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New reals: Can live with them, can live without them

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We give a self-contained proof of the preservation theorem for proper countable support iterations known as "tools-preservation", "Case A" or "first preservation theorem" in the literature. We do not assume that the forcings add reals.

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1 Introduction

In his book "Proper and Improper Forcing" [7, XVIII §3] Shelah gave several cases of general preservation theorems for countable support iterations of proper[1)] forcings (the proofs tend to be hard to digest, though). In this paper we deal with "Case A". A specific application is that proper countable support iterations of ω^ω-bounding forcings (see Example 2.2 here) are ω^ω-bounding.

A simplified version of Case A appeared in Section 5 of the first author's "Tools for your forcing constructions" [2]. This version uses the additional requirement that every iterand adds a new real. Note that this requirement is met in most applications, but the case of forcings "not adding reals" has important applications as well (and note that not adding reals is generally not preserved under proper countable support iterations).

A proof of the iteration theorem *without* this additional requirement appeared in [4] and was copied into "Set Theory of the Reals" [1] (as "first preservation theorem" 6.1.B), but Schlindwein pointed out a problem in this proof.[2)] In this paper, we generalize the proof of [2].

We thank Chaz Schlindwein for finding the problems in the existing proofs and bringing them to our attention.

2 The theorem

Fix a sequence of increasing arithmetical two-place relations $(R_j)_{j\in\omega}$ on ω^ω. Let R be the union of the R_j. Assume

1. $\mathcal{C} := \{f \in \omega^\omega : fR\eta \text{ for some } \eta \in \omega^\omega\}$ is closed;
2. $\{f \in \omega^\omega : fR_j\eta\}$ is closed for all $j \in \omega$, $\eta \in \omega^\omega$;
3. for every countable N there is η such that $fR\eta$ for all $f \in N \cap \mathcal{C}$ (in this case we say η *covers* N).

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1) P is *proper* if for all countable elementary submodels $N \prec H(\chi)$ containing P (χ a big regular cardinal) and all $p \in P \cap N$ there is $q \leq p$ which forces that G_P is N-generic (i. e. $G_P \cap D \cap N \neq \emptyset$ for all dense subsets $D \in N$). Such a q is called N-*generic*.

2) In [6], where Schlindwein gave a proof for the special case of ω^ω-bounding, following [7, VI]. However he later detected another problem in his own proof (C. Schlindwein, personal communication, April 2005), and he is preparing a new version [5].

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Definition 2.1 Let P be a forcing notion, $p \in P$.

1. $\bar{f}^* := (f_1^*, \dots, f_k^*)$ is a *P-interpretation of* $\underset{\sim}{\bar{f}} := (\underset{\sim}{f}_1, \dots, \underset{\sim}{f}_k)$ *under* p if $f_i^* \in \omega^\omega$, $\underset{\sim}{f}_i$ is a P-name for an element of $\mathcal{C}$, and there is a decreasing chain $p \geq p^0 \geq p^1 \geq \cdots$ of conditions in P such that p^i forces $\underset{\sim}{f}_1 \restriction i = f_1^* \restriction i \,\&\, \cdots \,\&\, \underset{\sim}{f}_k \restriction i = f_k^* \restriction i$.

2. A forcing notion P is *weakly preserving* if for all $N \prec H(\chi)$ countable, η covering N, $p \in N$, there is an N-generic $q \leq p$ which forces that η covers $N[G_P]$.

3. A forcing notion P is *preserving* if for all $N \prec H(\chi)$ countable, η covering N, $p \in N$, and $\bar{f}^*, \underset{\sim}{\bar{f}} \in N$ such that $\bar{f}^*$ is a P-interpretation of $\underset{\sim}{\bar{f}}$ under p, there is an N-generic $q \leq p$ which forces that η covers $N[G_P]$ and moreover that $f_i^* R_j \eta$ implies $\underset{\sim}{f}_i R_j \eta$ for all $i \leq k$, $j \in \omega$.

4. A forcing notion P is *densely preserving* if there is a dense subforcing $Q \subseteq P$ which is preserving.

Note that if $\bar{f}^*$ is an interpretation, then $f_l^* \in \mathcal{C}$ (since $\mathcal{C}$ is closed).

The simplest example is that of ω^ω-bounding:

Example 2.2 Set $f R_n \eta$ if $f(m) < \eta(m)$ for all $m > n$. So we have $\mathcal{C} = \omega^\omega$, and $f R \eta$ if there is n such that $f(m) < \eta(m)$ for all $m > n$. To cover a family of functions means to dominate it. P is weakly preserving iff P is proper and ω^ω-bounding[3)].

This example is typical in the sense that often R describes a covering property of the pair $(V, V[G])$.

The property "weakly preserving" is invariant under equivalent forcings. I. e. if P forces that there is a Q-generic filter over V and Q forces the same for P, then Q is weakly preserving iff P is weakly preserving.[4)] The notion "preserving" however does not seem to be invariant.[5)] It even seems that "densely preserving" does not imply "preserving". (Although we do not have an example. It is not important after all.) One direction however is clear:

Fact 2.3 *If P is preserving and $Q \subseteq P$ is dense, then Q is preserving.*

We could define $\bar{f}^*$ to be a "weak interpretation" of $\underset{\sim}{\bar{f}}$ under p by requiring that the truth value of $\underset{\sim}{f} \restriction m = f^* \restriction m$ is positive (under p) for all n. This would lead to a notion "strongly preserving". This notion is invariant under dense subforcings, and it is easy to see that Q is strongly preserving iff $\mathrm{ro}(Q)$ is preserving (which implies that Q is preserving by Fact 2.3).

