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# Preserving non-null with Suslin$^+$ forcings

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**Abstract** We introduce the notion of effective Axiom A and use it to show that some popular tree forcings are Suslin$^+$. We introduce transitive nep and present a simplified version of Shelah's "preserving a little implies preserving much": If $I$ is a Suslin ccc ideal (e.g. Lebesgue-null or meager) and $P$ is a transitive nep forcing (e.g. $P$ is Suslin$^+$) and $P$ does not make any $I$-positive *Borel* set small, then $P$ does not make *any* $I$-positive set small.

**Keywords** Forcing · Non-elementary proper · Definable forcing · Suslin proper

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## 1 Introduction

Properness is a central notion for countable support iterations: If a forcing $P$ is proper then it is "well behaved" in certain respects (most notably $P$ does not collapse $\omega_1$); and properness is preserved under countable support iterations. Properness can be defined by the requirement that the generic filter (over $V$) is generic for a countable elementary submodel $N$ as well (see 2.1).

It turns out that it can be useful to require genericity for non-elementary models $M$ as well.[1] The first notion of this kind was Suslin proper [6], with the

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[1]For this to make sense the forcing notion $P$ has to be definable; otherwise we do not know how to find $P$ in $M$, and therefore cannot formulate that $G$ is $P$-generic over $M$.

important special case Suslin ccc. This notion was generalized to Suslin$^+$ [4]. In this paper we recall these definitions, and introduce an effective version of Axiom A as a tool to show that all the usual Axiom A forcings are in fact Suslin$^+$.

In [13] Shelah introduced a further generalization: non-elementary proper (nep) forcing. Here, he applies the properness condition to certain models that are *neither* elementary *nor* transitive. This allows one to deal with long forcing-iterations (which can never be elements of a transitive countable model), but this also brings some unpleasant technical difficulties. To avoid some of these difficulties, [13] uses a set theory with ordinals as urelements.

In this paper we define a special case, the "transitive version", of nep. In this version we consider transitive candidates only, which makes the whole setting much easier.

As an example of how to apply non-elementary properness we give a simplified proof of Shelah's "preserving a little implies preserving much" [13, sec. 7]: If a forcing $P$ is provably nep and provably does not make the set of all old reals Lebesgue null, then $P$ does not make any positive set null. The proof uses the fact that we can find generic conditions for models of the form $N[G]$, where $N$ is (a transitive collapse of) an elementary submodel and $G$ an *internal* $N$-generic filter (i.e., $G \in V$).

The proof works in fact not only for the ideal of Lebesgue null sets, but also for all Suslin ccc ideals (e.g. the meager ideal). A couple of theorems of this kind lead up to the general case in [13]: for the meager case the result is due to Goldstern and Shelah [12, Lemma XVIII.3.11, p.920]; the Lebesgue null case in the special case of $P$ = Laver was done by Pawlikowski [10] (building on [7]). The definition and basic properties of Suslin ccc ideals have been used for a long time, for example in works of Judah, Bartoszyński and Rosłanowski, cited in [2]; also related is [14, Sect. 31].

The proof of preserving a little implies preserving much actually shows that generics are preserved (see Definition 4.2). This is useful for positivity preservation in limit-steps of proper countable support iterations $(P_\alpha)_{\alpha<\delta}$: while it is not clear how one could argue directly that $P_\delta$ still is positivity preserving, preservation of generics has a better chance of being iterable. In [12, Section XVIII.3.10] this iterability is claimed for $I$ = meager. For $I$ = Lebesgue null the result will appear in [9].

## 2 Suslin$^+$ and transitive nep forcing

### 2.1 A note on normal ZFC*

Let us recall the definition of properness:

**Definition 2.1** *$P$ is proper if for some sufficiently large regular cardinal $\chi$, for all $p \in P$ and all countable elementary submodels $N \prec H(\chi)$ containing $p$ and $P$ there is a $q \leq p$ which is $N$-generic.*

Intuitively, one would like to use elementary submodels of the universe instead of $H(\chi)$, but for obvious reasons this is not possible. So one has to show that the properness notion does not depend on the particular $\chi$ used in the definition, and that essential forcing constructions are absolute between $V$ and $H(\chi)$ (and $V[G]$ and $H^{V[G]}(\chi)$). So while the choice of $\chi$ is not important, it is not a good idea to fix a specific $\chi$ (say, $\beth_\omega^+$), since we might for example want to apply the properness notion to forcings larger than this specific $\chi$.

In Suslin forcing, instead of countable elementary submodels arbitrary countable transitive models of some theory ZFC*, so-called candidates, are used. Intuitively one would like to use ZFC, but this cannot be done for similar reasons. (For example, ZFC does not prove the existence of a ZFC-model.)

Again, it turns out that the choice of ZFC* is of no real importance (provided it is somewhat reasonable), but we should not fix a specific ZFC*.[2]

**Definition 2.2**

- *ZFC$^-$ denotes ZFC minus the powerset axiom plus "$\beth_\omega$ exists".*
- *An $\in$-theory ZFC* is called normal if $H(\chi) \vDash$ ZFC* for large regular $\chi$.*
- *A recursive theory ZFC* is strongly normal if ZFC proves*

$$(\exists \chi_0)\,(\forall \chi > \chi_0 \text{ regular})\, H(\chi) \vDash \text{ZFC}^*.$$

We will be interested in strongly normal theories only. Clearly, ZFC$^-$ is strongly normal. Also, if $T$ is strongly normal, then the theory $T$ plus "there is a $T$-candidate" is strongly normal, and a finite union of strongly normal theories is strongly normal.[3]

The importance of normality is the following: If ZFC* is normal, then forcings that are non-elementary proper with respect to ZFC* are proper (see Facts 2.4). However, normal does not necessarily mean "reasonable". For example, if in $V$ there is no inaccessible, then ZFC$^-$ plus the negation of the powerset axiom is normal.

As usual, we will (without further mentioning) assume that certain (finitely many) strongly normal sentences are in ZFC*. For example, we will state that Borel-relations are absolute between candidates and $V$, which of course assumes that ZFC* contains enough of ZFC$^-$ to guarantee this absoluteness.

## 2.2 Candidates, Suslin and Suslin$^+$ forcing

The following basic setting will apply to all versions of Suslin forcings used in this paper (Suslin proper, Suslin ccc, Suslin$^+$) as well as transitive nep:

We assume that the forcing $Q$ is defined by formulas $\varphi_{\in Q}(x)$ and $\varphi_{\leq}(x, y)$, using a real parameter $r_Q$. Fix a normal ZFC*. $M$ is called a "candidate" if it is

[2] We will sometimes require that every ZFC*-candidate $M$ thinks that there is a ZFC**-candidate $M'$ (and this fails for ZFC** = ZFC*), or that any forcing extension $M[G]$ of a ZFC*-candidate $M$ satisfies ZFC**.

[3] This is not true for countable unions, of course: by reflection, for every finite $T \subset$ ZFC, Con($T$) is strongly normal, but ZFC cannot prove $H(\chi) \vDash$ Con(ZFC).

a countable transitive ZFC* model and $r_Q \in M$. We denote the evaluation of $\varphi_{\in Q}$ and $\varphi_{\leq}$ in a candidate $M$ by $Q^M$ and $\leq^M$.

We further assume that in every candidate $Q^M$ is a set and $\leq^M$ a partial order on this set; and that $\varphi_{\in Q}$ and $\varphi_{\leq}$ are upwards absolute between candidates and $V$.[4]

A $q \in Q$ is called $M$-generic (or: $Q$-generic over $M$), if $q$ forces

"$G_Q \cap Q^M$ is $Q^M$-generic over $M$".

