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Even more simple cardinal invariants

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Abstract Using GCH, we force the following: There are continuum many simple cardinal characteristics with pairwise different values.

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1 Introduction

The union of countably many Lebesgue nullsets is again a nullset. On the other hand, there are $2^{\aleph_0}$ many nullsets with non-null union. If we assume $\neg$CH, i.e. $2^{\aleph_0} > \aleph_1$, then it is interesting to ask about the minimal size of a family of nullsets with non-null union. This is a cardinal number between (including) $\aleph_1$ and $2^{\aleph_0}$. Such cardinal numbers (or their definitions) are called cardinal characteristics.

There are numerous examples of such characteristics using notions from measure theory, topology or combinatorics. If a and b are such characteristics, on can learn something about the underlying notions by either proving dependencies (e.g. $a \leq b$) in ZFC, or by showing that a and b are independent (usually by finding forcing notions P and Q such that P forces $a < b$ and Q forces $b < a$, or by using MA).

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Blass [1] introduced a classification of cardinal characteristics, and in particular defined Π^0_1 characteristics. Goldstern and Shelah [2] showed that there are many Π^0_1 characteristics. In particular:

> Assume CH. Assume that $\kappa_\epsilon^{\aleph_0} = \kappa_\epsilon$ for all $\epsilon \in \omega_1$ and that the functions $f_\epsilon, g_\epsilon : \omega \to \omega$ ($\epsilon \in \omega_1$) are sufficiently different. Then there is a partial order P which preserves cardinals and forces that $c^\forall(f_\epsilon, g_\epsilon) = \kappa_\epsilon$ for all $\epsilon \in \omega_1$.

(The Π^0_1 cardinal characteristics $c^\forall(f, g)$ are defined in 2.1.)

If the κ_ϵ are pairwise different, then in the forcing extension the size of the continuum is at least $\aleph_{\omega_1}$. So $\aleph_1$, the number of different characteristics in the forcing extension, is smaller than the continuum.

In this paper, we assume GCH in the ground model and modify the construction to get a universe satisfying:

> There are continuum many pairwise different cardinal characteristics of the form $c^\forall(f_\epsilon, g_\epsilon)$.

We give a relatively simple proof for this result. A slightly stronger result was promised in [2] to appear in a paper called 448a, which never materialized: a "perfect set" of pairwise different characteristics. Shelah and the author are working on new creature forcing iteration techniques. One of the applications will hopefully be a proof of the perfect set result, as well as similar results for the dual notions $c^\exists$ (which require lim-inf constructions, cf. [3]). All these constructions are considerably more difficult than the ones in this paper.

2 The theorem and the forcing

Definition 2.1 Let $f, g : \omega \to \omega\setminus 1$ be such that $f(n) > g(n)$ for all n.

- $B : \omega \to \mathfrak{P}(\omega)$ is an (f, g)-slalom if $B(n) \subseteq f(n)$ and $|B(n)| < g(n)$ for all $n \in \omega$.
- A family $\mathfrak{B}$ of (f, g)-slaloms $\forall$-covers, if for all $\nu \in \prod_{n\in\omega} f(n)$ there is a $B \in \mathfrak{B}$ such that $\nu(n) \in B(n)$ for all $n \in \omega$.
- $c^\forall(f, g)$ is the minimal size of a $\forall$-covering family of (f, g)-slaloms.

See [2] for more about $c^\forall(f, g)$. We are going to prove the following:

Theorem 2.2 *Assume that CH holds, that $\mu = \mu^{\aleph_0}$, and for $\epsilon \in \mu$, $\kappa_\epsilon < \mu$ is a cardinal such that $\kappa_\epsilon^{\aleph_0} = \kappa_\epsilon$. Then there is a forcing notion P and there are P-names f_ϵ, g_ϵ such that P preserves cardinals and forces the following: $2^{\aleph_0} = \mu$, and $c^\forall(f_\epsilon, g_\epsilon) = \kappa_\epsilon$ for all $\epsilon \in \mu$.*

If we assume GCH, we can find such μ and κ_ϵ such that the κ_ϵ are pairwise different,[1] i.e., we get continuum many pairwise different invariants in the extension.

For the rest of the paper we assume that the conditions of the theorem are satisfied (in the ground model).

[1] Let $\mu = \aleph_\mu$ be the ω_1-th iterate of the function $\alpha \mapsto \aleph_\alpha$ (taking the union at limits), and pick cardinals $\kappa_\epsilon < \mu$ with uncountable cofinality.

We will use $\epsilon, \epsilon', \epsilon_1, \ldots$ for elements of μ.

Assumption 2.3 $(g_{n,l})_{n\in\omega, 0\leq l<2^n}$ and $(f_{n,l})_{n\in\omega,-1\leq l<2^n}$ are sufficiently fast growing sequences of natural numbers, such that $0 = f_{0,-1}$, $f_{n+1,-1} = f_{n,2^n-1}$ and $f_{n,l-1} \ll g_{n,l} \ll f_{n,l}$. We set $f_{\max}(m) = f_{m,2^m-1}$ and $g_{\min}(m) = g_{m,0}$.

Sufficiently fast growing means the following:[2] $g_{n,l} > 2 \cdot f_{n,l-1}^{n\cdot f_{\max}(n-1)^n}$, and $f_{n,l} > g_{n,l}^{n+1}$. ($f_{\max}(n-1)^n$ denotes the n-th power of $f_{\max}(n-1)$.)

We identify $[0, 2^n - 1]$ with the set of binary sequences of length n, ordered lexicographically. So for $s \in 2^n$, we can define $f_s = f_{n,s}$ and $g_s = g_{n,s}$. If $\eta \in 2^\omega$, then we can define $f : \omega \to \omega$ by $f(n) = f_{\eta\restriction n}$, and g analogously.