For some instances of R, weakly preserving is equivalent to preserving (and therefore to strongly preserving as well). Most notably this is the case for ω^ω-bounding (see [2, 6.5]).

For other instances of R (e. g. Lebesgue positivity, cf. [3]) "P is preserving" is equivalent to some other property invariant under equivalent forcings (and therefore again equivalent to "P is strongly preserving").

We will show that densely preserving is preserved under proper countable support iterations. This is our version of the theorem known as "tools preservation" [2, Section 5], "Case A" [7, XVIII §3] or the "first preservation theorem" [1, 6.1.B]:

Theorem 2.4 *Assume $(P_i^0, \underset{\sim}{Q}_i^0)_{i<\varepsilon}$ is a countable support iteration of proper, densely preserving forcings. Then P_ε^0 is densely preserving.*

---

3) P is *ω^ω-bounding* if for all P-names $\underset{\sim}{f} \in \omega^\omega$ and $p \in P$ there is $q \leq p$ and $g \in \omega^\omega$ such that $q \Vdash \underset{\sim}{f}(m) < g(m)$ for all m. So if P is ω^ω-bounding, η covers N, $\underset{\sim}{f} \in N$ and G is N-generic, then $\underset{\sim}{f}[G]$ is dominated by some $g \in N$ and therefore by η. If on the other hand P is weakly preserving, $\underset{\sim}{f}$ a P-name and $p \in P$, then there is $N \prec H(\chi)$ containing p and $\underset{\sim}{f}$. Pick an $\eta \in V$ covering N. So if $q \leq p$ is as in the definition of weakly preserving, then q forces that η dominates $\underset{\sim}{f}$.

4) This uses the following fact: P is weakly preserving (i. e. weakly preserving with respect to all $N \prec H(\chi)$) iff P is weakly preserving with respect to all $N \prec H(\chi)$ containing some fixed $x \in H(\chi)$.

5) The reason is that the notion of interpretation is not invariant. Given a forcing P and an interpretation f^* of a function $\underset{\sim}{f} \notin V$, we can find a dense subforcing $P' \subset P$ such that for every condition p' of P' there is $n(p')$ such that p' forces that $f^*(n(p')) \neq \underset{\sim}{f}(n(p'))$ (here we identify the P-name $\underset{\sim}{f}$ with the equivalent P'-name). So f^* cannot be a P'-interpretation of $\underset{\sim}{f}$.

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3 An outline of the proof

In this section, we describe the ideas used in the proof, without being too rigorous.

3.1 Use names

How can we show that the countable support limit of proper forcings is proper?

We have a countable support iteration $(P_\alpha, \underset{\sim}{Q}_\alpha)_{\alpha<\varepsilon}$ of proper forcings (ε limit), $N \prec H(\chi)$ countable, and $p \in P \cap N$. We want to find a $q_\omega \in P_\varepsilon$ which forces that G is N-generic, i. e. that $G \cap D \cap N \neq \emptyset$ for all dense subsets $D \in N$ of P.

So we fix an ω-sequence $0 = \alpha_0 < \alpha_1 < \cdots$ cofinal in $\varepsilon \cap N$, and enumerate all dense open sets of P that are in N as $(D_n)_{n\in\omega}$.

One unsuccessful attempt to construct q_ω could be the one illustrated in Figure 1: Set $p_{-1} := p$ and $q_{-1} := \emptyset$. Given $p_{n-1} \in N$ and q_{n-1}, choose (in N) a $p_n \leq p_{n-1}$ in $D_n \cap N$ and (in V) a $q_n \leq p_n{\restriction}\alpha_{n+1}$ which extends q_{n-1}. Set $q_\omega := \bigcup q_n$. Then q_ω is N-generic, since $q_\omega \leq p_n \in D_n \cap N$. Of course this does not work, since we generally cannot find a $p_n \leq p_{n-1}$ in D_n such that $q_{n-1} \leq p_n{\restriction}\alpha_n$.

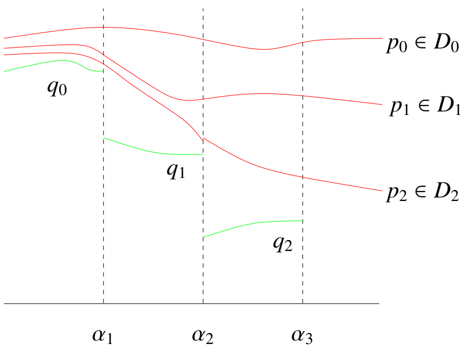

Fig. 1

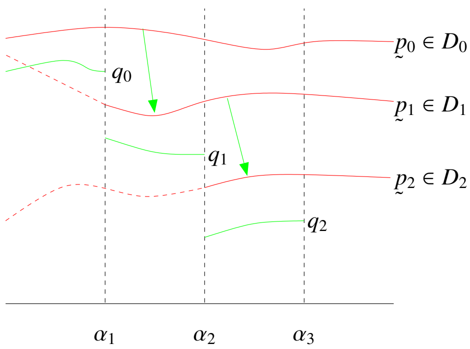

Fig. 2

What we actually do instead is the following (see Figure 2): The p_n will be P_{α_n}-names, and the q_n are $P_{\alpha_{n+1}}$-generic over N. So instead of choosing $p_n \in P_\varepsilon$, we choose (in N) a P_{α_n}-name $\underset{\sim}{p}_n$ for an element of P_ε such that the following is forced by P_{α_n}:

1. $\underset{\sim}{p}_n \in D_n$;
2. $\underset{\sim}{p}_n{\restriction}\alpha_n \in G_{\alpha_n}$;
3. if $\underset{\sim}{p}_{n-1}{\restriction}\alpha_n \in G_{\alpha_n}$, then $\underset{\sim}{p}_n \leq \underset{\sim}{p}_{n-1}$.