Usually (but not necessarily) it will be the case that $p \perp q$ is absolute between $M$ and $V$. In this case $q$ is $M$-generic iff $q \Vdash D \cap G_Q \neq \emptyset$ for all $D \in M$ such that $M \vDash$ "$D \subseteq Q$ dense". If $p \perp q$ is not absolute, then this is not enough, since it does not guarantee that $G_Q \cap Q^M$ is a filter on $Q^M$, i.e., that it does not contain elements $p, q$ such that $M \vDash$ "$p \perp q$". In this case, "$q$ is $M$-generic" is equivalent to: $q \Vdash |A \cap G_Q| = 1$ for all $A \in M$ such that $M \vDash$ "$A \subseteq Q$ is a maximal antichain".

We will only be interested in the case $Q \subseteq H(\aleph_1)$. Assume $\chi$ is regular and sufficiently large, and $N \prec H(\chi)$ is countable. Let $i : N \to M$ be the transitive collapse of $N$. Then $i \restriction Q$ is the identity, and $M$ is a candidate. If $Q$ is proper, then for every $p \in Q^M$ there is an $M$-generic $q \leq p$.

As already mentioned, sometimes it is useful to have generic conditions for other candidates (that are not transitive collapses of elementary submodels). The first notion of this kind was Suslin proper:

**Definition 2.3** *A (definition of a) forcing $Q$ is Suslin (or: strongly Suslin) in the parameter $r_Q \in \mathbb{R}$, if:*

1. *$r_Q$ codes three $\underset{\sim}{\Sigma}^1_1$ relations, $R^{\in}_Q$, $R^{\leq}_Q$ and $R^{\perp}_Q$.*
2. *$R^{\leq}_Q$ is a partial order on $Q = \{x \in \omega^\omega : R^{\in}_Q(x)\}$ and $p \perp_Q q$ iff $R^{\perp}_Q(p, q)$. $Q$ is Suslin proper with respect to some normal ZFC*, if in addition:*
3. *for every candidate $M$ and every $p \in Q^M$ there is an $M$-generic $q \leq p$.*

A forcing $Q$ (as a partial order) is called Suslin (proper), if there is a definition of $Q$ which is Suslin (proper).

**Facts 2.4**

- "$r_Q$ codes a Suslin forcing" is a $\underset{\sim}{\Pi}^1_2$ property. So if $Q$ is Suslin in $V$, then $Q$ is Suslin in all candidates and all forcing extensions of $V$ as well. In particular, in every candidate $M$, $\leq^M$ is a partial order on the set $Q^M$ and $p \perp q$ is equivalent to $R^{\perp}_Q(p, q)$.
  However, the formula "$(\in_Q, \leq_Q, r_Q, \mathrm{ZFC}^*)$ codes a Suslin proper forcing" is a $\underset{\sim}{\Pi}^1_3$ statement and in general not absolute.
- If $Q$ is Suslin, then $\perp$ is a Borel relation, and therefore the statement
  $\{q_i : i \in \omega\}$ is predense below $p$
  (i.e., $p \Vdash G \cap \{q_i : i \in \omega\} \neq \emptyset$) is $\underset{\sim}{\Pi}^1_1$ (i.e., relatively $\underset{\sim}{\Pi}^1_1$ in the $\underset{\sim}{\Sigma}^1_1$ set $Q^{(\omega+1)}$).

[4]This means that if $M_1$ and $M_2$ are candidates such that $M_1 \in M_2$, and if $q \leq^{M_2} p$, then $q \leq^{M_1} p$ and $q \leq^V p$.

- If $Q$ is Suslin proper with respect to ZFC$^*$, and ZFC$^{**}$ is stronger than ZFC$^*$, then $Q$ is Suslin proper with respect to ZFC$^{**}$ as well.
- If $Q$ is Suslin proper, then $Q$ is proper.
  (As mentioned already, the transitive collapse $M$ of a countable $N \prec H(\chi)$ is a candidate, $Q$ is not changed by the collapse, and $q \leq p$ is $M$-generic iff $q \leq p$ is $N$-generic.)

*Remark* The definition of Suslin proper forcing could be applied to non-normal $\{\in\}$ theories ZFC$^*$ as well. This could be useful in other contexts, but not for this paper. Obviously such a forcing $Q$ need not be proper any more. As an extreme example, ZFC$^*$ could contain "$0 = 1$". Then (3) is immaterial, since there are no candidates, and every forcing definition $Q$ satisfying (1) and (2) is Suslin proper.

In [6] it is proven that if a forcing $Q$ is Suslin and ccc (in short: Suslin ccc), then $Q$ is Suslin proper in a very absolute way:

**Lemma 2.5** *"$Q$ is Suslin ccc" is a $\utilde{\Pi}^1_2$ statement. So in particular, if $Q$ is Suslin ccc, then*

1. *$Q$ is Suslin ccc in every candidate $M$ and in every forcing extension of $V$.*
2. *$Q$ is Suslin proper: even $1_Q$ is generic for every candidate.*

The proof proceeds as follows: assume $Q$ is Suslin. Using the completeness theorem $\varphi^{\text{Keisler}}$ for the logic $L_{\omega_1\omega}(Q)$ (see [8]) it can be shown [6, 3.14] that "$Q$ is ccc" is a Borel statement. (This requires that $\varphi^{\text{Keisler}} \in$ ZFC$^*$, which we can assume since $\varphi^{\text{Keisler}}$ is strongly normal.) So if $M$ is a candidate and $M \vDash$ "$A \subseteq Q$ is a maximal antichain", then $M \vDash$ "$A$ is countable". And we have already seen that for $Q$ Suslin and $A$ countable, the statement "$A$ is predense" is $\utilde{\Pi}^1_1$ (and therefore absolute). So $A$ is predense in $V$, and $1_Q$ forces that $G_Q$ meets $A$.

*Remark* (1) and (2) of the lemma are trivially true for a $Q$ that is definable without parameters (e.g. Cohen, random, amoeba, Hechler), assuming that ZFC $\vdash Q$ is ccc and ZFC$^* \vdash Q$ is ccc.

For further reference, we repeat a specific instance of the last lemma here:

**Lemma 2.6** *If $Q$ is Suslin ccc, $M_1 \subseteq M_2$ are candidates, and $G$ is $Q$-generic over $M_2$ or over $V$, then $G$ is $Q$-generic over $M_1$.*

Cohen, random, Hechler and amoeba forcing are Suslin ccc and Mathias forcing is Suslin proper. Miller and Sacks forcing, however, are not, since incompatibility is not Borel.

This motivated a generalization of Suslin proper, Suslin$^+$ [4, p. 357]: here, we do not require $\perp$ to be $\utilde{\Sigma}^1_1$, so "$\{q_i : i \in \omega\}$ is predense below $p$" will generally not be $\utilde{\Pi}^1_1$ any more, just $\utilde{\Pi}^1_2$. However, we require that there is a $\utilde{\Sigma}^1_2$ relation epd ("effectively predense") that holds for "enough" predense sequences:

**Definition 2.7** *A (definition of a) forcing $Q$ is Suslin$^+$ in the parameter $r_Q$ with respect to ZFC*, if:*

1. *$r_Q$ codes two $\underset{\sim}{\Sigma}^1_1$ relations, $R^{\in}_Q$ and $R^{\leq}_Q$, and an $(\omega+1)$-place $\underset{\sim}{\Sigma}^1_2$ relation* epd.
2. *In $V$ and every candidate $M$, $\leq$ is a partial order on $Q$, and* $\mathrm{epd}(q_i, p)$ *implies "$\{q_i : i \in \omega\}$ is predense below $p$".*
3. *For every candidate $M$ and every $p \in Q^M$ there is a $q \leq p$ such that every dense subset $D \in M$ of $Q^M$ has an enumeration $\{d_i : i \in \omega\}$ such that* $\mathrm{epd}(d_i, q)$ *holds.*

Again, a partial order $Q$ is called Suslin$^+$ if it has a suitable definition.

Clearly, every Suslin proper forcing is Suslin$^+$: epd can just be defined by "$\{q_i : i \in \omega\}$ is predense below $p$", which is even a conjunction of $\underset{\sim}{\Pi}^1_1$ and $\underset{\sim}{\Sigma}^1_1$, and then the condition 2.7(3) is just a reformulation of 2.3(3).

