda > \aleph_0$ regular, and P a ccc forcing notion, let $\circledcirc_i(P, \lambda)$ stand for: "There is a sequence $(x_\alpha)_{\alpha\in\lambda}$ of P-names such that for every P-name y we have $(\exists \alpha \in \lambda)\, (\forall \beta \in \lambda \setminus \alpha)\, P \Vdash \neg x_\beta\, \mathrm{R}_i\, y$."

LEMMA 1.8. *$\circledcirc_{\mathrm{i}}(P,\lambda)$ implies $\mathfrak{b}_i \le \lambda$ and $\mathfrak{d}_i \ge \lambda$. In particular*:

1. *$\circledcirc_1(P,\lambda)$ implies $P \Vdash (\mathrm{add}(\mathcal{N}^{\bullet}) \le \lambda \,\&\, \mathrm{cof}(\mathcal{N}^{\bullet}) \ge \lambda)$.*
2. *$\circledcirc_2(P,\lambda)$ implies $P \Vdash (\mathrm{cov}(\mathcal{N}^{\bullet}) \le \lambda \,\&\, \mathrm{non}(\mathcal{N}^{\bullet}) \ge \lambda)$.*
3. *$\circledcirc_3(P,\lambda)$ implies $P \Vdash (\mathfrak{b} \le \lambda \,\&\, \mathfrak{d} \ge \lambda)$.*

PROOF. The set $\{x_\alpha : \alpha \in \lambda\}$ is certainly forced to be R_{i}-unbounded; and given a set $Y = \{y_j : j < \theta\}$ of $\theta < \lambda$ many P-names, each has a bound α_j, so for any $\beta \in \lambda$ above all α_j we get $P \Vdash \neg x_\beta \,\mathrm{R}_{\mathrm{i}}\, y_j$ for all j; i.e., Y cannot be dominating. ⊣

1.2. Ground model Borel functions, partial random forcing. The following lemma seems to be well known (but we are not aware of a good reference or an established notation):

DEFINITION 1.9. Let Q be a forcing notion, and let η be a Q-name for a real. We say that Q is "generically Borel determined (by η, via B)", if

- Q consists of reals,
- the Q-generic filter is determined by the real η, and moreover,
- $B \subseteq \mathbb{R}^2$ is a Borel relation such that for all $q \in Q$, $Q \Vdash (B(q,\eta) \leftrightarrow q \in G)$.

We investigate iterations of such forcings:

LEMMA 1.10. *Assume that $(P_\beta, Q_\beta)_{\beta<\alpha}$ is a FS ccc iteration such that each Q_β is generically Borel determined (in an absolute way already fixed in V). Then for each P_α-name r of a real, there is (in the ground model) a Borel function $F : \mathbb{R}^\omega \to \mathbb{R}$ and a sequence $(\alpha_i)_{i\in\omega}$ of ordinals in α such that P_α forces $r = F((\eta_{\alpha_i})_{i\in\omega})$.*

PROOF. We prove by induction on $\gamma \le \alpha$:

- For all $p \in P_\gamma$ there is a Borel relation $B^p \subseteq \mathbb{R}^\omega$ and a sequence $(\alpha^p_i)_{i\in\omega}$ of elements of γ such that $P_\gamma \Vdash B^p((\eta_{\alpha^p_i})_{i\in\omega}) \leftrightarrow p \in G_\gamma$.
- For each P_γ-name r of a real, there is a Borel function F^r and a sequence $(\alpha^r_i)_{i\in\omega}$ of elements of γ such that $P_\gamma \Vdash r = F^r((\eta_{\alpha^p_i})_{i\in\omega})$.

The second item follows from the first, as we can use the countable maximal antichains that decide $r(n) = m$.

If γ is a limit ordinal, then P_γ has no new elements, so there is nothing to do.

So assume $\gamma = \zeta + 1$. By our assumption, Q_ζ is generically Borel determined from η_ζ via a Borel relation B_ζ. Consider $(p,q) \in P_\zeta * Q_\zeta$. This is in G_γ iff $p \in G_\zeta$ (which, by induction, is Borel) and $q \in G(\zeta)$. As q is a real, it is forced that $q = B^q((\alpha^q_i)_{i\in\omega})$. Moreover, P_ζ forces that Q_ζ forces that $q \in G(\zeta)$ iff $B_\zeta(\eta_\zeta, q)$ iff $B_\zeta(\eta_\zeta, B^q((\alpha^q_i)_{i\in\omega}))$. ⊣

DEFINITION 1.11. Given $(P_\beta, Q_\beta)_{\beta<\alpha}$ as above, and some $w \subseteq \alpha$, we define the P_α-name $\mathbb{R}^w$ to consist of all reals r such that in the ground model there are a Borel function F and a sequence $(\alpha_i)_{i\in\omega}$ of elements of w such that $r = F((\eta_{\alpha_i})_{i\in\omega})$.

The following is straightforward:

FACT 1.12.
- *Set (in V) $\mu = (|w|+2)^{\aleph_0}$. Then it is forced that $\mathbb{R}^w$ has cardinality $\le\mu$.*
- *If $w' \supseteq w$, then (it is forced that) $\mathbb{R}^{w'} \supseteq \mathbb{R}^w$.*

- *If w is the increasing union of $(w_\alpha)_{\alpha\in\gamma}$ with $\mathrm{cf}(\gamma) \geq \omega_1$, then (it is forced that) $\mathbb{R}^w = \bigcup_{\alpha\in\gamma} \mathbb{R}^{w_\alpha}$.*
- *For every P_α-name r of a real there is a countable w such that (it is forced that) $r \in \mathbb{R}^w$.*

DEFINITION 1.13. Let $\mathbb{B}$ be (the definition of) random forcing, i.e., positive pruned trees T, ordered by inclusion. Given $(P_\beta, Q_\beta)_{\beta<\alpha}$ as above, $w \subseteq \alpha$, we define the P_α-name $\mathbb{B}^w := \mathbb{B} \cap \mathbb{R}^w$ and call it "partial random forcing defined from w".