It is clear that we can find such a name. So we first construct all the $\underset{\sim}{p}_n$ (each $\underset{\sim}{p}_n$ is in N, but the sequence is not). Then we construct $q_n \in P_{\alpha_{n+1}}$ satisfying the following:

1. q_n extends q_{n-1}.
2. q_n is $P_{\alpha_{n+1}}$-generic over N.
3. q_n is stronger than $\underset{\sim}{p}_n$ on the interval $[\alpha_n, \alpha_{n+1})$.[6)]

So (by induction) q_n forces that $\underset{\sim}{p}_n{\restriction}\alpha_{n+1} \in G_{\alpha_{n+1}}$ and that therefore $\underset{\sim}{p}_{n+1} \leq \underset{\sim}{p}_n$. So $q_\omega = \bigcup q_n$ forces that $\underset{\sim}{p}_n{\restriction}\alpha_n \in G_\alpha$ (by definition of $\underset{\sim}{p}_n$), that $\underset{\sim}{p}_n{\restriction}\alpha_{n+1} \geq \underset{\sim}{p}_{n+1}{\restriction}\alpha_{n+1} \in G_{\alpha_{n+1}}$ and generally that $\underset{\sim}{p}_n{\restriction}\alpha_m \in G_{\alpha_m}$ for all $m > n$. Therefore q_ω forces that $\underset{\sim}{p}_n \in G_\varepsilon$. Also, q_{n-1} is P_{α_n}-generic over N, and the P_{α_n}-name $\underset{\sim}{p}_n$ is in N, so q_ω forces that $\underset{\sim}{p}_n \in N \cap P_\varepsilon$ and therefore in $N \cap D_n \cap G_\varepsilon$, i. e. that G_ε is N-generic.

3.2 Interpolate approximations

First note that for every P_ε-name $\underset{\sim}{f} \in \mathcal{C}$ and for every $p \in P_\varepsilon$ we can find an approximation f^* of $\underset{\sim}{f}$ under p. If additionally $0 < \alpha < \varepsilon$ and P_α adds a new real $\underset{\sim}{r}$, then we can choose the witnesses of the approximation such that $\{p^m{\restriction}\alpha : m \in \omega\} \subseteq P_\alpha$ is inconsistent[7)] (just let $p^m{\restriction}\alpha$ decide $\underset{\sim}{r}(m)$).

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6) More formally (since $\underset{\sim}{p}_n$ is a name): For all $\alpha_n \leq \beta < \alpha_{n+1}$, $q_n{\restriction}\beta \Vdash_\beta \underset{\sim}{p}_n{\restriction}\beta \in G_\beta$ & $q_n(\beta) \leq \underset{\sim}{p}_n(\beta)$.

7) We call a set $A \subseteq P$ *inconsistent* if P forces that not every condition of A is in G.

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Now assume that f^* is a P_ε-approximation of $\underset{\sim}{f}$ witnessed by $(p_0^m)_{m\in\omega}$ and that $\{p_0^m\restriction\alpha : m\in\omega\}\subseteq P_\alpha$ is inconsistent. Then we can define P_α-names $(\underset{\sim}{p}_\alpha^m)_{m\in\omega}$ and $\underset{\sim}{f}^{**}$ such that the following is forced by P_α (see Figure 3):

1. $p_0^m\restriction\alpha\in G_\alpha$ implies $\underset{\sim}{p}_\alpha^0\le p_0^m$ (i. e. $\underset{\sim}{p}_\alpha^0$ is stronger than the strongest p_0^m whose restriction is in G_α).
2. $\underset{\sim}{p}_\alpha^m\restriction\alpha\in G_\alpha$, hence $\underset{\sim}{p}_\alpha^m\in P_\varepsilon/G_\alpha$.
3. $\underset{\sim}{f}^{**}$ is an approximation of $\underset{\sim}{f}$ witnessed by $(\underset{\sim}{p}_\alpha^m)_{m\in\omega}$.

Then $(p_0^m\restriction\alpha)_{m\in\omega}$ witnesses that f^* approximates $\underset{\sim}{f}^{**}$: $p_0^m\restriction\alpha$ forces that

1. $\underset{\sim}{p}_\alpha^m$ forces that $\underset{\sim}{f}^{**}\restriction m=\underset{\sim}{f}\restriction m$ and
2. $\underset{\sim}{p}_\alpha^m\le p_0^m$ and therefore that
3. $\underset{\sim}{p}_\alpha^m$ also forces (in P_ε/G_α) that $f^*\restriction m=\underset{\sim}{f}\restriction m$.

So $p_0^m\restriction\alpha\wedge\underset{\sim}{p}_\alpha^m$ forces $\underset{\sim}{f}^{**}\restriction m=\underset{\sim}{f}\restriction m=f^*\restriction m$, and since $\underset{\sim}{f}^{**}\restriction m$, $f^*\restriction m$ already live in $V[G_\alpha]$, $\underset{\sim}{f}^{**}\restriction m=f^*\restriction m$ is already forced by $p_0^m\restriction\alpha$.

So we can interpolate (or "factor") the interpretation $(f^*,\underset{\sim}{f})$ by the "composition" of the interpretations $(f^*,\underset{\sim}{f}^{**})$ and $(\underset{\sim}{f}^{**},\underset{\sim}{f})$.