## 2.3 Effective Axiom A

The usual tree-like forcings are Suslin$^+$. Here, we consider the following forcings consisting of trees on $\omega^{<\omega}$ ordered by $\subseteq$. (Usually, Sacks is defined on $2^{<\omega}$, but this is equivalent by a simple density argument.) For $s, t \in \omega^{<\omega}$ we write $s \leq t$ for "$s$ is an initial segment of $t$"; for a tree $T \subseteq \omega^{<\omega}$ $s \leq_T t$ means $s \leq t$ and $s, t \in T$; and $s^\frown n$ is the immediate successor of $s$ with last element $n$.

- Sacks, perfect trees: $(\forall s \in T)\ (\exists t \geq_T s)\ (\exists^{\geq 2} n)\ t^\frown n \in T$.
- Miller, superperfect trees: every node has either exactly one or infinitely many immediate successors, and $(\forall s \in T)\ (\exists t \geq_T s)\ (\exists^\infty n)\ t^\frown n \in T$.
- Rosłanowski: every node has either exactly one or all possible successors, and $(\forall s \in T)\ (\exists t \geq_T s)\ (\forall n \in \omega)\ t^\frown n \in T$.
- Laver: let $s$ be the stem of $T$. Then $(\forall t \geq_T s)\ (\exists^\infty n)\ t^\frown n \in T$.

In the following, we call Sacks, Miller and Rosłanowski "Miller-like". Clearly, "$p \in Q$" and "$q \leq p$" are Borel (but $p \perp q$ is not).[5]

For Sacks, there is a proof of the Suslin$^+$ property in [4] and [5] using games. However, in the same way as the "canonical" proof of properness of these forcings uses Axiom A, the most transparent way to prove Suslin$^+$ uses an effective version of Axiom A:

Baumgartner's Axiom A [3] for a forcing $(Q, \leq)$ can be formulated as follows: There are relations $\leq_n$ such that

1. $\leq_{n+1} \subseteq \leq_n \subseteq \leq$.
2. Fusion: if $(a_n)_{n\in\omega}$ is a sequence of elements of $Q$ such that $a_{n+1} \leq_n a_n$ then there is an $a_\omega$ such that $a_\omega \leq a_n$ for all $n$.
3. If $p \in Q$, $n \in \omega$ and $D \subseteq Q$ is dense then there is a $q \leq_n p$ and a countable subset $B$ of $D$ which is predense under $q$.

[5] Alternatively, $Q$ could of course be defined as the set of trees just *containing* a corresponding set, then $x \in Q$ is $\underset{\sim}{\Sigma}^1_1$, and for the Miller-like forcings two compatible elements $p$, $q$ have a canonical lower bound, $p \cap q$.

*Remarks*

- Actually, this is a weak version of Axiom A, usually something like $a_\omega \leq_n a_n$ will hold in (2).
- It is easy to see that in (3), instead of "and $D \subseteq Q$ is dense" we can equivalently use "and $D \subseteq Q$ is open dense" (or maximal antichain).

Now for effective Axiom A it is required that the $B \subseteq D$ in (3) is *effectively* predense below $q$, not just predense. Then Suslin$^+$ follows. To be more exact:

**Definition 2.8** *$Q$ satisfies effective Axiom A (in the parameter $r_Q$ with respect to ZFC$^*$), if*

1. *$r_Q$ codes $\underset{\sim}{\Sigma}^1_1$ relations, $R^{\in}_Q$, $R^{\leq}_Q$, and $\underset{\sim}{\Sigma}^1_2$ relations $\leq^n_Q$ ($n \in \omega$) and an $(\omega+1)$-place $\underset{\sim}{\Sigma}^1_2$ relation* epd.
2. *In $V$ and every candidate $M$, $\leq$ is a partial order on $Q$ and $epd(q_i, p)$ implies that $\{q_i : i \in \omega\}$ is predense below $p$.*
3. *Fusion: for all $(a_n)_{n\in\omega}$ such that $a_{n+1} \leq_n a_n$ there is an $a_\omega$ such that $a_\omega \leq a_n$.*
4. *In all candidates, if $p \in Q$, $n \in \omega$ and $D \subseteq Q$ is dense then there is a $q \leq_n p$ and a sequence $(b_i)_{i\in\omega}$ of elements of $D$ such that* $\mathrm{epd}(b_i, q)$ *holds.*

Again, a partial order $Q$ satisfies effective Axiom A if it has a suitable definition.

**Lemma 2.9** *If the partial order $Q$ satisfies effective Axiom A, then $Q$ is Suslin$^+$.*

*Proof* First we define $\mathrm{epd}'(p'_i, q')$ by

$$(\exists q \geq q')\,(\exists \{p_i\} \subseteq \{p'_i\})\ \mathrm{epd}(p_i, q).$$

Clearly, this is a $\underset{\sim}{\Sigma}^1_2$ relation coded by $r_Q$ satisfying 2.7(2). Let $M$ be a candidate, and let $\{D_i : i \in \omega\}$ list the dense sets of $Q^M$ that are in $M$. Pick an arbitrary $a_0 = p \in Q^M$. We have to find a $q \leq p$ satisfying 2.7(3) with respect to epd$'$. Assume we have already constructed $a_n$. In $M$, according to (4) using $D_n$ as $D$, we find an $a_{n+1} \leq_n a_n$ and $\{b^n_i : i \in \omega\} \subseteq D_n$ such that $\mathrm{epd}(b^n_i, a_{n+1})$ holds (in $M$ and therefore by absoluteness in $V$). In $V$ pick $q = a_\omega$ according to (3). □

The usual proofs that the forcings defined above satisfy Axiom A also show that they satisfy the effective version. To be more explicit: let $Q$ be one of the forcings. We define (for $p, q \in Q, n \in \omega$):

- $\mathrm{split}(p) = \{s \in p : (\exists^{\geq 2} n \in \omega)\ s^\frown n \in p\}$.
- $\mathrm{split}(p, n) = \{s \in \mathrm{split}(p) : (\exists^{=n} t \leq s)\ t \in \mathrm{split}(p)\}$.
  (So $s \in \mathrm{split}(p, n)$ means that $s$ is the $n$-th splitting node along the branch $\{t \leq s\}$. In particular, $\mathrm{split}(p, 0)$ is the singleton containing the stem of $p$.)
- $q \leq_n p$, if $q \leq p$ and $\mathrm{split}(q, n) = \mathrm{split}(p, n)$.
  (So $q \leq_0 p$ if $q \leq p$ and $q$ has the same stem as $p$.)
- For $s \in p$, $p^{[s]} = \{t \in p : t \leq s \vee s \leq t\}$.
- $F \subseteq p$ is a front (or: $F$ is a front in $p$), if it is an antichain meeting every branch of $p$.

- $\mathrm{epd}(q_i,p)$ is defined by: there is a front $F \subseteq p$ such that $(\forall t \in T)\,(\exists i \in \omega)\, q_i = p^{[t]}$.
- For Miller-like forcings, effectively predense could also be defined as

$$\mathrm{epd}'(q_i,p) \;\;:\leftrightarrow\;\; \exists n\,\forall s \in \mathrm{split}(p,n)\,\exists i:\; q_i = p^{[s]}.$$

Clearly, $\mathrm{split}(p)$, $\mathrm{split}(p,n)$, $p^{[s]}$ and $\mathrm{epd}'$ are Borel, "$F$ is a front" is $\underset{\sim}{\Pi}^1_1$, therefore epd is $\underset{\sim}{\Sigma}^1_2$. The following facts are easy to check ($p,q \in Q$):

- If $s \in p$, then $p^{[s]} \in Q$.
- If $F \subset p$ is a front and $q|p$, then $q|p^{[s]}$ for some $s \in F$.
- $\mathrm{split}(p,n)$ is a front in $p$.
- For $(q_n)_{n\in\omega}$ such that $q_{n+1} \leq_n q_n$, there is a canonical limit $q_\omega \in Q$ and $q_\omega \leq_n q_n$.
- Assume that $Q$ is Miller-like, $p \in Q$, $F \subset p$ a front, and for all $s \in F$ pick some $p_s \in Q$ such that $p_s \subseteq p^{[s]}$. Then $\bigcup_{s\in F} p_s \in Q$, and $\bigcup_{s\in F} p_s \subseteq p$.
- Let $Q$ be Laver, $p \in Q$, $F \subset p$ a front. Pick for all $s \in F$ a $p_s \in Q$ with stem $s$. Then $\bigcup_{s\in F} p_s \in Q$, and $\bigcup_{s\in F} p_s \subseteq p$.