Clearly $\mathbb{B}^w$ is a subforcing (not necessarily a complete one) of $\mathbb{B}$, and if p, q in $\mathbb{B}^w$ are incompatible in $\mathbb{B}^w$ then they are incompatible in random forcing. In particular $\mathbb{B}^w$ is ccc.

Note that $\mathbb{B}^w$ is forced to be generically Borel determined, in way already fixed in V: The generic real η is defined by $\{\eta\} = \bigcap\{[s] \in G : s \in 2^{<\omega}\}$, and the Borel relation by "$\eta \in [T]$".

REMARK 1.14. In this section, we have provided a very explicit notion of "partial random", using Borel functions. The use of Borel functions is not essential, we could use any other method of calculating reals from generic reals at certain restricted positions, provided this method satisfies Facts 1.12. One such alternative definition has been used in [6]: We can define the subforcing $P_\alpha \restriction w$ of P_α in a natural way, and require that it is a complete subforcing (which is a closure property of w). Then we can take Q_α to be random forcing as evaluated in the $P_\alpha \restriction w$-extension.

While this approach is basically equivalent (and may seem slightly more natural than the artificial use of Borel functions), it has the disadvantage that we have to take care of the closure property of w.

DEFINITION 1.15. Analogously to "partial random", we define the "partial Hechler" and "partial amoeba" forcings.

These forcings are generically Borel determined as well.

1.3. The initial forcing P^4. Assume that λ is regular uncountable and $\mu < \lambda$ implies $\mu^{\aleph_0} < \lambda$. Then $|w| < \lambda$ implies that the size of a partial forcing defined by w is $<\lambda$.

DEFINITION 1.16. Assume GCH and let $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ be regular cardinals. Set $\delta_4 = \lambda_4 + \lambda_4$. Partition $\delta_4 \setminus \lambda_4$ into unbounded sets S^1, S^2, and S^3. Fix for each $\alpha \in \delta_4 \setminus \lambda_4$ some $w_\alpha \subseteq \alpha$ such that each $\{w_\alpha : \alpha \in S^i\}$ is cofinal in $[\delta_4]^{<\lambda_i}$.[1]

We now define $P^4 = (P_\alpha, Q_\alpha)_{\alpha\in\delta_4}$ to be the FS ccc iteration which first adds λ_4 many Cohen reals, and such that for each $\alpha \in \delta_4 \setminus \lambda_4$,

$$\text{if } \alpha \text{ is in } \left\{\begin{matrix} S^1 \\ S^2 \\ S^3 \end{matrix}\right\}, \text{ then } Q_\alpha \text{ is the partial } \left\{\begin{matrix} \text{amoeba} \\ \text{random} \\ \text{Hechler} \end{matrix}\right\} \text{ forcing defined from } w_\alpha.$$

The forcing results in $2^{\aleph_0} = \lambda_4$, which follows from the following easy and well-known fact:

[1] I.e., if $\alpha \in S^i$ then $|w_\alpha| < \lambda_i$, and for all $u \subseteq \delta_4$, $|u| < \lambda_i$ there is some $\alpha \in S^i$ with $w_\alpha \supseteq u$.

LEMMA 1.17. *Let $(P_\alpha, Q_\alpha)_{\alpha<\delta}$ be a FS ccc iteration of length δ such that each Q_α is forced to consist of real numbers, and set $\lambda(\delta) := (2+\delta)^{\aleph_0}$. Then $P_\delta \Vdash 2^{\aleph_0} \leq \lambda(\delta)$.*

PROOF. By induction on δ, we show that there is a dense subforcing of $D_\delta \subseteq P_\delta$ of size $\leq\lambda(\delta)$. Then the continuum has size at most $\lambda(\delta)$ (as each name of a real corresponds to a countable sequence of antichains, labeled with 0, 1, in P_δ, without loss of generality in D_δ).

For $\delta+1$, $D_\delta \subseteq P_\delta$ is dense and has size $\leq\lambda(\delta)$, and Q_δ is forced to have size $\leq\lambda(\delta)$. Without loss of generality we can identify Q_δ with a subset of $\lambda(\delta)$. Let $D_{\delta+1}$ consist of $(p, \check{\alpha}) \in P_{\delta+1}$ such that $p \in D_\delta$ forces $\alpha \in Q_\delta$.

For δ limit, the union of D_α is dense in $P_\delta = \bigcup_{\alpha\in\delta} P_\alpha$. ⊣

According to Lemma 1.5 P^4 is $(\mathrm{R}_\mathrm{i}, \lambda_i)$-good for $i = 1, 2, 3$, so Lemmas 1.6 and 1.8 gives us:

LEMMA 1.18. *$\circledcirc_\mathrm{i}(P^4, \kappa)$ holds for $i = 1, 2, 3$ and each regular cardinal κ in $[\lambda_i, \lambda_4]$.*

So in particular, P^4 forces $\mathrm{add}(\mathcal{N}) \leq \lambda_1$, $\mathrm{cov}(\mathcal{N}) \leq \lambda_2$, $\mathfrak{b} \leq \lambda_3$ *and* $\mathrm{cof}(\mathcal{N}) = \mathrm{non}(\mathcal{N}) = \mathfrak{d} = 2^{\aleph_0}$.

THEOREM 1.19 ([11], Theorem 2). *P^4 forces* $\mathrm{add}(\mathcal{N}) = \lambda_1$, $\mathrm{cov}(\mathcal{N}) = \lambda_2$, $\mathfrak{b} = \lambda_3$, *and* $\mathfrak{d} = \lambda_4 = 2^{\aleph_0}$.

PROOF. It is easy to see that the partial amoebas take care of $\mathrm{add}(\mathcal{N}) \geq \lambda_1$: Let $(N_i)_{i\in\mu}$, $\aleph_1 \leq \mu < \lambda_1$ be a family of P^4-names of null sets. Each N_i is a Borel code, i.e., a real, and therefore Borel-computed from some countable set $w^i \subseteq \delta_4$. The union of the w^i is a set w^* of size $\leq\mu$ that already Borel-decides all N_i. There is some $\beta \in S^1$ such that $w_\beta \supseteq w^*$, so the partial amoeba forcing at β sees all the null sets N_i and therefore covers their union.