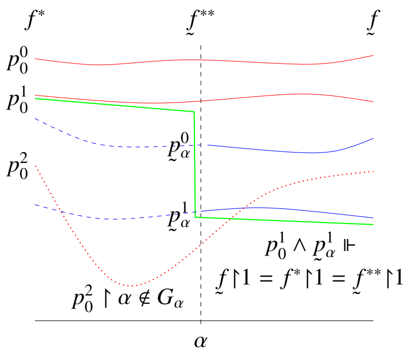

Fig. 3

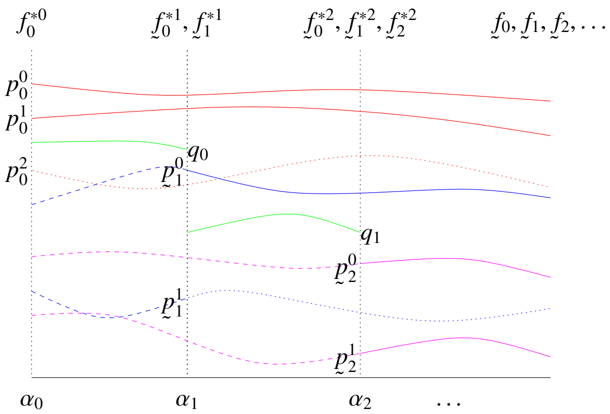

Fig. 4

Moreover, if $\underset{\sim}{g}$ is another P_ε-name for an element of $\mathcal{C}$, we may choose the names $\underset{\sim}{p}_\alpha^m$ such that P_α forces that $(\underset{\sim}{p}_\alpha^m)_{m\in\omega}$ is a witness not only for f^{**} approximating $\underset{\sim}{f}$, but also for some g^* approximating $\underset{\sim}{g}$.

3.3 Approximate more and more functions better and better

In addition to all the dense sets D_n of N – as in Subsection 3.1 – we also list all the P_ε-names $\underset{\sim}{f}_n$ in N for elements of $\mathcal{C}$. We have to make sure that q_ω forces that $\underset{\sim}{f}R\eta$. We assume that every element of D_n decides $\underset{\sim}{f}_m\restriction n$ for $m\le n$.

We start with an approximation f_0^{*0} for $\underset{\sim}{f}_0$ witnessed by $(p_0^m)_{m\in\omega}$. We assume that $\{p_0^m\restriction\alpha_1 : m\in\omega\}$ is inconsistent. We can find (in N) P_{α_1} names $(\underset{\sim}{p}_1^m)_{m\in\omega}$ and $\underset{\sim}{f}_0^{*1},\underset{\sim}{f}_1^{*1}$ (see Figure 4) such that the following is forced:

1. $p_1^m\restriction\alpha_1\in G_{\alpha_1}$, hence $p_1^m\in P_\varepsilon/G_{\alpha_1}$.
2. $\underset{\sim}{f}_0^{*1},\underset{\sim}{f}_1^{*1}$ are interpretations of $\underset{\sim}{f}_0,\underset{\sim}{f}_1$ witnessed by $(p_1^m)_{m\in\omega}$.
3. $p_0^m\restriction\alpha_1\in G_{\alpha_1}$ implies $p_1^0\le p_0^m$ (i. e. $\underset{\sim}{f}_0^{*1}$ interpolates $(f_0^{*0},\underset{\sim}{f}_0)$ as in Subsection 3.2).
4. $\underset{\sim}{p}_1^0\in D_1$ (in particular, $\underset{\sim}{p}_1^0$ decides $\underset{\sim}{f}_0\restriction 1$, $\underset{\sim}{f}_1\restriction 1$).
5. We again assume that $\{\underset{\sim}{p}_1^m\restriction\alpha_2 : m\in\omega\}$ is inconsistent.

Because of the last assumption, we can iterate this construction.

Now we choose (in V) a $q_0\in P_{\alpha_1}$ such that $q_0\le p_0^0\restriction\alpha_1$ and q_0 is P_{α_1}-generic over N and forces that η covers $N[G_{\alpha_1}]$ and that $f_0^{*0}R_j\eta$ implies $\underset{\sim}{f}^{*1}R_j\eta$ for all m. Inductively, we get a sequence $(q_n)_{n\in\omega}$ such that $q_n\in P_{\alpha_{n+1}}$ extends q_{n-1} and forces

1. $G_{\alpha_{n+1}}$ is N-generic and η covers $N[G_{\alpha_{n+1}}]$;
2. $\underset{\sim}{f}_m^{*n}R_j\eta$ implies $\underset{\sim}{f}_m^{*n+1}R_j\eta$ for $m\le n$ and all j.

Let q_ω be the union of all q_n. Then q_ω forces the following: For $m \geq n$, $\underaccent{\tilde}{f}_n \restriction m = \underaccent{\tilde}{f}^{*m}_n \restriction m$ (since $\underaccent{\tilde}{p}^0_m \in D_m$ decides $\underaccent{\tilde}{f}_n \restriction m$). Also, $\underaccent{\tilde}{f}^{*n}_n R_j \eta$ for some $j \in \omega$ (since $\underaccent{\tilde}{f}^{*n}_n \in N[G_{\alpha_n}]$ and η covers $N[G_{\alpha_n}]$). $\underaccent{\tilde}{f}_n$ is the limit of functions $\underaccent{\tilde}{f}^{*m}_n$ which all satisfy $\underaccent{\tilde}{f}^{*m}_n R_j \eta$. Since $\{f \in \omega^\omega : f R_j \eta\}$ is closed, $\underaccent{\tilde}{f}_n R_j \eta$. Also, q_ω is N-generic just as in Subsection 3.1.