We will define P so that P adds Sacks generics η_ϵ ($\epsilon \in \mu$) and forces that $c^\forall(f_\epsilon, g_\epsilon) = \kappa_\epsilon$ for the (f_ϵ, g_ϵ) defined by η_ϵ.

Fix $s \in 2^n$. If a is a subset of f_s (i.e. of the interval $[0, f_s - 1]$), we set $\mu_s(a) = \ln_{g_s}(|a|)$. (Alternatively, We could use any other g_s-big norm as well, i.e. a norm satisfying the following:)

Lemma 2.4 *$\mu_s : \mathfrak{P}(f_s) \to \mathbb{R}$ satisfies: $(a, b \subseteq f_s)$*

- *If $b \subseteq a$, then $\mu_s(a) \geq \mu_s(b)$.*
- *$\mu_s(f_s) \geq n$.*
- *$\mu_s(\{t\}) < 1$ for all $t \in f_s$.*
- *If F is a function from a to g_s, then there is a $b \subseteq a$ such that $F \restriction b$ is constant and $\mu_s(b) \geq \mu_s(a) - 1$.*

Note that $\mu_s(b) \geq 2$ implies that $|b| > g_s$.

Set $\omega^{\leq n} = \bigcup_{l\leq n} \omega^l$. We will use trees $T \subseteq \omega^{<\omega}$ (or $2^{<\omega}$ or $\omega^{\leq n}$). For a node $s \in T \cap \omega^n$, n is called the height of s. A branch b in T is a maximal chain (i.e. a maximal set of pairwise comparable nodes). We can identify b with an element of ω^ω (or ω^n), and denote with $b \restriction h$ the element of b of height h (for all $h < \omega$ or $h < n$, respectively). A front F in T is a set of pairwise incomparable nodes such that every branch of T hits a node in F. When talking about nodes, we use the terms "comparable" and "compatible" interchangeably. We use the symbol $\perp$ for incompatible (i.e. incomparable, when talking about nodes), and we use $\parallel$ for compatible. A splitting node s is a node with at least two immediate successors. The first splitting node is called $\mathrm{stem}(T)$.

A Sacks condition T is a perfect tree, i.e. $T \subseteq 2^{<\omega}$ is such that for every $s \in T$ there is a splitting node $s' > s$. Equivalently, along every branch of T there are infinitely many splitting nodes. So the set of the n-th splitting nodes forms a front.

We will use Sacks conditions as well as other "lim-sup" finite splitting tree forcings. Actually we will use finite approximations to such trees, but it might be useful to first specify the objects we are approximating: For $\eta \in 2^\omega$, T is an η-tree, if $T \subseteq \omega^{<\omega}$ is a tree without leaves ("dead ends") such that $s(n) < f_{\eta\restriction n}$ for all $s \in T$. For an η-tree T and $s \in T \cap \omega^n$, we set $\mu_T(s) = \mu_s(A)$, where A is the set of immediate

[2] The second inequality guarantees that there is a g-big norm (cf. 2.4), and the first one is extracted from the proof of 4.3. Obviously one can try to find weaker conditions, but we do not try to find optimal bounds in this paper.

T-successors of s. T is fat if $\limsup_{n\to\infty}(\mu_T(b \restriction n)) = \infty$ for every branch b of T. Q_η is the partial order of fat trees ordered by inclusion.[3]

It is easy to see (and analogous to Sacks forcing) that all forcing notions Q_η are proper and ω^ω-bounding.[4] In [2], Goldstern and Shelah picked ω_1 many different η_ϵ, defined P_ϵ to be the countable support product of κ_ϵ many copies of Q_{η_ϵ}, and defined P to be the countable support product of the P_ϵ. Then P forces $c^\forall(f_\epsilon, g_\epsilon) = \kappa_\epsilon$.

We need $\mu > 2^{\aleph_0}$ many different η, so η_ϵ will be a name (for a Sacks real). Then we again want to use κ_ϵ many copies of Q_{η_ϵ}. Instead of using a composition of forcings, we more explicitly use finite approximations to fat trees:

Definition 2.5 Assume $s \in 2^n$.

- T is an s-tree if $T \subseteq \omega^{\leq n+1}$ is a tree, every branch has length $n+1$ and $t(m) < f_{s\restriction m}$ for each $m \leq n$ and $t \in T \cap \omega^{m+1}$.
- For $m \leq n$ and $t \in T \cap \omega^m$, t is an l-large splitting node, if $\mu_{s\restriction m}(A) \geq l$ for the set A of immediate T-successors of t.
- T has l-large splitting if the set of l-large splitting nodes forms a front.

Definition 2.6
- For every ϵ in μ, pick some I_ϵ of size κ_ϵ such that μ and all the I_ϵ are pairwise disjoint. Set $I = \mu \cup \bigcup_{\epsilon\in\mu} I_\epsilon$.
- We define $\varepsilon : I \to I$: If $\alpha \in I_\epsilon$, then $\varepsilon(\alpha) = \epsilon$. If $\epsilon \in \mu$, then $\varepsilon(\epsilon) = \epsilon$.

I will be the index set of the product forcing. We will use $\alpha, \beta, \dots$ for elements of I.

Definition 2.7 $p \in P$ consists of the following objects, satisfying the following properties:

1. $\mathrm{dom}(p) \subseteq I$ is countable and closed under ε.
2. If $\epsilon \in \mathrm{dom}(p) \cap \mu$, then $p(\epsilon)$ is a Sacks condition.
3. If $\epsilon_1 \neq \epsilon_2 \in \mathrm{dom}(p) \cap \mu$, then $\mathrm{stem}(p(\epsilon_1))$ and $\mathrm{stem}(p(\epsilon_2))$ are incompatible.
4. If $\alpha \in \mathrm{dom}(p) \cap I_\epsilon$, then $p(\alpha)$ is a function from $p(\epsilon)$ to the power set of $\omega^{<\omega}$ satisfying the following:
   (a) If $s \in p(\epsilon) \cap 2^n$, then $p(\alpha, s) \subseteq \omega^{\leq n+1}$ is an s-tree.
   (b) If $s < t$ are in $p(\epsilon)$ and $s \in 2^n$, then $p(\alpha, s) = p(\alpha, t) \cap \omega^{\leq n+1}$.
   (c) For $l \in \omega$ and $s \in p(\epsilon)$ there is an $s' > s$ in $p(\epsilon)$ such that $p(\alpha, s')$ has l-large splitting.