Analogously one proves $\mathrm{cov}(\mathcal{N}) \geq \lambda_2$ and $\mathfrak{b} \geq \lambda_3$. ⊣

We will reformulate the proof for $\mathrm{cov}(\mathcal{N})$ in a cumbersome manner that can be conveniently used later on:

LEMMA 1.20. *Let $\boxplus_2(P, \lambda, \mu)$ stand for*: *"P is a ccc forcing notion, and there is a $<\lambda$-directed partial order $(S, \prec)$ of size μ and a sequence $(r_s)_{s\in S}$ of P-names for reals such that for each P-name N of a null set $(\exists s \in S)\,(\forall t \succ s)\; P \Vdash r_t \notin N$."*

- *$\boxplus_2(P, \lambda, \mu)$ implies $P \Vdash (\,\mathrm{cov}(\mathcal{N}) \geq \lambda \;\&\; \mathrm{non}(\mathcal{N}) \leq \mu\,)$.*
- *$\boxplus_2(P^4, \lambda_2, \lambda_4)$ holds.*

PROOF. $\mathrm{cov}(\mathcal{N}) \geq \lambda$: Fix $<\lambda$ many P-names N_α of null sets. Each real has a "lower bound" $s_\alpha \in S$, i.e., $P \Vdash r_t \notin N_\alpha$ whenever $t \succ s_\alpha$. Let $t \succ s_\alpha$ for all α (this is possible as S is directed). So $P \Vdash r_t \notin N_\alpha$ for every α, i.e., the union doesn't cover the reals.

$\mathrm{non}(\mathcal{N}) \leq \mu$, as the set of all r_s is not null: For every name N of a null set there is some $s \in S$ such that $P \Vdash r_s \notin N$.

For P^4, we set $S = S^2$, $s \prec t$ if $w_s \subseteq w_t$, and we let r_s be the partial random real added at s. A P^4 name for a null set N depends (in a Borel way) on a countable index set $w^* \subseteq \delta_4$. Fix some $s \in S^2$ such that $w_s \supseteq w^*$, and pick any $t \succ s$. Then w_t contains all information to calculate the null set N, and therefore the partial random r_t over w_t will avoid N. ⊣

§2. The Boolean ultrapower of the forcing.

2.1. Boolean ultrapowers. Boolean ultrapowers generalize regular ultrapowers by using arbitrary Boolean algebras instead of the power set algebra.

ASSUMPTION 2.1. *κ is strongly compact, B is a κ-distributive, κ^+-cc, atomless complete Boolean algebra.*

LEMMA 2.2 ([9]). *Every κ-complete filter on B can be extended to a κ-complete ultrafilter U.*[2]

PROOF. List the required properties of U as a set of propositional sentences in $\mathcal{L}_\kappa$ (a propositional language allowing conjunctions and disjunctions of any size $<\kappa$), using atomic formulas coding $b \in U$ and $b \notin U$ for $b \in B$. ⊣

ASSUMPTION 2.3. *U is a κ-complete ultrafilter on B.*

LEMMA 2.4. *There is a maximal antichain A_0 in B of size κ such that $A_0 \cap U = \emptyset$. In other words, U is not κ^+-complete.*

PROOF. Let A_0 be a maximal antichain in the open dense set $B \setminus U$. As B is κ^+-cc, A_0 has size $\leq\kappa$. It cannot have size $<\kappa$, as U is κ-complete and therefore meets every antichain of size $<\kappa$. ⊣

The Boolean algebra B can be used as forcing notion. As usual, V denotes the universe we start with, sometimes called the ground model. In the following, we will not actually force with B (or any other p.o.); we always remain in V, but we still use forcing notation. In particular, we call the usual B-names "forcing names".

DEFINITION 2.5. A *BUP-name* (or: labeled antichain) x is a function $A \to V$ whose domain is a maximal antichain. We may write $A(x)$ to denote A.

Each BUP-name corresponds to a forcing-name[3] for an element of V. We will identify the BUP-name and the corresponding forcing-name. In turn, every forcing name τ for an element of V has a forcing-equivalent BUP-name.

In particular, we can calculate, for two BUP-names x and y, the Boolean value $[\![x = y]\!]$.[4]

DEFINITION 2.6.
- Two BUP-names x and y are *equivalent*, if $[\![x = y]\!] \in U$.
- For $v \in V$, let $\check{v}$ be a BUP-name-version of the standard name for v (unique up to equivalence).
- The *Boolean ultrapower* M^- consists of the equivalence classes $[x]$ of BUP-names x; and we define $[x] \in^- [y]$ by $[\![x \in y]\!] \in U$.
- $j^- : V \to M^-$ maps v to $[\check{v}]$.

We are interested in the $\in$-structure $(M^-, \in^-)$.

Given BUP-names $x_1, \dots, x_n$ and an $\in$-formula φ, the truth value $[\![\varphi^V(x_1, \dots, x_n)]\!]$ is well defined (it is the weakest element of B forcing that in the ground model $\varphi(x_1, \dots, x_n)$ holds, which makes sense as $x_1, \dots, x_n$ are guaranteed to be in the ground model).[5]

[2]For this, neither κ^+-cc nor atomless is required, and it is sufficient that B is κ-complete.

[3]More specifically, to the forcing-name $\{(\widetilde{x(a)}, a) : a \in A(x)\}$.

[4]We can calculate $[\![x = y]\!]$ more explicitly as follows: Pick some common refinement A' of $A(x)$ and $A(y)$. This defines in an obvious way BUP-names x' and y' both with domain A': For $a \in A'$ we set $x'(a) = x(\tilde{a})$ for $\tilde{a}$ the unique element of $A(x)$ above a. Then $[\![x = y]\!]$ is $\bigvee\{a \in A' : x'(a) = y'(a)\}$ (which is independent of the refinement A').

[5]Equivalently, we can explicitly calculate $[\![\varphi^V(x_1, \dots, x_n)]\!]$ as follows: Chose a common refinement A' of $A(x_1), \dots, A(x_n)$, and set $[\![\varphi^V(x_1, \dots, x_n)]\!]$ to be $\bigvee\{a \in A' : \varphi(x'_1(a), \dots, x'_n(a))\}$; where again the BUP-names x'_i are the canonically defined BUP-names with domain A' that are equivalent to x_i.