3.4 Decide when we are σ-complete

The proof so far relies on the fact that we can always find approximations whose witnesses are inconsistent (see item 4. in Subsection 3.3). We already know that this is the case if the iteration between α_n and α_{n+1} adds a new real. Actually we just need that the iterands are "nowhere σ-complete", i. e. that below every p we can find an inconsistent decreasing sequence.

If no reals are added, it might seem that we do not have anything to do (since Case A preservation is vacuous without new reals). The problem is that the countable support iteration of proper forcings which do not add reals can add a real in the limit. So it might be that we do not have new reals in the intermediate steps (we would like to use such reals to get inconsistent witnesses for approximations), but we get new reals in the limit (which could be a problem for preservation). On the other extreme, if all iterands are σ-complete, then the limit is σ-complete as well, and therefore adds no reals, so there is nothing to do.

So what to do?

First note that we can split every forcing into a σ-complete and a nowhere σ-complete part. However, that does not solve our problem, since we can not split the index set ε of the iteration into $\varepsilon_1, \varepsilon_2$ such that P_α forces that $\underaccent{\tilde}{Q}_\alpha$ is σ-complete if $\alpha \in \varepsilon_1$ and nowhere σ-complete otherwise. For example, Q_0 could add a Cohen real $\underaccent{\tilde}{c}$, and $\underaccent{\tilde}{Q}_n$ could be defined to be σ-complete iff $\underaccent{\tilde}{c}(n) = 0$.

So we will do the following: Given a condition $p \in P_\varepsilon$, there is a maximal $\gamma \leq \varepsilon$ such that P_α forces that $\underaccent{\tilde}{Q}_\alpha$ is σ-complete (below $p(\alpha)$) for all $\alpha < \gamma$. So if $\gamma = \varepsilon$, then the rest of the iteration is σ-complete. If $\gamma < \varepsilon$, then we strengthen p such that P_γ forces that $\underaccent{\tilde}{Q}_\gamma$ is nowhere σ-complete (below $p(\gamma)$).

We will only be interested in honest approximations, that is an approximation witnessed by $(p^m)_{m\in\omega}$, where p^0 (and therefore all p^m) will know the index γ where $\underaccent{\tilde}{Q}_\gamma$ stops to be σ-complete (in the way just described).

Since in Subsection 3.3 the conditions $\underaccent{\tilde}{p}^m_n$ are P_{α_n}-names, the corresponding γ will be a P_{α_n}-name as well. In the iteration at stage n, we will have to distinguish three cases:

1. $\{\underaccent{\tilde}{p}^m_{n-1} \restriction \alpha_n : m \in \omega\}$ is inconsistent. Then continue as in Subsection 3.2.

2. The γ corresponding to $\underaccent{\tilde}{p}^0_{n-1}$ is bigger than α_n but less than ε. Then just "do nothing", i. e. wait in the iteration until α_m is above γ and therefore the witnesses are inconsistent.

3. Otherwise, we know that the rest of the iteration is σ-complete.

Again, we do not know from the beginning which case we will use at a given stage. In the example above, we will do nothing at stage n iff $\underaccent{\tilde}{c}(n) = 0$ (so it will never happen that the rest of the iteration is σ-complete).

Also, when we "do nothing", we cannot increase the number of functions we approximate. In Subsection 3.3, the number k_n of functions which we approximate in step n was $n+1$ ($\underaccent{\tilde}{f}^{*n}_0, \dots, \underaccent{\tilde}{f}^{*n}_n$ approximates $\underaccent{\tilde}{f}_0, \dots, \underaccent{\tilde}{f}_n$). So in the proof this number $\underaccent{\tilde}{k}_n$ will be a P_{α_n}-name which is $\underaccent{\tilde}{k}_{n-1}$ in case "do nothing" and $n+1$ otherwise.

4 The proof

Definition 4.1 Let Q be a forcing, $q \in Q$.

1. q is *σ-complete in* Q if $Q_q := \{r \in Q : r \leq q\}$ is σ-complete. In this case we write $q \in Q^\sigma$.

2. q is *nowhere σ-complete in* Q if there is no $q' \leq_Q q$ such that $q' \in Q^\sigma$. In this case we write $q \in Q^{\neg\sigma}$.

3. Q is *decisive* if every $q \in Q$ is either 1_Q (the weakest element of Q) or σ-complete or nowhere σ-complete.[8)]

Fact 4.2 *For every P the set of conditions that are either σ-complete or nowhere σ-complete is open dense. I. e. for every P there is a dense subforcing $Q \subseteq P$ which is decisive.*

[8)] Of course it is possible to have $1_Q \in Q^\sigma$ or $1_Q \in Q^{\neg\sigma}$.

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Fact 4.3 *If $(P_\alpha, \underset{\sim}{Q}_\alpha)_{\alpha<\varepsilon}$ is an iteration and P_α forces that $\underset{\sim}{Q}''_\alpha \subseteq \underset{\sim}{Q}_\alpha$ is dense (for every $\alpha \in \varepsilon$), then there are an iteration $(P'_\alpha, \underset{\sim}{Q}'_\alpha)_{\alpha<\varepsilon}$ and dense embeddings $\varphi_\alpha : P'_\alpha \longrightarrow P_\alpha$ ($\alpha \leq \varepsilon$) such that for $\alpha \leq \beta \leq \varepsilon$ the following hold:*

1. *If $p \in P'_\beta$, then $\varphi_\alpha(p{\restriction}\alpha) = \varphi_\beta(p){\restriction}\alpha$.*
2. *In particular φ_β is an extension of φ_α.*
3. *P'_α forces that $\underset{\sim}{Q}'_\alpha = \underset{\sim}{Q}''_\alpha[G_{P_\alpha}]$*[9]*.*

Because of Fact 2.3, Fact 4.2 and Fact 4.3 we can modify the original iteration $(P^0_\alpha, \underset{\sim}{Q}^0_\alpha)_{\alpha<\varepsilon}$ of Theorem 2.4 to get an iteration $(P_\alpha, \underset{\sim}{Q}_\alpha)_{\alpha<\varepsilon}$ satisfying "P_ε is a dense subforcing of P^0_ε" and:

Assumption 4.4 P_α forces that $\underset{\sim}{Q}_\alpha$ is proper, decisive and preserving.