Note that item 3 is a real restriction in the sense that P is not dense in the product defined as above but without item 3.

Item 4c implies also the following seemingly stronger variant (in 3.5 we will use yet another one): If $p \in P$, $\alpha \in I_\epsilon \cap \mathrm{dom}(p)$, $l \in \omega$ and $s \in p(\epsilon)$, then there is an $s' > s$ in $p(\epsilon)$ such that every branch in $p(\alpha, s')$ has l many l-large splitting nodes. (Any

[3] Q_η is a special case of a lim-sup finite splitting tree forcing Q, informally defined as follows: Q is defined by a finite splitting tree T_0 and a norm on the successor sets. $T \subseteq T_0$ is a condition of Q if for all branches b of T, the T-norm of $b \restriction n$ gets arbitrarily large.
Sacks forcing is a simple example of such a forcing: T_0 is $2^{<\omega}$. Pick $s \in 2^{<\omega}$ and set $A = \{s^\frown 0, s^\frown 1\}$. Then we set $\mu(A) = 1$ and $\mu(B) = 1$ for all proper subsets B of A.

[4] This holds of course for all lim-sup finite splitting tree forcings.

finite s-tree can be l-large for finitely many l only, so we can first extend s to some s'_0 witnessing l-largeness, then to some s'_1 witnessing l_1-largeness for some sufficiently large l_1 etc.)

The order on P is the natural one:

Definition 2.8 For $p, q \in P$, we define $q \leq p$ by:

- $\mathrm{dom}(q) \supseteq \mathrm{dom}(p)$.
- If $\alpha \in \mathrm{dom}(p) \cap \mu$, then $q(\alpha) \subseteq p(\alpha)$.
- If $\alpha \in \mathrm{dom}(p) \cap I_\epsilon$ and $s \in q(\alpha) \cap \omega^n$, then $q(\alpha, s) \subseteq p(\alpha, s)$.

Definition 2.9 • For $\alpha \in I$, η_α is the P-name of the generic at α.[5]

- $f_\epsilon : \omega \to \omega$ is the P-name for the function defined by $f_\epsilon(n) = f_{\eta_\epsilon \restriction n}$, and analogously for g_ϵ.

It is straightforward to check[6] that $\leq$ is transitive and that η_α is indeed the name of an element of ω^ω. If $\alpha \in \mu$, then $\eta_\alpha \in 2^\omega$, otherwise $\eta_\alpha(n) < f_{\varepsilon(\alpha)}(n)$ for all $n \in \omega$.

3 Preservation of cardinals, $\kappa_\epsilon \leq c^\forall(f_\epsilon, g_\epsilon)$

Lemma 3.1 *P is $\aleph_2$-cc.*

Proof Assume towards a contradiction that A is an antichain of size $\aleph_2$. Without loss of generality $\{\mathrm{dom}(p) : p \in A\}$ forms a Δ-system with root $u \subseteq I$. We fix enumerations $\{\alpha_0^p, \alpha_1^p, \dots\}$ of $\mathrm{dom}(p)$ for all $p \in A$. We can assume that the following are independent of $p \in A$ (for $i, j \in \omega$ and $\beta \in u$): $p \restriction u$; the statements "$\alpha_i^p = \beta$", "$\alpha_i^p \in \mu$", "$\alpha_i^p = \varepsilon(\alpha_j^p)$"; and the sequence of Sacks conditions $(p(\alpha_i^p) : \alpha_i^p \in \mu)$.

Pick elements p, q of A. We will show $p \parallel q$. Take $p \cup q$ and modify it the following way: If $i \in \omega$ is such that $\alpha_i^p \in \mu$ and $\alpha_i^p \neq \alpha_i^q$, then we extend the stems of (the identical Sacks conditions) $p(\alpha_i^p)$ and $q(\alpha_i^q)$ in an incompatible way (e.g. at the first split, we choose the left node for p and the right one for q). We call the result of this r. Then $r \in P$ and $r \leq p, q$: Assume that $\alpha_i^p \neq \alpha_j^q$ are in $\mathrm{dom}(r) \cap \mu$. If $i \neq j$, then $q(\alpha_j^q) = p(\alpha_j^p)$ has an incompatible stem with $p(\alpha_i^p)$, so the (possibly longer) stems in r are still incompatible. If $i = j$, we made the stems in r incompatible. □

Lemma 3.2 *P has fusion and pure decision. In particular P has continuous reading of names, and P is is proper and ω^ω-bounding. Therefore P preserves all cardinals and forces $2^{\aleph_0} = \mu$.*

The proof is straightforward, but the notation a bit cumbersome:

Definition 3.3 • $\mathrm{pos}(p, \leq n)$ is the set of sequences $a = (a(\alpha))_{\alpha \in \mathrm{dom}(p)}$ such that $a(\alpha) \in \omega^{n+1}$, $a(\alpha) \in p(\alpha)$ for $\alpha \in \mu$, and $a(\alpha) \in p(\alpha, a(\varepsilon(\alpha)))$ otherwise.

5 More formally: If $\epsilon \in \mu$, then $\eta_\epsilon = \bigcup_{p \in G} \mathrm{stem}(p(\epsilon))$.
If $\alpha \notin \mu$, then $\eta_\alpha = \bigcup \{\mathrm{stem}(p(\alpha, s)) : p \in G, s \in \mathrm{stem}(p(\varepsilon(\alpha)))\}$.