LEMMA 2.7.
- *Łoś's theorem*: $(M^-, \in^-) \vDash \varphi([x_1], \dots, [x_n])$ *iff* $[\![\varphi^V(x_1, \dots, x_n)]\!] \in U$.
- $j^- : (V, \in) \to (M^-, \in^-)$ *is an elementary embedding.*
- *In particular,* $(M^-, \in^-)$ *is a ZFC model.*

PROOF. Straightforward by the definition of equivalence and of $[x] \in^- [y]$, and by induction (using that U is a filter for $\varphi \wedge \psi$ and for $\exists v\, \varphi(v)$, and that it is an ultrafilter for $\neg\varphi$). For elementarity, note that $M^- \vDash \varphi([\check{x}_1], \dots, [\check{x}_n])$ iff $[\![\varphi^V(\check{x}_1, \dots, \check{x}_n)]\!] \in U$ iff $V \vDash \varphi(x_1, \dots, x_n)$. ⊣

LEMMA 2.8. $(M^-, \in^-)$ *is wellfounded.*

PROOF. This is the standard argument, using the fact that U is σ-complete:

Assume $[x_{n+1}] \in^- [x_n]$ for $n \in \omega$. Choose a common refinement A of the antichains $A(x_n)$, Again, let x'_n be the BUP-names with domain A equivalent to x_n. So, by our assumption, $u_n := [\![x_{n+1} \in x_n]\!] = \bigvee\{a \in A : x'_{n+1}(a) \in x'_n(a)\}$ is in U for each n. As U is σ-complete, there is some $u \in U$ stronger than all u_n. This implies: If $a \in A$ is compatible with u, then a is compatible with u_n (for all n), and therefore $x'_{n+1}(a) \in x'_n(a)$ for all n, a contradiction. ⊣

DEFINITION 2.9. Let M be the transitive collapse of $(M^-, \in^-)$, and let $j : V \to M$ be the composition of j^- with the collapse. We denote the collapse of $[x]$ by x^U. So in particular $\check{v}^U = j(v)$.

LEMMA 2.10.
- $M \vDash \varphi(x_1^U, \dots, x_n^U)$ *iff* $[\![\varphi^V(x_1, \dots, x_n)]\!] \in U$. *In particular,* $j : V \to M$ *is an elementary embedding.*
- *If* $|Y| < \kappa$, *then* $j(Y) = j''Y$. *In particular,* j *restricted to* κ *is the identity.* M *is closed under* $<\kappa$*-sequences.*
- $j(\kappa) \neq \kappa$. *I.e.,* $\kappa = \mathrm{cr}(j)$.

PROOF. If $[x] \in j^-(Y)$, then we can refine the antichain $A(x)$ to some A' such that each $a \in A'$ either forces $x = y$ for some $y \in Y$, or $x \notin Y$. Without loss of generality (by taking suprema), we can assume different elements a of A' giving different values $y(a)$; i.e., A' has size $|Y|+1 < \kappa$. So U selects an element a of A', and as $[\![x \in Y]\!] \in U$, this element a proves that $[x] = j^-(y(a))$.

We have already mentioned that there is a maximal antichain $A_0 = \{a_i : i \in \kappa\}$ of size κ such that $A_0 \cap U = \emptyset$. The BUP-name x with $A(x) = A_0$ and $x(a_i) = i$ satisfies $[x] \in^- j^-(\kappa)$, but is not equivalent to any $\check{\nu}$; so $\kappa \leq x^U < j(\kappa)$. ⊣

As we have already mentioned, an arbitrary forcing-name for an element of V has a forcing-equivalent BUP-name, i.e., a maximal antichain labeled with elements of V. If τ is a forcing-name for an element of Y ($Y \in V$), then without loss of generality τ corresponds to a maximal antichain labeled with elements of Y. We call such an object y a "BUP-name for an element of $j(Y)$" (and not "for an element of Y", for the obvious reason: unlike in the case of a forcing extension, y^U is generally not in Y, but, by definition of $\in^-$, it is in $j(Y)$).

2.2. The algebra and the filter. We will now define the concrete Boolean algebra we are going to use:

DEFINITION 2.11. Assume GCH, let κ be strongly compact, and $\theta > \kappa$ regular.

$P_{\kappa,\theta}$ is the forcing notion adding θ Cohen subsets of κ. More concretely: $P_{\kappa,\theta}$ consists of partial functions from θ to κ with domain of size $<\kappa$, ordered by extension. Let $f^* : \theta \to \kappa$ be the name of the generic function.

$\mathcal{B}_{\kappa,\theta}$ is the complete Boolean algebra generated by $P_{\kappa,\theta}$.

Clearly $\mathcal{B}_{\kappa,\theta}$ is κ^+-cc and κ-distributive, as $P_{\kappa,\theta}$ is even κ-closed.

LEMMA 2.12. *There is a κ-complete ultrafilter U on $B = \mathcal{B}_{\kappa,\theta}$ such that*

(a) *The Boolean ultrapower gives an elementary embedding $j : V \to M$. M is closed under $<\kappa$-sequences.*
(b) *The elements x^U of M are exactly (the collapses of equivalence classes of) B-names x for elements of V; more concretely, a function from an antichain (of size κ) to V. We sometimes say "x^U is a mixture of κ many possibilities".*
Similarly, for $Y \in V$, the elements x^U of $j(Y)$ correspond to the B-names x of elements of Y, i.e., antichains labeled with elements of Y.
(c) *If $|A| < \kappa$, then $j''A = j(A)$. In particular, j restricted to κ is the identity.*
(d) *j has critical point κ, $\mathrm{cf}(j(\kappa)) = \theta$, and $\theta \leq j(\kappa) \leq \theta^+$.*
(e) *If $\lambda > \kappa$ is regular, then $\max(\theta, \lambda) \leq j(\lambda) < \max(\theta, \lambda)^+$.*
(f) *If S is a $<\lambda$-directed partial order, and $\kappa < \lambda$, then $j''S$ is cofinal in $j(S)$.*
(g) *If $\mathrm{cf}(\alpha) \neq \kappa$, then $j''\alpha$ is cofinal in $j(\alpha)$, so in particular $\mathrm{cf}(j(\alpha)) = \mathrm{cf}(\alpha)$.*

PROOF. We have already seen (a)–(c).