**Lemma 2.10** *The tree forcings defined above satisfy the effective Axiom A.*

*Proof* We show that $\leq_n$ and epd defined above satisfy 2.8.

(1)–(3) are clear.

For Miller-like forcings, (4) is proven as follows: Assume $D \subseteq Q$ is dense and $p \in Q$. For all $s \in \mathrm{split}(p,n+1)$, $p^{[s]} \in Q$, so there is a $q^s \subseteq p^{[s]}$ such that $q^s \in D$. Now set $q := \bigcup_{s\in F} q^s \in Q$. Then $q \leq_n p$, and the set $\{q^s : s \in F\} \subseteq D$ is effectively predense below $q$ according to the definition of $\mathrm{epd}'$ (or epd).

To show (4) for Laver, we have to define a rank of nodes: Assume $D$ is dense, and $p_0$ a condition with stem $s_0$, $s \geq s_0$, and $s \in p_0$. We define $\mathrm{rk}_D(p_0,s)$ as follows:

- If there is a $q \subseteq p_0$ such that $q \in D$ and $q$ has stem $s$, then $\mathrm{rk}_D(p_0,s) = 0$.
- Otherwise $\mathrm{rk}_D(p_0,s)$ is the minimal $\alpha$ such that for infinitely many immediate successors $t$ of $s$ the following holds: $t \in p_0$ and $\mathrm{rk}_D(p_0,t) < \alpha$.

$\mathrm{rk}_D$ is well-defined for all nodes $\geq s_0$ in $p_0$:

Assume towards a contradiction that $\mathrm{rk}_D(p_0,s)$ is undefined. Then

$$q := \{s' \in p_0^{[s]} : s' \leq s \text{ or } \mathrm{rk}_D(p_0,s') \text{ undefined}\}$$

is a Laver condition stronger than $p_0$. Pick a $q' \leq q$ such that $q' \in D$. Let $s'$ be the stem of $q'$. Then $\mathrm{rk}_D(p,s') = 0$, $s' \geq s$ and $s' \in q$, a contradiction.

Now define $q' \leq p_0$ inductively. First add all $s \leq s_0$ to $q'$. Assume $s \in q'$ and $s \geq s_0$. Then we add infinitely many immediate successors $t \in p_0$ of $s$ to $q'$. If $\mathrm{rk}_D(p,s) \neq 0$, we additionally require that $\mathrm{rk}_D(p,t) < \mathrm{rk}_D(p,s)$ for each of these $t$ (this is possible by the definition of $\mathrm{rk}_D(p,s)$). So the $q'$ constructed this way is a Laver condition with the same stem $s_0$ as $p_0$. Also, along every branch of $q'$, $\mathrm{rk}_D(p,s)$ is strictly decreasing (until it gets 0); therefore, there is a front $F_0$ in $q'$ such that for all $s \in F_0$, $\mathrm{rk}_D(p,s) = 0$. That means that for all $s \in F_0$ there is a

$q^s \leq p_0$ such that $q^s \in D$ and $q^s$ has stem $s$. Define $q_0$ to be $\bigcup_{s\in F_0} q^s$. Clearly $q_0 \leq p_0$, $q_0$ has the same stem $s_0$ as $p_0$, $F_0$ is a front in $q_0$ and for every $s \in F_0$, $q_0^{[s]} \in D$.

Given a Laver condition $p$ and $n \in \omega$, define for every $p_0 \in \text{split}(p, n)$ a $q_0$ as above, and let $q$ be the union of these $q_0$, and $F$ the union of the corresponding $F_0$. Then $q \leq_n p$, and for every $s$ in the front $F \subset q$, $q^{[s]} \in D$. This finishes the proof of effective Axiom A for Laver. □

*Remark* It is clear that the same proof of effective Axiom A works for other tree forcings as well, for example for all finite-splitting lim-sup tree forcings. (In [11, 1.3.5] such forcings are called $\mathbb{Q}_0^{\text{tree}}$.)

### 2.4 Transitive nep

So we have seen that Suslin ccc implies Suslin proper, which implies Suslin$^+$. For the proof of the main theorem 4.4, even less than Suslin$^+$ is required:[6] A forcing definition $Q$ (using the parameter $r_Q$) is transitive nep (non-elementary proper), if

- "$p \in Q$" and "$q \leq p$" are upwards absolute between candidates and $V$.
- In $V$ and all candidates, $Q \subseteq H(\aleph_1)$ and "$p \in Q$" and "$q \leq p$" are absolute between the universe and $H(\chi)$ (for large regular $\chi$).
- For all candidates $M$ and $p \in Q^M$ there is a $q \leq p$ forcing that $G_Q \cap Q^M$ is $Q^M$-generic over $M$.

Recall our initial consideration: In proper forcing, we get the properness condition for (collapses of) elementary submodels only, but we would like to have it for non-elementary models as well. (This is the reason for the name "non-elementary proper".) So transitive nep captures this consideration with only few additional assumptions.

There is also a (technically more complicated) version of nep for non-elementary and non-transitive candidates, defined in [13], which makes it possible for long iterations to be nep (transitive nep requires $Q \subseteq H(\aleph_1)$). The main theorem 4.4 of this paper holds for this general notion of nep as well (with nearly the same proof).

For every countable transitive model, $M \vDash$ "$p \Vdash \varphi(\tau)$" iff for all $M$-generic $G$ containing $p$, $M[G] \vDash$ "$\varphi(\tau[G])$". If $Q$ is nep and $M$ a candidate, then $M \vDash$ "$p \Vdash \varphi(\tau)$" iff for all $M$- and $V$-generic $G$ containing $p$, $M[G] \vDash$ "$\varphi(\tau[G])$": One direction is clear. For the other, assume $M \vDash$ "$p' \leq p, p' \Vdash \neg\varphi(\tau)$". Let $q \leq p'$ be $M$ generic. Then for any $V$-generic $G$ containing $q$, $G$ is $M$-generic as well and $M[G] \vDash$ "$\neg\varphi(\tau[G])$".

We will use the following

[6]Actually, for the main theorem even less than nep would be sufficient: we need generic conditions only for candidates $M$ that are internal set forcing extensions of transitive collapses of elementary submodels only. However, this restriction does not seem to lead to a natural nep notion.

**Fact 2.11** Let $x \in H(\aleph_1)$. Then "there is a candidate $M$ containing $x$ such that $M \vDash \varphi(x)$" is $\underset{\sim}{\Sigma}{}^1_2$ (and therefore absolute between universes with the same $\omega_1$).

All in all we get the following implications:

$$\begin{array}{ccccccc} \text{Suslin ccc} & \longrightarrow & \text{Suslin proper} & & & & \\ \downarrow & & \downarrow & & & & \\ \text{effective Axiom A} & \longrightarrow & \text{Suslin}^{+} & \longrightarrow & \text{transitive nep} & \longrightarrow & \text{proper} \end{array}$$

## 3 Suslin ccc ideals

The set of Borel codes (or Borel definitions) will be denoted by "BC". So BC is a set of reals. For $A \in \mathrm{BC}$ we denote the set of reals that satisfy the definition $A$ (in the universe $V$) with $A^V$.