We will show that in this case P_ε is densely preserving,[10] so P^0_ε is densely preserving as well, proving Theorem 2.4.

From now on we fix the iteration $(P_\alpha, \underset{\sim}{Q}_\alpha)_{\alpha<\varepsilon}$ satisfying Assumption 4.4. We also fix a regular $\chi \gg 2^{|P_\varepsilon|}$, a countable $N \prec H(\chi)$ containing $(P_\alpha, \underset{\sim}{Q}_\alpha)_{\alpha<\varepsilon}$, and an η covering N.

Definition 4.5 We will use the following notation ($\alpha \leq \beta$):

1. For $p \in P_\alpha$, $p \Vdash_\alpha \varphi$ means $p \Vdash_{P_\alpha} \varphi$.
2. If $p \in P_\beta$, $r \in P_\alpha$ and $r \leq p{\restriction}\alpha$, then we can define $r \wedge p \in P_\beta$, *the weakest condition stronger than* r *and* p.
3. G_α is the P_α-generic filter over V (or its canonical name). So $\Vdash_\beta G_\alpha = G_\beta \cap P_\alpha$. We set $V_\alpha := V[G_\alpha]$.
4. P_β/G_α is the P_α-name for the forcing consisting of those P_β-conditions p such that $p{\restriction}\alpha \in G_\alpha$ (with the same order as P_β).
5. In V_α: If $p \in P_\beta/G_\alpha$, then $p \Vdash_{(\alpha,\beta)} \varphi$ means $p \Vdash_{P_\beta/G_\alpha} \varphi$. We also say p *(α,β)-forces* φ.

Fact 4.6 *Let $0 \leq \alpha \leq \beta \leq \varepsilon$.*

1. *The function $P_\beta \longrightarrow P_\alpha * P_\beta/G_\alpha$ defined by $p \longmapsto (p{\restriction}\alpha, p)$ is a dense embedding.*
2. *If $p_1 \in P_\alpha$ and $\underset{\sim}{p}_2$ is a P_α-name for an element of P_β/G_α, then $p_1 \Vdash_\alpha \underset{\sim}{p}_2 \Vdash_{(\alpha,\beta)} \varphi$ is equivalent to*

$$\Vdash_\beta (p_1 \in G_\beta \,\&\, \underset{\sim}{p}_2 \in G_\beta) \rightarrow \varphi.$$

3. *If D is an (open) dense subset of P_β, then $D \cap P_\beta/G_\alpha$ is a P_α-name for an (open) dense subset of P_β/G_α.*

Note: If $\underset{\sim}{p}$ is a P_α-name for an element of P_β/G_α, then $\Vdash_\alpha \underset{\sim}{p} \Vdash_{(\alpha,\beta)} \varphi$ does not imply that $\underset{\sim}{p}[G_\alpha]$ (which is an element of P_β and therefore of V) forces φ in V (as element of P_α). I. e. $V \vDash (\Vdash_\alpha \underset{\sim}{p} \Vdash_{(\alpha,\beta)} \varphi)$ does not imply $\Vdash_\alpha (V \vDash \underset{\sim}{p} \Vdash_\beta \varphi)$.

We will use the following straightforward technical facts:

Lemma 4.7 *Let $0 \leq \alpha \leq \gamma \leq \beta \leq \varepsilon$. P_α forces:*

1. *If $p \in P_\beta/G_\alpha$, $q \in P_\gamma/G_\alpha$, and $q \Vdash_{(\alpha,\gamma)} p{\restriction}\gamma \in G_\gamma$, then we can define $q \wedge (p{\restriction}\beta \setminus \gamma)$ to be a condition $p' \in P_\beta/G_\alpha$ such that $p'{\restriction}\gamma = q$ and*

$$p'{\restriction}\xi \Vdash_{(\alpha,\xi)} p'(\xi) = p(\xi)$$

for $\gamma \leq \xi < \beta$. If $q \leq p{\restriction}\gamma$, then $q \wedge (p{\restriction}\beta \setminus \gamma) \leq p$, and if $p_2 \leq p_1$, then $q \wedge (p_2{\restriction}\beta \setminus \gamma) \leq q \wedge (p_1{\restriction}\beta \setminus \gamma)$.

2. *If $p^0 \geq p^1 \geq \cdots$ is a decreasing sequence in P_γ/G_α, and for every $\alpha \leq \zeta < \gamma$, $p^0{\restriction}\zeta \Vdash_{(\alpha,\zeta)} p^0(\zeta) \in \underset{\sim}{Q}^\sigma_\zeta$, then there is $p^\omega \leq p^0 \in P_\gamma/G_\alpha$ such that $p^\omega \Vdash_{(\alpha,\gamma)} p^m \in G_\gamma$ for all $m \in \omega$. (Here we actually use that P_α is proper.)*

P r o o f. To show 1., set $A := \mathrm{dom}(q) \cup (\mathrm{dom}(p) \setminus \alpha)$. Note that $A \in V$. Fix a P_α-name for p. Define for $\xi \in A$ (in V) $p'(\xi) = q(\xi)$ if $\xi < \gamma$, and for $\xi \geq \gamma$ let $p'(\xi)$ be $p(\xi)$ provided that $p{\restriction}\xi \in G_\xi$ ($1_{\underset{\sim}{Q}_\xi}$ otherwise).