6 This uses e.g. the fact that for every $p \in P$, $\alpha \in I$ and $h \in \omega$ there is a $q \leq p$ such that $\alpha \in \mathrm{dom}(q)$ and all stems in q have height at least h. To see that 2.7.3 does not prevent us to increase the domain, use the argument in the proof of 4.2.

- For $a \in \text{pos}(p, \leq n)$, $p \wedge a$ is the result of extending the stems in p to a.[7]
- Let τ be a P-name. τ is $(\leq n)$-decided by p, if for all $a \in \text{pos}(p, \leq n)$, $p \wedge a$ decides τ (i.e. there is some $x \in V$ such that $p \wedge a$ forces $\tau = \check{x}$).
- Assume $q \leq p$. $\text{pos}(p, \leq n) \equiv \text{pos}(q, \leq n)$ means that for all $a \in \text{pos}(p, \leq n)$ there is exactly one $b \in \text{pos}(q, \leq n)$ such that a is b restricted to $\text{dom}(p)$. In other words: On $\text{dom}(p)$, p and q are identical up to height $n+1$, and the stems of q outside of $\text{dom}(p)$ have height at least $n+1$. If $\text{dom}(q) = \text{dom}(p)$, then $\text{pos}(p, \leq n) \equiv \text{pos}(q, \leq n)$ is equivalent to $\text{pos}(p, \leq n) = \text{pos}(q, \leq n)$.
- $p \in P$ is finitary if $\text{pos}(p, \leq n)$ is finite for all $n \in \omega$.

Lemma 3.4 *The set of finitary conditions is dense in P.*

(Enumerate $\text{dom}(p)$ as $(\alpha_i)_{i\in\omega}$, and extend all stems at α_i to height at least i.)

The set of finitary conditions is not open, but we get the following: If $p \in P$ is finitary and $q \leq p$ is such that $\text{dom}(q) = \text{dom}(p)$, then q is finitary.

We now consider a strengthening of the property 2.7.4c of conditions in P:

Definition 3.5 p is uniform, if for all $\alpha \in I_\epsilon$ and $l \in \omega$ there is a $h \in \omega$ such that $p(\alpha, s)$ is l-large for all $s \in p(\epsilon) \cap \omega^h$.

First, we briefly comment on the connection between fronts and maximal antichains in Sacks conditions:[8] Let T be a perfect tree. "A is a front" is stronger than "A is a maximal antichain". In particular, it is possible that $p \in P$ is not uniform, e.g. that for $\alpha \in I_\epsilon$ the set of nodes $s \in p(\epsilon)$ such that $p(\alpha, s)$ has 1-large splitting contains a maximal antichain, but not a front. (For example, we can assume that $p(\epsilon) = 2^{<\omega}$, $p(\alpha, 0^n)$ has a trunk of length at least $n+1$, but that $p(\alpha, 0^n{}^\frown 1)$ has 1-large splitting. So the nodes that guarantee 1-large splitting contain the maximal antichain $\{1, 01, 001, \dots\}$, but no front.) However, if $A_1, A_2, \dots$ are maximal antichains in T, we can find a perfect tree $T' \subseteq T$ such that $A_i \cap T'$ is a front in T'. (Construct finite approximations T_i to T': For every leaf $s \in T_{i-1}$, extend s to some s' above some element of A_i and further to some splitting node s''. Let T_i contain the successors of all these splitting nodes.)

This implies that the uniform conditions are dense:

Lemma 3.6 *Assume $p \in P$. Then there is a uniform $q \leq p$ such that* $\text{dom}(q) = \text{dom}(p)$.

Proof Fix $\epsilon \in \mu$. Enumerate $\text{dom}(p) \cap I_\epsilon$ as $\alpha_0, \alpha_1, \dots$. For $i, l \in \omega$ and $s \in p(\epsilon)$ and there is an $s' > s$ such that $p(\alpha_i, s')$ has l-large splitting. This gives (open) dense sets $D_{i,l} \subseteq p(\epsilon)$. Choose maximal antichains $A_{i,l} \subseteq D_{i,l}$. Then there is a perfect tree $q(\epsilon) \subseteq p(\epsilon)$ such that $A_{i,l} \cap q$ is a front in q for all $i, l \in \omega$. □

We can also fix p up to some height h and do the construction starting with h. Then we get:

7 More formally: $[p \wedge a](\epsilon)$ is $\{s \in p(\epsilon) : s \parallel a(\epsilon)\}$ for $\epsilon \in \mu$, and $[p \wedge a](\alpha, s)$ is $\{t \in p(\alpha, s) : t \parallel a(\alpha)\}$ for $\alpha \in I_\epsilon$. $p \wedge a$ is again a condition in P.

8 Of course, the same applies to all lim-sup finite splitting tree forcings.

Lemma 3.7 *Assume that* $p \in P$, $h \in \omega$ *and that* $\mathrm{pos}(p, \leq h)$ *is finite. Then there is a finitary, uniform* $q \leq p$ *such that* $\mathrm{dom}(p) = \mathrm{dom}(q)$ *and* $\mathrm{pos}(p, \leq h) = \mathrm{pos}(q, \leq h)$.

Using this notation, we can finally prove continuous reading of names:

Proof of Lemma 3.2. **Pure decision:** Fix $p \in P$ finitary, $h \in \omega$ and a P-name τ for an ordinal. We can find a finitary, uniform $q \leq p$ which $(\leq h)$-decides τ, such that $\mathrm{pos}(p, \leq h) \equiv \mathrm{pos}(q, \leq h)$.