If $Q \subseteq H(\aleph_1)$ is ccc, then a name $\underset{\sim}{\tau}$ for an element of $\omega^\omega$ can be transformed into an equivalent hereditarily countable name $\eta$: for every $n$, pick a maximal antichain $A_n$ deciding $\underset{\sim}{\tau}(n)$, then $\underset{\sim}{\eta} := \{\widetilde{(p,(n,m))} : p \in A_n, p \Vdash \underset{\sim}{\tau}(n) = m\}$ is equivalent to $\underset{\sim}{\tau}$.

If $Q$ is a partial order, then we denote the complete Boolean algebra of regular open sets by $\mathrm{ro}(Q)$.

If $B$ is a Boolean algebra, then we sometimes write $B$ when we mean $B \setminus \{0\}$ (i.e., when we use $B$ as forcing).

From now on, we will assume the following:

**Assumption 3.1** $Q$ is a Suslin ccc forcing, $\underset{\sim}{\eta}$ is a hereditarily countable name coded by $r_Q$, $\Vdash_Q \underset{\sim}{\eta} \in \omega^\omega \setminus V$, and in all candidates: $\{[\![\underset{\sim}{\eta}(n) = m]\!] : n, m \in \omega\}$ generates $\mathrm{ro}(Q)$.

"$X$ generates $\mathrm{ro}(Q)$" means that there is no proper sub-Boolean-algebra $B \supseteq X$ of $\mathrm{ro}(Q)$ such that $\sup_{\mathrm{ro}(Q)}(Y) \in B$ for all $Y \subseteq B$.

**Lemma 3.2** *This assumption is a* $\underset{\sim}{\Pi}{}^1_2$ *statement.*

*Proof* "$Q$ is Suslin ccc" is $\underset{\sim}{\Pi}{}^1_2$ according to 2.5. For $x \in H(\aleph_1)$, a statement of the form "every candidate thinks $\varphi(x)$" is $\underset{\sim}{\Pi}{}^1_2$ (cf. 2.11). $\Vdash_Q (\underset{\sim}{\eta} \in \omega^\omega \setminus V)$ holds in $V$ iff it holds in every candidate: If $M \vDash p \Vdash \underset{\sim}{\eta} = r$, then this holds in $V$ as well: For Suslin ccc forcings, every $V$-generic filter is $M$-generic, and $\underset{\sim}{\eta} = r$ is absolute. The other direction follows from normality. □

**Lemma 3.3** *For $A \in BC$, "$q \Vdash \underset{\sim}{\eta} \in A^{V[G_Q]}$" is* $\underset{\sim}{\Delta}{}^1_2$.

*Remark* [1, 2.7] gives a general result for $\underset{\sim}{\Sigma}{}^1_n$ formulas.

*Proof* For any candidate $M$ containing $q$ and $A$, "$q \Vdash \underset{\sim}{\eta} \in A$" is absolute between $V$ and $M$: If $G$ is $V$-generic, then $G$ is $M$-generic as well (since $Q$ is Suslin ccc), and $\underset{\sim}{\eta}[G] \in A$ is absolute between $M[G]$ and $V[G]$.

So $q \Vdash \underset{\sim}{\eta} \in A$ iff for all candidates $M$, $M \vDash q \Vdash \underset{\sim}{\eta} \in A$ (a $\underset{\sim}{\Pi}{}^1_2$ statement) iff for some candidate $M$: $M \vDash q \Vdash \underset{\sim}{\eta} \in A$ (a $\underset{\sim}{\Sigma}{}^1_2$ statement). □

**Lemma 3.4** *The statement*

$$\{[\![\underset{\sim}{\eta}(n) = m]\!] : n,m \in \omega\} \text{ generates } \mathrm{ro}(Q)$$

*holds in M iff the following holds (in V):*
*if $G_1, G_2 \in V$ are Q-generic over M and $G_1 \cap M \neq G_2 \cap M$, then $\underset{\sim}{\eta}[G_1] \neq \underset{\sim}{\eta}[G_2]$.*

*Proof* If $\{[\![\underset{\sim}{\eta}(n) = m]\!] : n,m \in \omega\}$ generates ro$(Q)$, then $G \cap Q^M$ can be calculated (in $M[G]$) from $\underset{\sim}{\eta}[G]$. On the other hand, let (in $M$) $B = \mathrm{ro}(Q)$, $C$ the proper complete sub-algebra generated by $[\![\underset{\sim}{\eta}(n) = m]\!]$. Take $b_0 \in B$ such that no $b' \leq b_0$ is in $C$, and set

$$c = \inf\{c' \in C : c' \geq b_0\}, \quad b_1 = c \setminus b_0.$$

So for all $c' \in C$, $c' \| b_0$ iff $c' \| b_1$. Let $G_0$ be $B$-generic over $M$ such that $b_0$ in $G$. Then $H = G_0 \cap C$ is $C$-generic. In $M[H]$, $b_1 \in B/H$. So there is a $G_1 \supset H$ containing $b_1$. □

**Definition 3.5** *The Suslin ccc ideal corresponding to $(Q, \underset{\sim}{\eta})$:*

- $I_{BC} = \left\{A \in BC : \Vdash_Q \underset{\sim}{\eta} \notin A^{V[G_Q]}\right\}.$
- $I = \{X \subseteq \omega^\omega : \exists A \in I_{BC} : A^V \supseteq X\}.$
- *$X \in I^+$ (or: X is positive) means $X \notin I$, and X is of measure 1 means $\omega^\omega \setminus X \in I$.* $I^+_{BC} := BC \setminus I_{BC}$.

Note that we use the phrases “of measure 1”, “null” and “positive” for all Suslin ccc ideals, not just for the Lebesgue null ideal. For example, if $\mathbb{C}$ is Cohen forcing, then the null sets are the meager sets, and a set has “measure 1” if it is co-meager.

Clearly $A \in I_{\mathrm{BC}}$ iff $A^V \in I$.

An immediate consequence of Lemma 3.3 is

**Corollary 3.6** *For $A \in BC$, “$A \in I_{BC}$” is $\underset{\sim}{\Delta}^1_2$.*

So for Borel sets, being null is absolute.

**Lemma 3.7** *I is a $\sigma$-complete ccc ideal containing all singletons, and there is a surjective $\sigma$-Boolean-algebra homomorphism $\phi$ : Borel $\to$* ro$(Q)$ *with kernel I, i.e.,* ro$(Q)$ *is isomorphic to Borel/I as a complete Boolean algebra.*

ccc means: there is no uncountable family $\{A_i\}$ such that $A_i \in I^+$ and $A_i \cap A_j \in I$ for $i \neq j$ (or equivalently: $A_i \cap A_j = \emptyset$).

*Proof* $\sigma$-complete is clear: If $X_i \subseteq A_i \in I$, and $\Vdash \underset{\sim}{\eta} \notin A_i$ for all $i \in \omega$, then $\Vdash \underset{\sim}{\eta} \notin \bigcup A_i \supseteq \bigcup X_i$.
For $A \in$ BC, define $\phi(A) = [\![\underset{\sim}{\eta} \in A^{V[G]}]\!]_{\mathrm{ro}(Q)}$. Then $\phi(\omega^\omega \setminus A) = \neg\phi(A)$, $\phi(\bigcup A_i) = \sup\{\phi(A_i)\}$, and if $A \subseteq B$, then $\phi(A) \leq \phi(B)$. If $\phi(A) \leq \phi(B)$, then $\Vdash \underset{\sim}{\eta} \notin (A \setminus B)$, so $A \setminus B \in I$. Since $\underset{\sim}{\eta}$ generates ro$(Q)$ (in all candidates, and therefore in $V$ as well by normality) and since $Q$ is ccc, ro$(Q) = \phi''$Borel. So $\phi$ : Borel $\to$ ro$(Q)$ is a surjective $\sigma$-Boolean-algebra homomorphism. The kernel is the $\sigma$-closed ideal $I$, so Borel$/I$ is isomorphic to ro$(Q)$ as a $\sigma$-Boolean-algebra, and (since ro$(Q)$ is ccc), even as complete Boolean algebra. □

**Definition 3.8** *$\eta^*$ is called generic over $M$ ($\eta^* \in \mathrm{Gen}(M)$), if there is an $M$-generic $G \in V$ such that $\underset{\sim}{\eta}[G] = \eta^*$.*

According to 3.4, this $G$ is unique (on $Q \cap M$). For example, if $Q$ is random, then $\mathrm{Gen}(M)$ is the set of random reals over $M$.