2. is similar: There is $A \in V$ countable in V such that $A \supseteq \bigcup_{m\in\omega} \mathrm{dom}(p^m)$ (since P_α is proper). Fix a P_α-name (in V) for the sequence $(p^m)_{m\in\omega}$.

Now define p^ω in V: Set $p^\omega{\restriction}\alpha := p^0{\restriction}\alpha$. For $\alpha \leq \zeta < \gamma$, $\zeta \in A$ define $p^\omega(\zeta) \in \underset{\sim}{Q}_\zeta$ to be a lower bound of $\{p^m(\zeta) : m \in \omega\}$ if such a lower bound exists, and $p^0(\zeta)$ otherwise. □

[9] Where $G_{P_\alpha} := \{p \in P_\alpha : (\exists p' \in G_{P'_\alpha})\,(\varphi_\alpha(p') \leq p)\}$ is the canonical P_α-generic filter over V.

[10] Note that we do not claim that P_ε is preserving.

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From now on, to distinguish between P_β-names and P_α-names for some $\alpha < \beta$, we denote P_β-names (in V as well as P_α-names for such names) with a tilde under the symbol (e. g. $\underset{\sim}{\tau}$) and we denote P_α-names for V_α objects that are not P_β-names (but could be P_β conditions) with a dot under the symbol (e. g. $\underset{\cdot}{\tau}$). In particular we write $(P_\alpha, \underset{\cdot}{Q}_\alpha)_{\alpha<\varepsilon}$.

Definition 4.8 Let $\alpha \le \beta \le \varepsilon$. Work in V_α.

1. $(p^m)_{m\in\omega}$ is an *honest* (α, γ, β)*-sequence* if
   (a) $p^m \in P_\beta/G_\alpha$;
   (b) $p^{m+1} \le p^m$;
   (c) $\alpha \le \gamma \le \beta$;
   (d) for all $\alpha \le \zeta < \gamma$, $p^0{\restriction}\zeta \Vdash_{(\alpha,\zeta)} p^0(\zeta) \in \underset{\cdot}{Q}^\sigma_\zeta$;[11)]
   (e) $p^m{\restriction}\gamma = p^0{\restriction}\gamma$ for all m;
   (f) if $\gamma < \beta$, then $p^0{\restriction}\gamma$ (α,γ)-forces that $p^0(\gamma) \in \underset{\cdot}{Q}^{\neg\sigma}_\gamma$, and that $\{p^m(\gamma) : m \in \omega\} \subseteq \underset{\cdot}{Q}_\gamma$ is inconsistent.

2. Let k be a natural number, $\bar{f}^* = (f^*_i)_{i<k}$ a k-sequence of elements of ω^ω, and $\underset{\sim}{\bar{f}} = (\underset{\sim}{f}_i)_{i<k}$ a k-sequence of P_β-names of elements of $\mathcal{C}$. We say $\bar{f}^*$ *is an honest* (α,γ,β)*-approximation of* $\underset{\sim}{\bar{f}}$ *witnessed by* $(p^m)_{m\in\omega}$ if $(p^m)_{m\in\omega}$ is an honest (α,γ,β)-sequence and $p^m \Vdash_{(\alpha,\beta)} \underset{\sim}{f}_i{\restriction}m = f^*_i{\restriction}m$ for all $m \in \omega$ and $i < k$.

3. $\bar{f}^*$ *is an honest* (α,β)*-approximation of* $\underset{\sim}{\bar{f}}$ *under* p means that there are γ and $(p^m)_{m\in\omega}$ such that $p^0 \le p$ and $\bar{f}^*$ is an honest (α,γ,β)-approximation of $\underset{\sim}{\bar{f}}$ witnessed by $(p^m)_{m\in\omega}$.

Lemma 4.9 *Let $\alpha \le \zeta \le \beta \le \varepsilon$. P_α forces:*

1. *If $(p^m)_{m\in\omega}$ is an honest (α,γ,β)-sequence, then $(p^m{\restriction}\zeta)_{m\in\omega}$ is an honest $(\alpha,\min(\zeta,\gamma),\zeta)$-sequence.*

2. *Assume that p is an element of P_β/G_α, k a natural number, $(\underset{\sim}{f}_i)_{i<k}$ a k-sequence of P_β-names for elements of $\mathcal{C}$, and D a dense subset of P_β/G_α. Then there are $p' \le p$ in D and $(f^*_i)_{i<k}$ such that $(f^*_i)_{i<k}$ is an honest (α,β)-approximation of $(\underset{\sim}{f}_i)_{i<k}$ under p'.*

P r o o f. We just show 2. Work in V_α.

Let $\alpha \le \gamma < \beta$ be minimal such that $p{\restriction}\gamma \nVdash_{(\alpha,\gamma)} p(\gamma) \in \underset{\cdot}{Q}^\sigma_\gamma$. If there is no such γ, set $\gamma = \beta$ and $p_2 = p$. Otherwise pick an $r \le p{\restriction}\gamma$ in P_γ/G_α such that $r \Vdash_{(\alpha,\gamma)} p(\gamma) \in \underset{\cdot}{Q}^{\neg\sigma}_\gamma$, and set $p_2 = p \wedge r$.

Pick $p' \le p_2$ in D.