Proof: Enumerate $\mathrm{pos}(p, \leq h)$ as $a_0, \dots, a_{l-1}$. We just strengthen each $p \wedge a_i$ to decide τ and glue back together the resulting conditions. More formally: Set $p_0 = p$. Let $0 \leq i < l$. We assume that we have constructed $p_i \leq p$ such that $\mathrm{pos}(p_i, \leq h) \equiv \mathrm{pos}(p, \leq h)$. Let $b \in \mathrm{pos}(p_i, \leq h)$ correspond to $a_i \in \mathrm{pos}(p, \leq h)$, and find a finitary $p' \leq p_i \wedge b$ deciding τ, so that the length of all stems are at least $h + 1$. Define p_{i+1} the following way: $\mathrm{dom}(p_{i+1}) = \mathrm{dom}(p')$.

- If $\alpha \in \mathrm{dom}(p') \setminus \mathrm{dom}(p_i)$, then $p_{i+1}(\alpha) = p'(\alpha)$.
- If $\epsilon \in \mathrm{dom}(p_i) \cap \mu$, then $p_{i+1}(\epsilon) = p'(\epsilon) \cup \{s \in p_i : s \perp b(\epsilon)\}$.
- Assume that $\alpha \in \mathrm{dom}(p_i) \cap I_\epsilon$. If $s \in p_i(\epsilon) \setminus p'(\epsilon)$, or if $s \in p'(\epsilon)$ is incompatible with $b(\epsilon)$, then $p_{i+1}(\alpha, s) = p_i(\alpha, s)$. Otherwise, $p_{i+1}(\alpha, s) = p'(\alpha, s) \cup \{t \in p_i(\alpha, s) : t \perp b(\alpha)\}$.

Note that $p_{i+1} \leq p_i$, $\mathrm{pos}(p_{i+1}, \leq h) \equiv \mathrm{pos}(p_i, \leq h)$ and $p_{i+1} \wedge b = p'$. Let $q \leq p_l$ be finitary and uniform such that $\mathrm{pos}(q, \leq h) \equiv \mathrm{pos}(p_l, \leq h)$. Then $q \leq p$, $\mathrm{pos}(q, \leq h) \equiv \mathrm{pos}(p, \leq h)$ and $q \wedge b$ decides τ for each $b \in \mathrm{pos}(q, \leq h)$.

Fusion: Assume the following:

- $p_0 \geq p_1 \geq \dots$ is a sequence of finitary, uniform conditions in P.
- $h_0, h_1, \dots$ is an increasing sequence of natural numbers.
- $\mathrm{pos}(p_{n+1}, \leq h_n) \equiv \mathrm{pos}(p_n, \leq h_n)$.
- $u_n \subseteq \mathrm{dom}(p_n)$ is finite and ε-closed for $n \in \omega$. Every $\alpha \in \bigcup_{n \in \omega} \mathrm{dom}(p_n)$ is contained in infinitely many u_i.
- If $\epsilon \in u_n \cap \mu$, then the height of the front of n-th splitting nodes in $p_n(\alpha)$ is below h_n (i.e. the front is a subset of $2^{\leq h_n}$).
  If $\alpha \in u_n \cap I_\epsilon$ and $s \in p_n(\epsilon) \cap \omega^{h_n}$, then $p_n(\epsilon, s)$ has n-large splitting.

Then there is a canonical limit $q \leq p_i$ in P.

Proof: $q(\epsilon)$ is defined by $\mathrm{dom}(q) = \bigcup_{n \in \omega} \mathrm{dom}(p_n)$, $q(\epsilon) \cap 2^{h_i + 1} = p_i(\epsilon)$, and analogously for $q(\alpha, s)$. Pick $\alpha \in P_\epsilon$, $s \in q(\epsilon)$ and $l \in \omega$. Pick $n > l$ such that $\alpha \in u_n$. Then $p_n(\alpha, s')$ has l-large splitting for some $s' \parallel s$ in $p_n(\epsilon)$.

Continuous reading of names, ω^ω-bounding: Let ν be the name of a function from ω to ω and $p \in P$. Then there is an increasing sequence $(h_i)_{i \in \omega}$ and a finitary $q \leq p$ which $(\leq h_i)$-decides $\nu \restriction h_i$ for all $i \in \omega$.[9]

Proof: Pick $p_0 \leq p$ finitary and uniform. Construct a sequence $p_0 \geq p_1 \geq \dots$ suitable for fusion the following way: Given p_i, find (by some bookkeeping) $u_i \subseteq \mathrm{dom}(p_i)$, pick h_i large enough to witness largeness of p_i u_i, and then (using pure decision) find p_{i+1} which $(\leq h_i)$-decides $\nu \restriction h_i$.

[9] Or $\nu \restriction 2 \cdot h_i$ or just $\nu(i)$ etc., that does not make any difference at that stage.

Properness: Let χ be a sufficiently large regular cardinal, and let $N \prec H(\chi)$ be a countable elementary submodel, $p \in P \cap N$. We have to show that there is a $q \leq p$ forcing $\tau \in \check{N}$ for every P-name $\tau \in N$ for an ordinal. We can enumerate (in V) all the names τ_i of ordinals in N. As above, we pick an sequence $p \geq p_0 \geq p_1 \geq \dots$ suitable for fusion such that $p_i \in N$ is $(\leq h_i)$-deciding τ_i (for the h_i used for fusion). In V, we fuse the sequence to some $q \leq p$. Then q is N-generic.

Preservation of cardinals follows from $\aleph_2$-cc and properness.

Continuum is forced to be μ: Let τ be the name of a real, and $p \in P$. There is a $q \leq p$ continuously reading τ. I.e. τ can be read off $q \in P$ in a recursive manner (using a real parameter in the ground model). The size of P is $\mu^{\aleph_0} = \mu$, so there are only μ many reals that can be read continuously from some q. On the other hand, the η_ϵ are forced to be pairwise different. □

Lemma 3.8 *P forces that $\kappa_\epsilon \leq c^\forall(f_\epsilon, g_\epsilon)$.*

Proof Assume the following towards a contradiction: $\aleph_1 \leq \lambda < \kappa_\epsilon$, B_i $(i \in \lambda)$ are P-names, and p forces that $\{B_i : i \in \lambda\}$ is a covering family of (f_ϵ, g_ϵ)-slaloms.