$[\![\underset{\sim}{\eta} \in B]\!] = q$ is equivalent to

$$q \Vdash \underset{\sim}{\eta} \in B \quad \text{and} \quad \text{if } p \perp q \text{ then } p \Vdash \underset{\sim}{\eta} \notin B,$$

which is $\underset{\sim}{\Pi}^1_2$ (because of Lemma 3.3 and the fact that $p \perp q$ is Borel). For $q \in Q$ we denote a $B$ such that $[\![\underset{\sim}{\eta} \in B]\!] = q$ by $B_q$. Of course $B_q$ is not unique, just unique modulo $I$. $q \Vdash \underset{\sim}{\eta} \in A$ iff $\Vdash (\underset{\sim}{\eta} \in B_q \rightarrow \underset{\sim}{\eta} \in A)$, i.e., iff $\Vdash \underset{\sim}{\eta} \notin B_q \setminus A$. So we get $q \Vdash \underset{\sim}{\eta} \notin A$ iff $A \cap B_q \in I$, and $q \Vdash \underset{\sim}{\eta} \in A$ iff $B_q \setminus A \in I$.

If $M$ is a candidate and $q \in M$, then because of Lemma 3.2 the Assumption 3.1 holds in $M$, so $M$ knows about the isomorphism $\mathrm{ro}(Q) \rightarrow \mathrm{Borel}/I$ and in $M$ there is a $B^M_q$ as above.

**Lemma 3.9** *Let $M$ be a candidate and $q \in Q \cap M$. Then*

1. $\mathrm{Gen}(M) = \omega^\omega \setminus \bigcup\{A^V : A \in I_{BC} \cap M\}$.
2. $\{\underset{\sim}{\eta}[G] : G \in V$ *is $M$-generic and* $q \in G\} =$
   $= \omega^\omega \setminus \bigcup \left\{A^V : A \in BC \cap M, q \Vdash \underset{\sim}{\eta} \notin A^{V[G_Q]}\right\} = \mathrm{Gen}(M) \cap B^M_q$.
3. $\mathrm{Gen}(M)$ *is a Borel set of measure 1.*

For example, if $Q$ is random forcing, this just says that $\eta^*$ is generic (i.e., random) over $M$ iff for all Borel codes $A \in M$ of null sets, $\eta^* \notin A^V$.

*Proof* (1) is just a special case of (2).

(2) Set

$$X := \omega^\omega \setminus \bigcup \left\{A^V : A \in \mathrm{BC} \cap M, q \Vdash \underset{\sim}{\eta} \notin A^{V[G_Q]}\right\}, \quad \text{and}$$
$$Y := \{\underset{\sim}{\eta}[G] : G \in V \text{ is } M\text{-generic and } q \in G\}.$$

Assume $\eta^* \in Y$. Let $G$ be $M$-generic such that $q \in G$ and $\underset{\sim}{\eta}[G] = \eta^*$. If $M \vDash q \Vdash \underset{\sim}{\eta} \notin A^{V[G_Q]}$, then $M[G] \vDash \eta^* \notin A^{M[G]}$, i.e., $\eta^* \notin A^V$. So $\eta^* \in X$.

If $\eta^* \in X$, use (in $M$) the mapping $\phi : \mathrm{Borel} \rightarrow \mathrm{ro}(Q)$ ($A \mapsto [\![\underset{\sim}{\eta} \in A]\!]$). If $\phi(A) \leq \phi(B)$, then $\Vdash \underset{\sim}{\eta} \notin (A \setminus B)$, so by our assumption, $\eta^* \notin (A \setminus B)$. Given $\eta^*$, define $G$ by $\phi(A) \in G$ iff $\eta^* \in A$. $G$ is well defined: If $\eta^* \in A \setminus B$, then $\phi(A) \neq \phi(B)$. We have to show that $G$ is a generic filter over $M$: If $\phi(A_1), \phi(A_2) \in G$, then $\eta^* \in A_1 \cap A_2$, so $\phi(A_1) \wedge \phi(A_2) \in G$. If $\phi(A) \leq \phi(B)$, then $\eta^* \notin (A \setminus B)$, so $\phi(A) \in G \rightarrow \phi(B) \in G$. Since $\phi(\emptyset) = 0$, and $\eta^* \notin \emptyset$, $0 \notin G$. If $\sup(\phi(A_i)) \in G$, $(A_i) \in M$, then $\eta^* \in \bigcup A_i$, i.e., for some $i$, $\phi(A_i) \in G$. Since $q \Vdash \underset{\sim}{\eta} \notin \omega^\omega \setminus B^M_q$, $\eta^* \notin \omega^\omega \setminus B^M_q$, i.e., $\eta^* \in B^M_q$, and since $\phi(B^M_q) = q$, $q \in G$, so $\eta^* \in Y$. So we have seen that $Y = X \subseteq \mathrm{Gen}(M) \cap B^M_q$.

If $\eta^* \in \mathrm{Gen}(M) \cap B^M_q$, witnessed by $G$, then $\underset{\sim}{\eta}[G] \in B^M_q$, so $q \in G$ (since $q = [\![\underset{\sim}{\eta} \in B^M_q]\!]$), i.e., $\eta^* \in Y$.

(3) follows from (1), since $I$ is $\sigma$-complete. □

*Remark* If $Q$ is not ccc, then our definition of $I$ does not lead to anything useful. For example, if $Q$ is Sacks forcing, then $I_Q$ is the ideal of countable sets, and clearly Lemma 3.9 does not hold any more. There are a few possible definitions for ideals generated by non-ccc forcings, see for example [2]. For tree-forcings $Q$, a popular ideal is the following: A set of reals $X$ is in $I$, if for every $T \in Q$ there is a $S \leq_Q T$ such that $\lim(S) \cap X = \emptyset$. In the case of Sacks forcing this ideal is called the Marczewski ideal, it is not ccc, and a Borel set $A$ is in $I$ iff $A$ is countable.

## 4 Preservation

Note: A slightly stonger form of the result of this section (with a similar proof) is presented in [9].

**Definition 4.1**

- *$P$ is Borel $I^+$-preserving, if for all $A \in I^+_{BC}$, $\Vdash_P A^V \in I^+$.*
- *$P$ is $I^+$-preserving, if for all $X \in I^+$, $\Vdash_P \check{X} \in I^+$.*

For example, if $Q$ = random, then random forcing is $I^+$-preserving, and Cohen forcing is not Borel $I^+$-preserving. If $Q$ = Cohen, then Cohen forcing is $I^+$-preserving, and random forcing is not Borel $I^+$-preserving.

Note that being Borel $I^+$-preserving is stronger than just "$\Vdash_P V \cap \omega^\omega \notin I$". For example, set $X := \{x \in \omega^\omega : x(0) = 0\}$ and $Y := \omega^\omega \setminus X$. Let $Q$ be the forcing that adds a real $\eta$ such that $\eta$ is random if $\eta \in X$ and $\eta$ is Cohen otherwise. Clearly, $Q$ is Suslin ccc. $A \in \tilde{I}$ iff ($A \cap X$ is null and $A \cap Y$ is meager). So if $P$ is random forcing, then $\Vdash_P (\omega^{\omega V} \notin I \ \&\ Y^V \in I)$. Note that in this case a $Q$-generic real $\eta^*$ over $M$ will still be generic after forcing with $P$ if $\eta^* \in X$, but not if $\eta^* \in Y$.

However, if $P$ is homogeneous in a certain way with respect to $Q$, then Borel $I^+$-preserving and "$\Vdash_P V \cap \omega^\omega \notin I$" are equivalent (see [13] or [9, 3.2] for more details).