(d): For each $\delta \in \theta$, $f^*(\delta)$ is a forcing-name for an element of κ, and thus a BUP-name for an element of $j(\kappa)$. Let x be some other BUP-name for an element of $j(\kappa)$, i.e., an antichain A of size κ labeled with elements of κ. Let $\delta \in \theta$ be bigger than the supremum of $\mathrm{supp}(a)$ for each $a \in A$. We call such a pair (x, δ) "suitable", and set $b_{x,\delta} := [\![f^*(\delta) > x]\!]$. We claim that all these elements form a basis for a κ-complete filter. To see this, fix suitable pairs (x_i, δ_i) for $i < \mu$ where $\mu < \kappa$; we have to show that $\bigwedge_{i\in\mu} b_{x_i,\delta_i} \neq \mathbb{0}$. Enumerate $\{\delta_i : i \in \mu\}$ increasing (and without repetitions) as δ^j for $j \in \gamma \leq \mu$. Set $A_j = \{i : \delta_i = \delta^j\}$. Given q_j, define $q_{j+1} \in P_{\kappa,\theta}$ as follows: $q_{j+1} \leq q_j$; $\delta^j \in \mathrm{supp}(q_{j+1}) \subseteq \delta^j \cup \{\delta^j\}$; and $q_{j+1} \restriction \delta^j$ decides for all $i \in A_j$ the values of x_i to be some α_i; and $q_{j+1}(\delta^j) = \sup_{i\in A_j}(\alpha_i) + 1$. For $j \leq \gamma$ limit, let q_j be the union of $\{q_k : k < j\}$. Then q_γ is stronger than each b_{x_i,δ_i}.

As κ is strongly compact, we can extend the κ-complete filter generated by all b_{x_i,δ_i} to a κ-complete ultrafilter U. Then the sequence $(f^*(\delta)^U)_{\delta\in\theta}$ is strictly increasing (as $(f^*(\delta), \delta')$ is suitable for all $\delta < \delta'$) and cofinal in $j(\kappa)$ (as we have just seen); so $\mathrm{cf}(j(\kappa)) = \theta$.

(e): We count all BUP-names for elements of $j(\lambda)$. As we can assume that the antichains are subsets of $P_{\kappa,\theta}$, which has size θ, and as λ is regular and GCH holds, we get $|j(\lambda)| \leq [\theta]^\kappa \times \lambda^\kappa = \max(\theta, \lambda)$.

(f): An element x^U of $j(S)$ is a mixture of κ many possibilities in S. As $\kappa < \lambda$, there is some $t \in S$ above all the possibilities. Then $j(t) > x^U$.

(g): Set $\mu = \mathrm{cf}(\alpha)$, and pick an increasing cofinal sequence $\bar\beta = (\beta_i)_{i\in\mu}$ in α. $j(\bar\beta)$ is increasing cofinal in $j(\alpha)$ (as this is absolute between M and V). If $\mu < \kappa$, then $j''\bar\beta = j(\bar\beta)$, otherwise use (f). ⊣

2.3. The ultrapower of a forcing notion. We now investigate the relation of a forcing notion $P \in V$ and its image $j(P) \in M$, which we use as a forcing notion over V.

(Think of P as being one of the forcings of Section 1; it has no relation with the Boolean algebra B.)

Note that as $j(P) \in M$ and M is transitive, every $j(P)$-generic filter G over V is trivially generic over M as well, and we will use absoluteness between $M[G]$ and $V[G]$ to prove various properties of $j(P)$.

LEMMA 2.13. *If P is κ-cc, then j gives a complete embedding from P into $j(P)$. I.e., $j''P$ is a complete subforcing of $j(P)$, and j is an isomorphism from P to $j''P$.*

PROOF. It is clear that j is an isomorphism onto $j''P$: By definition the order $<_{j(P)}$ on $j(P)$ is $j(<_P)$, and by elementarity $p \leq_P q$ iff $j(q) <_{j(P)} j(p)$. Also, $p \perp q$ is preserved: $M \vDash p \perp_{j(P)} q$ by elementarity, so $p \perp_{j(P)} q$ holds in V (as $j(P) \in M$ and M is transitive).

It remains to be shown that each maximal antichain A of P is preserved, i.e., $j''A \subseteq j(P)$ is predense.

By our assumption, $|A| < \kappa$, so $j''A = j(A)$ (by Lemma 2.12(c)), which is maximal in M (by elementarity) and thus maximal in V (by absoluteness). ⊣

Accordingly, we can canonically translate P-names into $j(P)$-names, etc.

For later reference, let us make this a bit more explicit: Let g be a P-name for a real (i.e., an element of ω^ω). Each $g(n)$ is decided by a maximal antichains A_n, where $a \in A_n$ forces $g(n) = g_{n,a} \in \omega$. Then the $j(P)$-name $j(g)$ corresponds to the antichains

$$j(A_n) = j''A_n, \text{ and } j(a) \text{ forces } j(g)(n) = g_{n,a} \text{ for each } a \in A_n. \tag{2.1}$$

LEMMA 2.14. *If $P = (P_\alpha, Q_\alpha)_{\alpha<\delta}$ is a finite support (FS) ccc iteration of length δ, then $j(P)$ is a FS ccc iteration of length $j(\delta)$ (more formally: it is canonically equivalent to one).*

PROOF. M certainly thinks that $j(P) = (P^*_\alpha, Q^*_\alpha)_{\alpha<j(\delta)}$ is a FS iteration of length $j(\delta)$.