For every B_i, find a maximal antichain A_i of conditions that read B_i continuously. Because of $\aleph_2$-cc, $X = \bigcup_{i\in\lambda, a\in A_i} \mathrm{dom}(a)$ has size $\lambda < \kappa_\epsilon$, so there is an $\alpha \in I_\epsilon \setminus X$. Find a $q \leq p$ and an $i \in \lambda$ such that q forces that $\eta_\alpha(n) \in B_i(n)$ for all n. Without loss of generality, q is uniform and stronger than some $a \in A_i$, i.e. $q \restriction \mathrm{dom}(q)\setminus\{\alpha\}$ continuously reads B_i. (And $q \restriction \{\epsilon\}$ continuously reads $\eta_\epsilon \restriction n$ and therefore $g_\epsilon(n)$.)

Pick some h big enough such that $q(\alpha, s)$ has 2-large splitting for all $s \in q(\epsilon) \cap \omega^h$. Increase the stems of $q(\beta)$ for $\beta \in \mathrm{dom}(q)\setminus\{\alpha\}$ to some height $h' > h$ to decide $g_\epsilon \restriction h+1$ as well as $B_i \restriction h+1$. So the resulting condition r decides for all $m \leq h$ the values of $B_i(m)$ and $g_\epsilon(m)$. B is the name of an (f_ϵ, g_ϵ)-slalom, and therefore $|B_i(m)| < g_\epsilon(m)$. Also, $r(\alpha, \eta_\epsilon \restriction h)$ has a 2-large splitting node at some $m \leq h$. But that implies that there are more than $g_\epsilon(m)$ many possibilities for $\eta_\epsilon(m)$. So we can extend the stem or r at α and choose some $\eta_\alpha(m) \notin B_i(m)$, a contradiction. □

4 The complete subforcing P_ϵ, $\kappa_\epsilon \geq c^\forall(f_\epsilon, g_\epsilon)$

Definition 4.1 $P_\epsilon \subseteq P$ consists of conditions with domain in $\{\epsilon\} \cup I_\epsilon$.

Lemma 4.2 *P_ϵ is a complete subforcing of P, and also has continuous reading of names. In particular, P_ϵ forces $2^{\aleph_0} = \kappa_\epsilon$.*

Proof Continuous reading is analogous to the case of P. To see that P_ϵ is a complete subforcing, it is enough to show that for all $p \in P$ there is a reduction $p' \in P_\epsilon$ (i.e. for all $q \leq p'$ in P_ϵ, q and p are compatible in P). Set $p' = p \restriction (\{\epsilon\} \cup I_\epsilon)$, pick $q \leq p'$ in P_ϵ, and set $r = q \cup p \restriction I\setminus(I_\epsilon \cup \{\epsilon\})$. If $\epsilon \in \mathrm{dom}(p)$, then r is a condition in P (and stronger than q, p). Otherwise, it could happen that $\mathrm{stem}(q, \epsilon)$ is compatible with $\mathrm{stem}(p, \epsilon')$ for some $\epsilon' \in \mu$. We can assume without loss of generality that $\mathrm{stem}(q, \epsilon) \supseteq \mathrm{stem}(p, \epsilon')$. Increase the stems of both $q(\epsilon)$ and $p(\epsilon')$ to be incompatible. Then for any ϵ'', $\mathrm{stem}(q, \epsilon)$ and $\mathrm{stem}(p, \epsilon'')$ are incompatible as well. □

To complete the proof of the main theorem, it remains to be shown:

Lemma 4.3 *P forces that the (f_ϵ, g_ϵ)-slaloms in $V[G_{P_\epsilon}]$ form a cover, in particular that $c^\forall(f_\epsilon, g_\epsilon) \leq \kappa_\epsilon$.*

For the proof, we need more notation:
Let $q \in P$.

- For $\epsilon \in \mu$, n is a splitting level of $q(\epsilon)$ if there is some splitting node $s \in q(\epsilon) \cap \omega^n$. n is a unique splitting level if there is exactly one such s.
- Let $\alpha \in I_\epsilon$. n is a splitting level of $q(\alpha)$ if there is some $s \in q(\epsilon) \cap \omega^n$ such that some $t \in q(\alpha, s) \cap \omega^n$ is a splitting node. n is a unique splitting level of $q(\alpha)$ if there is exactly one such s, and if moreover for this s there is exactly one t as well.
- q has unique splitting below h if for all $n < h$ there is at most one $\alpha \in I$ such that n is splitting level of $q(\alpha)$, and in this case n is a unique splitting level of $q(\alpha)$. q has unique splitting if q has unique splitting below all h.
- If q has unique splitting below h, we enumerate (in increasing order) the splitting levels below h (for any α) by $(m_i^{\text{split}})_{i \in l}$ and the corresponding α by $(\alpha_i^{\text{split}})_{i \in l}$. If q has unique splitting, we get the corresponding infinite sequences.[10]
- q has unique, large splitting if it has unique splitting and if for $\alpha_i^{\text{split}} \notin \mu$, the splitting node t of height m_i^{split} is i-large.
- Let ν be a P-name for a sequence in $\prod_{n \in \omega} f_{\max}(n)$. q rapidly reads ν below h if:
  - q has unique, large splitting below h.
  - If $\alpha \in I_\epsilon$, then all splits at α are higher than some split at ϵ, i.e.: If $\alpha_i^{\text{split}} = \alpha$, then $\alpha_j^{\text{split}} = \epsilon$ for some $j < i$.
  - $\nu \restriction m_i^{\text{split}}$ is $(\leq m_i^{\text{split}})$-decided by q.
  - If $\alpha_i^{\text{split}} \notin \mu$, then $\nu \restriction m_i^{\text{split}}$ is even $(\leq m_i^{\text{split}} - 1)$-decided.[11]

  q rapidly reads ν if this is the case below all h.