Also, Borel $I^+$-preserving and $I^+$-preserving are generally not equivalent, not even if $P$ is ccc. The standard example is the following: let $Q$ be $\mathbb{C}$ (i.e., Cohen forcing, so $I$ is the ideal of meager sets). We will construct a forcing extension $V'$ of $V$ and a ccc forcing $P \in V'$ such that $P$ is Borel $I^+$-preserving but not $I^+$-preserving (in $V'$):

Let $\mathbb{C}_{\omega_1}$ be the forcing adding $\aleph_1$ many Cohen reals $(c_i)_{i \in \omega_1}$, i.e., $\mathbb{C}_{\omega_1}$ is the set of all finite partial functions from $\omega \times \omega_1$ to 2. Then in any $\mathbb{C}_{\omega_1}$-extension $V[(c_i)_{i \in \omega_1}]$ the Cohen reals $\{c_i : i \in \omega_1\}$ are a Luzin set[7] and for all non-meager Borel sets $A$, $A \cap \{c_i : i \in \omega_1\}$ is uncountable. If $r$ is random over $V$, and $(c_i)_{i \in \omega_1}$ is $\mathbb{C}_{\omega_1}$-generic over $V[r]$, then $(c_i)_{i \in \omega_1}$ is $\mathbb{C}_{\omega_1}$-generic over $V$ as well. So the ccc forcing $\mathbb{B} * \mathbb{C}_{\omega_1}$ can be factored as $\mathbb{C}_{\omega_1} * \underset{\sim}{P}$, where $\underset{\sim}{P}$ is (a name for a) ccc forcing. Set $V' := V[(c_i)_{i \in \omega_1}]$ and $V'' = V'[G_P] = V[r][(c_i)_{i \in \omega_1}]$. Then in $V'$, $P = \underset{\sim}{P}[(c_i)_{i \in \omega_1}]$ is ccc and Borel $I^+$-preserving, $\omega^\omega \cap V \notin I$, but $P \Vdash \omega^\omega \cap V \in I$.

[7] $C$ is a Luzin set if $C$ is uncountable and the intersection of $C$ with any meager set is countable.

**Definition 4.2**

- *For $p \in P^M$, $\eta^*$ is called absolutely $(Q, \underset{\sim}{\eta})$-generic with respect to p, or: $\eta^* \in \mathrm{Gen}^{\mathrm{abs}}(M,p)$, if there is an M-generic $p' \leq p$ forcing that $\eta^* \in \mathrm{Gen}(M[G])$.*
- *P preserves generics for M if for all $p \in P^M$, $\mathrm{Gen}(M) = \mathrm{Gen}^{\mathrm{abs}}(M,p)$. (I.e. every M-generic real could still be $M[G]$-generic for some V- and M-generic G.)*

Note that $\mathrm{Gen}^{\mathrm{abs}}(M,p) \subseteq \mathrm{Gen}(M)$ by 2.6 (or 3.9).

**Lemma 4.3** *If P preserves generics for (the transitive collapse of) unboundedly many countable $N \prec H(\chi)$, then P is $I^+$-preserving.*

Here, unboundedly many means that for all countable $X \subset \omega^\omega$ there is an $N \prec H(\chi)$ countable containing $X$ and $P$ with the required property.

*Remark* The lemma still holds if $Q$ is any ccc forcing, i.e., not Suslin ccc. (Then $N$ is not collapsed but used directly as in usual proper forcing theory).

*Proof* Assume $p \Vdash_P X \subseteq \underset{\sim}{A}[G_P] \in I$, i.e., $p \Vdash_P \Vdash_Q \underset{\sim}{\eta} \notin \underset{\sim}{A}[G_P]^{V[G_P][G_Q]}$. Let $N \prec H(\chi)$ contain $P, X, \underset{\sim}{A}, \underset{\sim}{Q}, p$. Let $M$ be the collapse of $N$ and $\eta^* \in \mathrm{Gen}(M)$, $p' \leq p$ $M$-generic such that $p' \Vdash \eta^* \in \mathrm{Gen}(M[G_P])$. Let $G$ be $V$-generic, $p' \in G$.
Then $V[G] \models M[G_P][G_Q] \models \eta^* \notin A \supseteq X$, so $V \models \eta^* \notin X$. Therefore $\mathrm{Gen}(M) \cap X = \emptyset$. $\mathrm{Gen}(M)$ is of measure 1, therefore $V \models X \in I$. □

**Theorem 4.4** *Assume that P is transitive nep (with respect to a strongly normal ZFC*) and Borel $I^+$-preserving in V and every forcing extension of V. Then P preserves generics (for unboundedly many candidates) and therefore P is $I^+$-preserving.*

We will start with showing that for all candidates $M$ and $p \in P^M$, $\mathrm{Gen}^{\mathrm{abs}}(M,p)$ is nonempty:

**Lemma 4.5** *If P is Borel $I^+$-preserving, $A \in I^+_{BC}$, M a candidate and $p \in P^M$, then* $\mathrm{Gen}^{\mathrm{abs}}(M,p) \cap A^V \neq \emptyset$.

*Proof* Let $G$ be $P$-generic over $M$ and $V$ and contain $p$. In $V[G]$, $\mathrm{Gen}(M[G])$ is of measure 1, and $A^V$ is positive (since $P$ is Borel $I^+$-preserving). So there is an $\eta^* \in \mathrm{Gen}(M[G]) \cap A^V$. Let $p' \leq p$ force all this (in particular "$G$ is $P$-generic over $M$", so $p'$ is $M$-generic). Then $p'$ witnesses that $\eta^* \in \mathrm{Gen}^{\mathrm{abs}}(M,p)$. □

Before we proceed, we take a look once more at strongly normal theories, to make sure that the models we will be using in the proof really are ZFC*-candidates. Intuitively, the reader can think of ZFC models instead of ZFC* (formally that would require a few inaccessibles) and elementary submodels of the universe instead of $H(\chi)$ (that would be more complicated to justify formally).

The ZFC* is strongly normal, so for any forcing notion $R$, $\chi'$ regular and large, $1_R \Vdash H(\chi')^{V[G]} \models$ ZFC*. For $p \in R \subseteq H(\chi)$, $\chi' \gg \chi$ regular, $\underset{\sim}{\tau} \in H(\chi')$, the following are equivalent: $H(\chi') \vDash$ "$p \Vdash_R \varphi(\underset{\sim}{\tau})$" and $p \Vdash_R (H(\chi')^{V[G]} \vDash \varphi(\underset{\sim}{\tau}))$. So in $H(\chi')$ the following holds: For all small forcings $R$, $1_R \Vdash_R$ ZFC*.

"$P$ is Borel $I^+$-preserving" is absolute between $V$ and $H(\chi)$ for $\chi > 2^{\aleph_0}$ regular, since for every $A \in I^+_{\mathrm{BC}} \subset H(\chi)$, $p \Vdash_P A^V \in I$ iff $p \Vdash_P H(\chi)^{V[G_P]} \vDash A^V \in I$ iff $H(\chi) \vDash p \Vdash_P A^V \in I$. Also, "$P$ is transitive nep" is absolute: every countable transitive candidate $M$ and every $p \in P$ is in $H(\chi)$, and $p \Vdash_P (G_P \cap P^M$ is $M$-generic) is absolute by the same argument. In the same way we see the following: If $R \in H(\chi)$, $\chi \ll \chi'$, then "$\Vdash_R P$ is transitive nep and Borel $I^+$-preserving" is absolute between $V$ and $H(\chi')$, and therefore true in $H(\chi')$ according to our assumption.

So every forcing extension $M'$ (by a small forcing) of $H(\chi')$ (or a transitive collapse of an elementary submodel of $H(\chi')$) as well as $H(\chi)^{M'}$ (for $\chi$ large with respect to the forcing) will satisfy ZFC* and think that $P$ is transitive nep and Borel $I^+$-preserving.

Now we can proceed with the proof of the theorem: Fix $\chi_1 \ll \chi_2 \ll \chi_3$ regular such that $H(\chi_i) \vDash$ ZFC*. Let $N \prec H(\chi_3)$ be countable and contain $P, \chi_1, \chi_2$. Clearly there are unboundedly many such $N$. Let $M$ be the transitive collapse of $N$. We want to show that $P$ preserves generics for $M$.