By induction on α we define the FS ccc iteration $(\tilde{P}_\alpha, \tilde{Q}_\alpha)_{\alpha<j(\delta)}$ and show that P^*_α is a dense subforcing of $\tilde{P}_\alpha$: Assume this is already the case for P^*_α. M thinks that Q^*_α is a P^*_α-name, so we can interpret it as a $\tilde{P}_\alpha$-name and use it as $\tilde{Q}_\alpha$. Assume that (p, q) is an element (in V) of $\tilde{P}_\alpha * \tilde{Q}_\alpha$. So p forces that q is a name in M; we can increase p to some p' that decides q to be the name $q' \in M$. By induction we can further increase p' to $p'' \in P^*_\alpha$, then $(p'', q') \in P^*_{\alpha+1}$ is stronger than (p, q). (At limits there is nothing to do, as we use FS iterations.)

$j(P)$ is ccc, as any $A \subseteq j(P)$ of size $\aleph_1$ is in M (and M thinks that $j(P)$ is ccc). ⊣

Similarly, we get the following:

- If $\tau = x^U$ is in M a $j(P)$-name for an element of $j(Z)$, then τ is a mixture of κ many P-names for an element of Z (i.e., the BUP-name x consists of an antichain $A \subseteq B$ labeled, without loss of generality, with P-names for elements of Z).
  (This is just the instance of "each $x^U \in j(Y)$ is a mixture of elements of Y", where we set Y to be the set[6] of P-names for elements of Z.)
- A $j(P)$-name τ for an element of $M[G]$ has an equivalent $j(P)$-name in M.
  (There is a maximal antichain A of $j(P)$ labeled with $j(P)$-names in M. As M is countably closed, this labeled antichain is in M, and gives a $j(P)$-name in M equivalent to τ.)

[6]Formally: We set Y to be some set that contains representatives of each equivalence class of P-names of elements of Z.

- In $V[G]$, $M[G]$ is closed under $<\kappa$ sequences.
  (We can assume the names to be in M and use $<\kappa$-closure.)
- In particular, every $j(P)$-name for a real, a Borel-code, a countable sequence of reals, etc., is in M (more formally: has an equivalent name in M).
- If each iterand is forced to consist of reals, then $j(P)$ forces the continuum to have size at most $|2 + j(\delta)|^{\aleph_0}$.
  (This follows from Lemma 1.17 as $j(P)$ also satisfies that each iterand consists of reals.)

2.4. Preservation of values of characteristics.

LEMMA 2.15. *Let λ be a regular uncountable cardinal and P a ccc forcing.*

(a) *Let $\mathfrak{x}$ be either* $\mathrm{add}(\mathcal{N})$ *or* $\mathfrak{b}$. *If $P \Vdash \mathfrak{x} = \lambda$ and $\kappa \neq \lambda$, then $j(P) \Vdash \mathfrak{x} = \lambda$.*
(b) *Let $\mathfrak{y}$ be either* $\mathrm{cof}(\mathcal{N})$ *or* $\mathfrak{d}$. *If $P \Vdash \mathfrak{y} \geq \lambda$ and $\kappa < \lambda$, then $j(P) \Vdash \mathfrak{y} \geq \lambda$.*
(c) *Let $(\mathfrak{x}, \mathfrak{y})$ be either $(\mathfrak{b}, \mathfrak{d})$ or $(\mathrm{add}(\mathcal{N}), \mathrm{cof}(\mathcal{N}))$. Then we get the following:*
If $P \Vdash (\kappa < \mathfrak{x} \,\&\, \mathfrak{y} \leq \lambda)$ then $j(P) \Vdash \mathfrak{y} \leq \lambda$.

PROOF. (a) We formulate the proof for $\mathrm{add}(\mathcal{N})$; the proof for $\mathfrak{b}$ is the same.

Let $\bar{N} = (N_i)_{i<\lambda}$ be P-names for an increasing sequence of null sets such that $\bigcup_{i<\lambda} N_i$ is not null. So in particular for every P-name N of a null set: $(\exists i_0 \in \lambda)\,(\forall i \in \lambda \setminus i_0)\; P \Vdash N_i \not\subseteq N$. (We can choose the i_0 in V due to ccc.)

Therefore M thinks that the same holds for the sequence $j(\bar{N})$ of $j(P)$-names of length $j(\lambda)$. So whenever N is a $j(P)$-name of a null set, we can assume without loss of generality that $N \in M$, so M thinks that from some i_0 on it is forced that $N_i \not\subseteq N$, which is absolute.

As $\kappa \neq \lambda$, we know that $j''\lambda$ is cofinal in $j(\lambda)$. So (since the sequence $j(\bar{N})$ is increasing) we can use $(j(N_i))_{i\in\lambda}$ and get the same property.

This shows that $j(P) \Vdash \mathrm{add}(\mathcal{N}) \leq \lambda$.

For the other inequality, fix some $\chi < \lambda$, and $(N_i)_{i<\chi}$ a family of $j(P)$-names for null sets (without loss of generality each name is in M), and $p \in j(P)$.

- CASE 1: $\kappa \geq \lambda$. Then the sequence $(N_i)_{i<\chi}$ (as well as p) is in M, and $M \models \big(p \Vdash \bigcup N_i \text{ null}\big)$; which is absolute.
- CASE 2: $\kappa < \lambda$. Every N_i is a "mixture" of κ many P-names for null sets, so there is a single P-name N'_i such that P forces N'_i is superset of all the names involved. Therefore, $j(P)$ forces that $j(N'_i) \supseteq N_i$. And P forces that $\bigcup_{i<\chi} N'_i$ is null, i.e., covered by some null set N^*. Then $j(P)$ forces that $j(N^*)$ covers $\bigcup_{i<\chi} N_i$.

(b) We show that a small set cannot be dominating: Fix a sequence $(f_i)_{i<\chi}$ of $j(P)$-names of reals, with $\chi < \lambda$. Each f_i corresponds to $\kappa < \lambda$ many possible P-names. As $\chi < \lambda$, there is a P-name g unbounded by all $\chi \times \kappa < \lambda$ many possible P-names. So if f is any of the possibilities, then P forces $g \not\leq^* f$; and thus $j(P)$ forces $j(g) \not\leq^* f_i$ for all i. So $j(P)$ forces $\mathfrak{d} \geq \lambda$.

The same proof works for $\mathrm{cof}(\mathcal{N})$ (using "the null set g is not a subset of any of the possible null sets").