If q has unique splitting, then q is finitary.

Lemma 4.4 *Assume that $p \in P$ and that ν is a P-name for a sequence in $\prod_{n \in \omega} f_{\max}(n)$. Then there is a $q \leq p$ rapidly reading ν.*

Proof We use the following notion of unique extension: Fix $p \in P$ finitary, $m \in \omega$, and a splitting node s (or (s, t)) in p of height $h > m$.[12] Then we can extend p uniquely above m up to s (or s, t), i.e. there is a r satisfying:

- $r \leq p$, $\text{dom}(r) = \text{dom}(p)$.
- $\text{pos}(r, \leq m) = \text{pos}(p, \leq m)$.
- If $m < n < h$, then n is not a splitting level of r.

10 In this case, each each $\alpha \in \text{dom}(q)$ will appear infinitely often in the sequence $(\alpha_i^{\text{split}})_{i \in \omega}$, to allow for sufficiently large splitting.

11 And therefore $(\leq m_{i-1}^{\text{split}})$-decided, since every $\eta \in \text{pos}(q, \leq m_{i-1}^{\text{split}})$ extend uniquely to an $\eta' \in \text{pos}(q, \leq m_i^{\text{split}} - 1)$.

12 This means: Either $\epsilon \in \mu$ and $s \in p(\epsilon)$ is a splitting node, or $\alpha \in I_\epsilon$, $s \in p(\epsilon)$ and $t \in p(\alpha, s)$ is a splitting node.

- h is a unique splitting level of r.
- If $a \in \mathrm{pos}(p, \leq h)$ extends s (or s, t), then $a \in \mathrm{pos}(r, \leq h)$.

In other words, we eliminate all splits between m and h, and at h we leave only the split s (or t) with all its successors.

We use this fact to define an increasing sequence $(p_i)_{i\in\omega}$ and show that the limit q has the desired properties.

Set $p_{-1} = p$ and $m^{\mathrm{split}}_{-1} = -1$. Assume we already have p_i as well as m^{split}_j and $\alpha^{\mathrm{split}}_j$ for all $j \leq i$, such that p_i rapidly reads ν below $m^{\mathrm{split}}_i + 1$. For the final limit, we will keep all elements of $\mathrm{pos}(p_i, \leq m^{\mathrm{split}}_i + 1)$.

We use some bookkeeping to choose $\alpha \in \mathrm{dom}(p_i)$ and $s \in p_i(\varepsilon(\alpha)) \cap \omega^{m^{\mathrm{split}}_i+1}$. If $\alpha \in \mu$, we pick some splitting node $s' > s$ in $p_i(\alpha)$. Otherwise we again use the bookkeeping to choose $t \in p_i(\alpha, s) \cap \omega^{m^{\mathrm{split}}_i+1}$, and pick some $s' > s$ in $p_i(\varepsilon(\alpha))$ and an $i+2$-big splitting node $t' > t$ in $p_i(\alpha, s')$. Let h be the height of the splitting node s' (or t'). We extend p_i uniquely above m^{split}_i to s' (or s', t'). Call the result r. Set $m^{\mathrm{split}}_{i+1} = h$. Then, using pure decision, we can find some $p' \leq r$ which is $(\leq h)$-deciding $\nu \restriction h$ so that $\mathrm{pos}(p', \leq h) \equiv \mathrm{pos}(r, \leq h)$ and the stems of p' outside of $\mathrm{dom}(r)$ are higher than h.

If $\alpha \in \mu$, set $p_{i+1} = p'$. Otherwise, let A be the set of successors of t'. There are less than $f_{\max}(h-1)^h$ many possibilities for $\nu \restriction h$, and at most h many splitting nodes below h, each with at most $f_{\max}(h-1)$ many successors. This gives a function

$$f_{\max}(h-1)^h \times A \rightarrow f_{\max}(h-1)^h$$

or

$$A \rightarrow f_{\max}(h-1)^{h\cdot f_{\max}(h-1)^h} < g_{\min}(h).$$

So we can use bigness to thin out A to some homogeneous B that has norm at least $i+1$. Call the result p_{i+1}. In this case. p_{i+1} already $(\leq h{-}1)$-decides $\nu \restriction h$.

Let q be the limit of $(p_i)_{i\in\omega}$. We have to show that $q \in P$. It is enough to require from the bookkeeping that the following is satisfied:

- For all $\epsilon \in \mathrm{dom}(q) \cap \mu$, and $s_0 \in q(\epsilon)$, there is an $s > s_0$ such that the bookkeeping chooses ϵ, s at some stage.
- For all $\alpha \in \mathrm{dom}(q) \cap I_\epsilon$, for all $s_0 \in q(\epsilon)$, and for all $t_0 \in q(\alpha, s_0)$, there are $s > s_0$ and $t > t_0$ such that α, s, t are chosen at some stage.
- For all $\alpha \in \mathrm{dom}(q) \cap I_\epsilon$, ϵ is chosen (for the first time) before α is chosen.