In $M$, let $H_1 := H(\chi_1) \vDash$ ZFC*. Let $R_i$ (in $M$) be the collapse of $H(\chi_i)$ to $\omega$. (I.e. $R_i$ consists of finite functions from $\omega$ to $H(\chi_i)$.) Let $\eta^* \in \mathrm{Gen}(M)$, $p_0 \in P^M$. We have to show that $\eta^* \in \mathrm{Gen}^{\mathrm{abs}}(M, p_0)$. Let $G_Q \in V$ be an $M$-generic filter such that $\underset{\sim}{\eta}[G_Q] = \eta^*$, and let $G_R \in V$ be $R_2$-generic over $M[G_Q]$, $M' = M[G_Q][G_R]$.

**Lemma 4.6** *$M' \vDash$ "$H_1$ is a ZFC*-candidate, $\eta^* \in \mathrm{Gen}^{\mathrm{abs}}(H_1, p_0)$".*

If this is correct, then Theorem 4.4 follows: assume $M' \vDash$ "$p' \leq p_0$ $H_1$-generic, $p' \Vdash \eta^* \in \mathrm{Gen}(H_1[G_P])$". $M'$ is a ZFC*-candidate, so we can find a $p'' \leq p'$ that is $M'$-generic. Then $p''$ is $H_1$ generic and therefore $M$ generic as well (since $\mathfrak{P}(P) \cap M = \mathfrak{P}(P) \cap H_1$), and $p'' \Vdash \eta^* \in \mathrm{Gen}(M[G_P])$.

*Proof (of Lemma 4.6)* It is clear that $H_1$ is a ZFC*-candidate in $M'$. Assume towards a contradiction, that $M' \vDash$ "$\eta^* \notin \mathrm{Gen}^{\mathrm{abs}}(H_1, p_0)$". Then this is forced by some $q \in G_Q$ and $r \in R_2$, but since $R_2$ is homogeneous, without loss of generality $r = 1$, i.e.,

$$M \vDash \text{“}q \Vdash_Q \Vdash_{R_2} \eta^* \notin \mathrm{Gen}^{\mathrm{abs}}(H_1, p_0)\text{”}. \qquad (*)$$

Now we are going to construct the models of Fig. 1: first, choose a $G_{R_1} \in V$ which is $R_1$-generic over $M$, and let $M_1 = M[G_{R_1}]$. In $M_1$, pick $\eta^\otimes \in \mathrm{Gen}^{\mathrm{abs}}(H_1, p_0) \cap B^M_q$. (We can do that by Lemma 4.5, since we know that $P$ is Borel $I^+$-preserving in $M_1$). Since $\mathrm{Gen}^{\mathrm{abs}} \subseteq \mathrm{Gen}$, $M_1 \vDash$ "$\exists\, G^\otimes_Q$ $Q$-generic over $H_1$ such that $q \in G^\otimes_Q, \underset{\sim}{\eta}[G^\otimes_Q] = \eta^\otimes$". This $G^\otimes_Q$ clearly is $M$-generic as well (since $M \cap \mathfrak{P}(Q) = H_1 \cap \mathfrak{P}(Q)$), so we can factorize $R_1$ as $R_1 = Q * R_1/Q$ such that $G_{R_1} = G^\otimes_Q * \tilde{G}_1$.

Now we look at the forcing $R_2 = R^M_2$ in $M[\eta^\otimes] = M[G^\otimes_Q]$. $R_2$ forces that $R_1$ is countable and therefore equivalent to Cohen forcing. $R_1/Q$ is a subforcing of

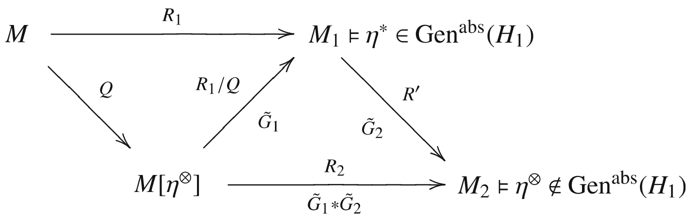


**Fig. 1** The models used in the proof of Lemma 4.6

$R_1$. Also, $R_2$ adds a Cohen real. So $R_2$ can be factorized as $R_2 = (R_1/Q) * R'$, where $R' = (R_2/(R_1/Q))$. We already have $\tilde{G}_1$, an $(R_1/Q)$-generic filter over $M[G^{\otimes}_Q]$; now choose $\tilde{G}_2 \in V$ $R'$-generic over $M_1$, and let $G_{R_2} = \tilde{G}_1 * \tilde{G}_2$ So $G_{R_2} \in V$ is $R_2$-generic over $M[G^{\otimes}_Q]$, $M_2 := M[\eta^{\otimes}][G_{R_2}]$.

Let $H_2$ be $H(\chi_2)^{M_1}$. $H_2 \vDash \mathrm{ZFC}^*$. Also, $H_2 \vDash$ "$p_1 \leq p_0$ is $H_1$-generic, $p_1 \Vdash \eta^{\otimes} \in \mathrm{Gen}(H_1[G_P])$" (since this is absolute between the universe $M_1$ and $H_2 = H(\chi_2)^{M_1}$). In $M_2$, $H_2$ is a ZFC$^*$-candidate. In $M_2$, let $p_2 \leq p_1$ be $H_2$-generic. Then (in $M_2$), $p_2$ witnesses that $\eta^* \in \mathrm{Gen}^{\mathrm{abs}}(H_1, p_0)$, a contradiction to $(*)$. □

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## References

1. Bagaria, J., Bosch, R.: Projective forcing. Ann. Pure Appl. Logic **86**(3), 237–266 (1997)
2. Bartoszyński, T., Judah, H.: Set theory: on the structure of the real line. In: Peters, A.K., (ed.) Wellesley (1995)
3. Baumgartner, J.E.: Iterated forcing. In: Mathias, A. (ed.) Surveys in set theory, London mathematical society lecture note series, vol. 87, pp. 1–59. Cambridge University Press, Cambridge (1983)
4. Goldstern, M.: Tools for your forcing construction. In: Judah, H. (ed.) Set Theory of The Reals, israel mathematical conference proceedings, vol. 6, pp. 305–360. American Mathematical Society (1993)
5. Goldstern, M., Judah, H.: Iteration of Souslin forcing, projective measurability and the Borel conjecture. Isr. J. Math. **78**, 335–362 (1992)
6. Ihoda (Haim Judah), J., Shelah, S.: Souslin forcing. J. Symb. Logic **53**, 1188–1207 (1988)
7. Judah, H., Shelah, S.: The kunen-miller chart (lebesgue measure, the baire property, laver reals and preservation theorems for forcing). J. Symb. Logic **55**, 909–927 (1990)
8. Keisler, H.J.: Logic with the quantifier "there exist uncountably many". Ann. Math. Logic **1**, 1–93 (1970)
9. Kellner, J., Shelah, S.: Preserving preservation. J. Symb. Logic **70**(3), 914–945 (2005). Math. LO/0405081
10. Pawlikowski, J.: Laver's forcing and outer measure. In: Bartoszyński, T., Scheepers, M. (eds.) Proceedings of BEST conferences 1991–1994. American Mathematical Society, Providence (1995)
11. Roslanowski, A., Shelah, S.: Norms on possibilities i: forcing with trees and creatures. Memoirs of the American Mathematical Society, vol. 141, xii + 167 (1999). Math.LO/9807172
12. Shelah, S.: Proper and improper forcing. Perspectives in Mathematical Logic. Springer, Berlin Heidelberg New York (1998)
13. Shelah, S.: Properness without elementaricity. J. Appl. Anal. **10**, 168–289 (2004). Math. LO/9712283
14. Sikorski, R.: Boolean Algebras. Springer, Berlin Heidelberg New York (1964)