(c) For $(\mathfrak{x}, \mathfrak{y}) = (\mathfrak{b}, \mathfrak{d})$: Fix a P-name of a dominating family $\bar{f} = (f_i)_{i\in\lambda}$.

We claim that $j(P)$ forces that $j''\bar{f} = (j(f_i))_{i<\lambda}$ is dominating. Let r be a $j(P)$-name of a real, i.e., a mixture of κ many possibilities (each possibility corresponding to a P-name for a real). As $P \Vdash \kappa < \mathfrak{b}$, P forces that these reals cannot be unbounded, i.e.,

there is a P-name $\alpha \in \lambda$ such that f_α is forced to dominate all the possibilities. By absoluteness, $j(P) \Vdash j(f_\alpha) >^* r$.

It remains to be shown that $j(P) \Vdash j(f_\alpha) \in j'' \bar{f}$. (Note that α is just a P-name.) Fix a maximal antichain A in P deciding α, i.e., $a \in A$ forces $\alpha = \alpha(a)$. As j maps P completely into $j(P)$, $j'' A$ is a maximal antichain in $j(P)$. So $j(P)$ forces that exactly on $j(a)$ for $a \in A$ is in the generic filter, cf. (2.1). Accordingly $j(f_\alpha) = j(f_{\alpha(a)}) \in j'' \bar{f}$.

The proof for $\mathrm{cof}(\mathcal{N})$ is the same. ⊣

For the other direction of the invariants, and the pair $(\mathrm{cov}(\mathcal{N}), \mathrm{non}(\mathcal{N}))$, we use the following two lemmas, which are reformulations of results of Shelah.[7]

Recall Definition 1.7 (which is useful because of Lemma 1.8 and satisfied for the initial forcing according to Lemma 1.18).

LEMMA 2.16. *Assume* $\circledcirc_{\mathrm{i}}(P, \lambda)$. *Then* $\circledcirc_{\mathrm{i}}(j(P), \mathrm{cf}(j(\lambda)))$.
So if $\kappa \neq \lambda$, *then* $\circledcirc_{\mathrm{i}}(j(P), \lambda)$, *and if* $\kappa = \lambda$, *then* $\circledcirc_{\mathrm{i}}(j(P), \theta)$.

PROOF. Let $\bar{y} = (y_\alpha)_{\alpha<\lambda}$ be the sequence of P-names witnessing $\circledcirc_{\mathrm{i}}(P, \lambda)$. Note that $j(\bar{y})$ is a sequence of length $j(\lambda)$; we denote the β-th element by $(j(\bar{y}))_\beta$. So M thinks: For every $j(P)$-name r of a real $(\exists \alpha \in j(\lambda))\, (\forall \beta \in j(\lambda) \setminus \alpha)\, \neg (j(\bar{y}))_\beta \, \mathrm{R_i} \, r$. This is absolute. In particular, pick in V a cofinal subset A of $j(\lambda)$ of order type $\mathrm{cf}(j(\lambda)) =: \mu$. Then $j(\bar{y}) \restriction A$ witnesses that $\circledcirc_{\mathrm{i}}(j(P), \mu)$ holds. ⊣

We have seen in Lemma 1.20 that $\boxplus_2(P^4, \lambda_2, \lambda_4)$ holds and implies that P^4 forces $\mathrm{cov}(\mathcal{N}) \geq \lambda_2$ and $\mathrm{non}(\mathcal{N}) \leq \lambda_4$ (the latter being trivial in the case of P^4).

LEMMA 2.17. *Assume* $\boxplus_2(P, \lambda, \mu)$. *If* $\kappa > \lambda$, *then* $\boxplus_2(j(P), \lambda, |j(\mu)|)$; *if* $\kappa < \lambda$, *then* $\boxplus_2(j(P), \lambda, \mu)$.

PROOF. Let $(S, \prec)$ and $\bar{r}$ witness $\boxplus_2(P, \lambda, \mu)$.

M thinks that

for each $j(P)$-name N of a null set

$$(\exists s \in j(S))\, (\forall t \in j(S))\, t \succ s \to j(P) \Vdash (j(\bar{r}))_t \notin N, \quad (*)$$

which is absolute.

If $\kappa > \lambda$, then $j(\lambda) = \lambda$, and $j(S)$ is λ-directed in M and therefore in V as well, and so we get $\boxplus_2(j(P), \lambda, |j(\mu)|)$.

So assume $\kappa < \lambda$. We claim that $j'' S$ and $j'' \bar{r}$ witness $\boxplus_2(j(P), \lambda, \mu)$. $j'' S$ is isomorphic to S, so directedness is trivial. Given a $j(P)$-name N, without loss of generality in M, there is in M a bound $s \in j(S)$ as in $(*)$. As $j'' S$ is cofinal in $j(S)$ (according to Lemma 2.12(f)), there is some $s' \in S$ such that $j(s') \succ s$. Then for all $t' \succ s'$, i.e., $j(t') \succ j(s')$, we get $j(P) \Vdash j(r_t) \notin N$. ⊣

2.5. The main theorem. We now have everything required for the main result:

THEOREM 2.18. *Assume GCH and that* $\aleph_1 < \kappa_7 < \lambda_1 < \kappa_6 < \lambda_2 < \kappa_5 < \lambda_3 < \lambda_4 < \lambda_5 < \lambda_6 < \lambda_7$ *are regular,* κ_i *strongly compact for* $i = 5, 6, 7$. *Then there is a ccc order* P^7 *forcing*

$$\mathrm{add}(\mathcal{N}) = \lambda_1 < \mathrm{cov}(\mathcal{N}) = \lambda_2 < \mathfrak{b} = \lambda_3 < \mathfrak{d} = \lambda_4 < \mathrm{non}(\mathcal{N}) = \lambda_5 < \mathrm{cof}(\mathcal{N}) = \lambda_6 < 2^{\aleph_0} = \lambda_7.$$

[7] S. Shelah, personal communication.