(It is easy to find a bookkeeping meeting these requirements.) Then q is indeed in P: Assume that $\alpha \in \mathrm{dom}(q) \cap I_\epsilon$, $s_0 \in q(\epsilon)$, and $l \in \omega$. We have to show that $q(\alpha, s)$ is l-large for for some $s > s_0$. First extend s to some s' of height at least m^{split}_l (defined from q). Enumerate the leaves in $q(\alpha, s')$ as $t^0, t^1, \ldots, t^{k-1}$. Increase s' to s'_0 such that in $q(\alpha, s'_0)$ there is a splitting node above t^0. Repeat that for the other t^i and set $s = s'_{k-1}$. If b is a branch through $q(\alpha, s)$, then there has to be some split in b above m^{split}_l, but each splitting node in q of this height is l-large. $\square$

So we get: If $\alpha_{i+1}^{\text{split}} \notin \mu$, then $\nu \restriction m_{i+1}^{\text{split}}$, and in particular $\nu(m_i^{\text{split}})$, is $(\leq m_i^{\text{split}})$-decided. Otherwise, it is $(\leq m_i^{\text{split}})$-decided only modulo the two possibilities left and right for the successor at the split at height m_{i+1}^{split} in the Sacks condition $q(\alpha_{i+1}^{\text{split}})$. So in both cases, and for all n, we can calculate $\nu(n)$ from $2 \times \text{pos}(q, \leq n)$. We can write this as a function:

$$G : 2 \times \text{pos}(q, \leq n) \to f_{\max}(n).$$

Proof (Proof of Lemma 4.3) Fix $p \in P$ and a P-name ν for a function in $\prod_{n\in\omega} f_\epsilon(n)$. We have to find $q \leq p$ and a P_ϵ-name B of an (f_ϵ, g_ϵ)-slalom such that q forces $\nu(n) \in B(n)$ for all $n \in \omega$.

Let $r \leq p$ rapidly read ν. We can assume that $\epsilon \in \text{dom}(r)$. We can also assume that the i-th splitting node is even $(i+1)$-large and not just i-large.[13] We will define, by induction on n, $B(n)$ as well as $q \leq r$ up to height $\leq n$.

q will be the result of thinning out some of the splitting nodes in r (in the non-Sacks part), in a such way that the norm of the node will be decreased by at most 1. So q will again have unique, large splitting, and q will be a condition in P.

If we already constructed q below n, and if there is no split at height n, we have no choice for q at height n but just take the unique extension given by r. If there is a split, we may thin out the successor set (reducing the norm by at most 1). Of course, this way we will loose former splits at higher levels (which extended the successors we just left out). So the splitting levels of q will be a proper subset of the splitting levels of r. In the following, m_i^{split} and α_i^{split} denote the splits of q.

If $\epsilon' \neq \epsilon$, $\alpha \in \text{dom}(r) \cap I_{\epsilon'}$, and h is a splitting level of $r(\alpha)$, then there is some splitting level $h' < h$ of $r(\epsilon')$. Also, $\text{trunk}(r, \epsilon)$ and $\text{trunk}(r, \epsilon')$ are incompatible, i.e. they differ below h. By the way we construct q, we get the same for q:

> $(*)$ If $\alpha \in I_{\epsilon'}$, $\epsilon' \neq \epsilon$, and if h is a splitting level of $q(\alpha)$, then either all $s \in q(\epsilon) \cap 2^h$ are lexicographically smaller than all $t \in q(\epsilon') \cap 2^h$, or the other way round.

We now define q at height n and $B(n)$: Assume that i is maximal such that $m = m_i^{\text{split}} \leq n$. Set $\alpha = \alpha_i^{\text{split}}$. By rapid reading there is a function G with domain $2 \times \text{pos}(r, \leq m)$ that calculates $\nu(n)$. Let A be the set of successors of the split of level m. $\text{pos}(r, \leq m-1)$ has size at most $f_{\max}(m-1)^m$. So we can write G as

$$G : 2 \times f_{\max}(m-1)^m \times A \to f_\epsilon(n).$$

Case A: $n > m$.
There are no splits on level n, so for q at level n we use the unique extensions given by r.

[13] It is clear we can get this looking at the proof of rapid reading, or we can get first a "standard" rapid reading r and then just remove the very first split by enlarging the trunk.

The size of A is at most $f_{\max}(m)$, so the domain of G has at most size

$$2 \cdot f_{\max}(m-1)^m \cdot f_{\max}(m) < g_{\min}(n),$$

and therefore is smaller than $g_\epsilon(n)$. So we can put all possible values for $\nu(n)$ into $B(n)$.

Case B: $n = m$, $\alpha \in \{\epsilon\} \cup I_\epsilon$.
q at level n contains all the successors of the split at level n.
In the P_ϵ-extension, we know which successor we choose.[14] Given this knowledge, the domain of G is again smaller than $g_{\min}(m)$, just as in Case A.

Case C: $n = m$, $\alpha \in \mu \setminus \{\epsilon\}$.
q at level n contains both successors of the split at level n.
$|A| = 2$, so there are again only

$$2 \cdot f_{\max}(n-1)^n \cdot 2 < g_{\min}(n)$$

many possible values for $\nu(n)$.

Case D: Otherwise $n = m$, $\alpha \in I_{\epsilon'}$, $\epsilon' \neq \epsilon$.
So for an $s \in r(\epsilon') \cap \omega^n$ there is a splitting node $t \in r(\alpha, s)$ of height n with successor set A. As stated in $(*)$ above, s is (lexicographically) either smaller or larger than all the nodes in $r(\epsilon) \cap \omega^n$.

Subcase D1: s is smaller.
We keep all the successors of the split at level n.
$|A| \leq f_s$, and $g_\epsilon(n) = g_{\eta_\epsilon \restriction n}$ has to be some $g_{n,k}$ for $k > s$ (in $[0, 2^n - 1]$). So we get

$$2 \cdot f_{\max}(n-1)^n \cdot f_s < g_\epsilon(n)$$

many possible values.

Subcase D2: s is larger.
Let k be $s - 1$ (in $[0, 2^n - 1]$). So $\nu(n)$ is less than $f_{n,k}$. We can transform G into a function

$$F : A \to f_{n,k}^{2 \cdot f_{\max}(n-1)^n} < g_{n,s}.$$

So we can thin out A to get an F-homogeneous set $B \subseteq A$, decreasing the norm by at most 1. q at height n contains only the successors in B. Modulo q, there remain only $2 \cdot f_{\max}(n-1)^n$ many possibilities for $\nu(m)$. □

[14] If any. Of course the filter could be incompatible with s (or s, t).

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