PROOF. Let $j_i : V \to M_i$ be the Boolean ultrapower embedding with $\mathrm{cf}(j(\kappa_i)) = \lambda_i$ (for $i = 5, 6, 7$). Recall that P^4 is an iteration of length δ_4. We set $P^5 := j_5(P^4)$, $P^6 := j_6(P^5)$, and $P^7 := j_7(P^6)$; and $\delta_5 := j_5(\delta_4)$, $\delta_6 := j_6(\delta_5)$ and $\delta_7 := j_7(\delta_6)$.

It is enough to show the following:

(a) P^i is a FS ccc iteration of length δ_i and forces $2^{\aleph_0} = \lambda_i$ for $i = 4, 5, 6, 7$.
(b) $P^i \Vdash (\mathrm{add}(\mathcal{N}) = \lambda_1 \,\&\, \mathfrak{b} = \lambda_3 \,\&\, \mathfrak{d} = \lambda_4)$ for $i = 4, 5, 6, 7$.
(c) $P^i \Vdash \mathrm{non}(\mathcal{N}) \geq \lambda_5$ for $i = 5, 6, 7$.
$P^i \Vdash \mathrm{cof}(\mathcal{N}) \geq \lambda_6$ for $i = 6, 7$.
$P^i \Vdash \mathrm{cov}(\mathcal{N}) \leq \lambda_2$ for $i = 4, 5, 6, 7$.
(d) $P^i \Vdash \mathrm{cof}(\mathcal{N}) = \lambda_6$ for $i = 6, 7$.
(e) $P^i \vDash (\mathrm{cov}(\mathcal{N}) \geq \lambda_2 \,\&\, \mathrm{non}(\mathcal{N}) \leq \lambda_5)$ for $i = 4, 5, 6, 7$.

(a) was shown in Section 2.3.

(b): For P^4 this is Theorem 1.19. For P^5 use Lemma 2.15 (using for $\mathfrak{d}$ that $\kappa_5 < \lambda_3$). Using the same lemma again we get the result for P^6 and P^7 (using that $\kappa_i < \lambda_3$ for $i = 6, 7$ as well.)

(c): As $\kappa_5 > \lambda_2$, we have $\circledcirc_2(P^4, \kappa_5)$ (by Lemma 1.18), and thus $\circledcirc_2(P^5, \lambda_5)$ (by Lemma 2.16, as $\mathrm{cf}(j_5(\kappa_5)) = \lambda_5$), so $P^5 \Vdash \mathrm{non}(\mathcal{N}) \geq \lambda_5$ (Lemma 1.8). Repeating the same argument we get $\circledcirc_2(P^i, \lambda_5)$ for $i = 6, 7$ (as $\kappa_i \neq \lambda_5$ for $i = 6, 7$).

Analogously, as $\kappa_6 > \lambda_1$, we start with $\circledcirc_1(P^4, \kappa_6)$, get $\circledcirc_1(P^5, \kappa_6)$ (as $\kappa_5 \neq \kappa_6$) and then $\circledcirc_1(P^6, \lambda_6)$ (as $\mathrm{cf}(j_6(\kappa_6)) = \lambda_6$) and $\circledcirc_1(P^7, \lambda_6)$ (again as $\kappa_7 \neq \lambda_6$). So we get thus $P^i \Vdash \mathrm{cof}(\mathcal{N}) \geq \lambda_6$ for $i = 6, 7$.

Similarly, $\circledcirc_2(P^4, \lambda_2)$ holds, which is preserved by all embeddings, so we get $\mathrm{cov}(\mathcal{N}) \leq \lambda_2$.

(d): As P^6 forces the continuum to have size λ_6, the previous item implies $P^6 \Vdash \mathrm{cof}(\mathcal{N}) = \lambda_6$. And as in (b), this implies the same for P^7 (as $\kappa_7 < \lambda_1$, the value of $\mathrm{add}(\mathcal{N})$).

(e): $\boxplus_2(P^4, \lambda_2, \lambda_4)$ holds (cf. Lemma 1.20). So by Lemma 2.17 for the case $\kappa > \lambda$, and as $|j_5(\lambda_4)| = \lambda_5$, according to Lemma 2.12(e), $\boxplus_2(P^5, \lambda_2, \lambda_5)$ holds. I.e., P^5 forces $\mathrm{cov}(\mathcal{N}) \geq \lambda_2$ and $\mathrm{non}(\mathcal{N}) \leq \lambda_5$ (the latter being trivial as the continuum has size λ_5). For $i = 6, 7$, the same lemma, now for the case $\kappa < \lambda$, gives $\boxplus_2(P^i, \lambda_2, \lambda_5)$, i.e., P^i forces $\mathrm{cov}(\mathcal{N}) \geq \lambda_2$ and $\mathrm{non}(\mathcal{N}) \leq \lambda_5$. ⊣

2.6. An alternative. In the same way we can prove the consistency of

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

(I.e., we can replace $\mathfrak{b}$ and $\mathfrak{d}$ by $\mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M})$, respectively.)

For this, we use the following relation as R_3:

$$f \,\mathrm{R}_3\, g, \text{ if } f, g \in \omega^\omega \text{ and } (\forall^* n \in \omega)\, f(n) \neq g(n).$$

By a result of [1, 12] (cf. [2, 2.4.1 and 2.4.7]) we have

$$\mathrm{non}(\mathcal{M}) = \mathfrak{b}_3 \text{ and } \mathrm{cov}(\mathcal{M}) = \mathfrak{d}_3.$$

As before, we use that an iteration where each iterand has size $<\lambda_3$ is $(\mathrm{R}_3, \lambda_3)$-good.

To define P^4, we use partial eventually different (instead of partial Hechler) forcings.

Unlike for $(\mathfrak{b}, \mathfrak{d})$, we do not know whether $\mathrm{non}(\mathcal{M}) = \lambda$ is generally preserved if $\kappa \neq \lambda$ and $\mathrm{cov}(\mathcal{M}) = \lambda$ is preserved if κ is small; but we can use the same argument for $(\mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M}))$ that we have used for $(\mathrm{cov}(\mathcal{N}), \mathrm{non}(\mathcal{N}))$. So we can get the analog

of Lemma 1.20 that proves that non($\mathcal{M}$) is large and cov($\mathcal{M}$) small; and $\circledcirc_3$ implies that non($\mathcal{M}$) is small and cov($\mathcal{M}$) large.

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