Let $\bar{f}^*$ approximate $\underset{\sim}{\bar{f}}$ witnessed by $p' = q^0 \ge q^1 \ge \cdots$ (in P_β/G_α). According to Lemma 4.7, 2., there is $q^\omega \in P_\gamma/G_\alpha$ such that $q^\omega \le p'{\restriction}\gamma$ and $q^\omega \Vdash_{(\alpha,\gamma)} q^m{\restriction}\gamma \in G_\gamma$ for all m. If $\gamma < \beta$, we can assume that q^ω decides whether $\{q^m(\gamma) : m \in \omega\}$ is consistent. Set $r^m = q^\omega \wedge (q^m{\restriction}\beta \setminus \gamma)$, cf. Lemma 4.7, 1. Assume $\gamma < \beta$ and q^ω forces consistency, i. e. $q^\omega \Vdash_{(\alpha,\gamma)} \underset{\cdot}{s} \le r^m(\gamma)$ for all m. Then q^ω forces that there is an inconsistent sequence $s = s^0 \ge s^1 \ge \cdots$ (since $s \in \underset{\cdot}{Q}^{\neg\sigma}_\gamma$). Modify r^m such that $r^m{\restriction}\gamma = q^\omega \Vdash r^m(\gamma) = s^m$. □

Induction Lemma 4.10 *Assume that $q \in P_\alpha$ and that the following are in N: $\alpha \le \beta \le \varepsilon$, the P_α-names $\underset{\cdot}{p}, \underset{\cdot}{k}$, $\underset{\cdot}{\bar{f}}^* = (\underset{\cdot}{f}^*_i)_{i\in\underset{\cdot}{k}}$ and the P_β-name $\underset{\sim}{\bar{f}} = (\underset{\sim}{f}_i)_{i\in\underset{\cdot}{k}}$ for elements of $\mathcal{C}$. Assume that q forces*

1. *$\underset{\cdot}{\bar{f}}^*$ is an honest (α,β)-approximation of $\underset{\sim}{\bar{f}}$ under $\underset{\cdot}{p}$ (in particular $\underset{\cdot}{p} \in P_\beta/G_\alpha$);*

2. *G_α is N-generic and η covers $N[G_\alpha]$.*

Then there is $q^+ \in P_\beta$ such that $q^+{\restriction}\alpha = q$ and q^+ forces

1. *$\underset{\cdot}{p} \in G_\beta$;*

2. *G_β is N-generic and η covers $N[G_\beta]$;*

3. *$\underset{\cdot}{f}^*_i R_j \eta$ implies $\underset{\sim}{f}_i R_j \eta$ for all $i \in k$, $j \in \omega$.*

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11) If $\zeta \notin \mathrm{dom}(p)$, then $p(\zeta)$ is defined to be 1_{Q_ζ}. In this case $p(\zeta) \in Q^\sigma_\zeta$ means that Q_ζ is σ-complete. Therefore it is possible that $\gamma \ge \alpha + \omega_1$, this is no contradiction to countable support.

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Proof. We prove the lemma by induction on β. For $\alpha = \beta$ there is nothing to do. We split the proof into two cases: β successor and β limit.

Suppose that $\beta = \zeta + 1$ is a successor. Let p^m be P_α-names for witnesses of the approximation.

First assume that $q \in G_\zeta$ (i. e. $q \in G_\zeta \cap P_\alpha = G_\alpha$) and work in V_ζ. Set $p^{-1} = 1_{P_\beta}$. Let $-1 \le m^* \le \omega$ be the supremum of $\{m : p^m \restriction \zeta \in G_\zeta\}$.

Case 1: $m^* = \omega$. In this case set $\bar f^{**} := \bar f^*$ and $r := p^0(\zeta) \in Q_\zeta$. Note that $p^m(\zeta) \Vdash_{Q_\zeta} \underset{\sim}{f}_i \restriction m = f_i^{**} \restriction m$, i. e. $\bar f^{**}$ is an interpretation of $\underset{\sim}{\bar f}$ (with respect to Q_ζ) under $r = p^0(\zeta)$.

Case 2: $m^* < \omega$. Find a Q_ζ-interpretation $\bar f^{**}$ of $\underset{\sim}{\bar f}$ under $r = p^{m^*}(\zeta) \in Q_\zeta$ (use the fact that Q_ζ is preserving). Note that $f_i^{**} \restriction m^* = f_i^* \restriction m^*$.

Now fix (in V) P_ζ-names $\bar f^{**}$ and r for this $\bar f^{**}$ and r (we do not care how these names behave if $q \notin G_\alpha$). Then we get $q \Vdash_\alpha p^m \restriction \zeta \Vdash_{(\alpha,\zeta)} f_i^{**} \restriction m = f_i^* \restriction m$ for all $i < k$. So by Lemma 4.9, 1., q forces that $\bar f^*$ is an honest (α, ζ)-approximation of $\bar f^{**}$ under $p \restriction \zeta$.

By the induction hypothesis there is an N-generic $q^+ \in P_\zeta$ which forces that $p^0 \restriction \zeta \in G_\zeta$, η covers $N[G_\zeta]$ and of course that Q_ζ is proper and preserving. Assume $q^+ \in G_\zeta$ and work in V_ζ. Since Q_ζ is preserving and $\bar f^{**}$ is an approximation of $\underset{\sim}{\bar f}$ under r, there is an $N[G_\zeta]$-generic $q' \le r$ which forces that η covers $N[G_\zeta][G(\zeta)]$. Let (in V) q' be a name for this q, and set $q^{++} := q^+ \wedge q'$. This q^{++} is as required. (To see that $q^{++} \Vdash p \in G_\beta$, note that $q^+ \Vdash (p \restriction \zeta \in G_\zeta \;\&\; q' \le p(\zeta))$.)

Suppose now that β is limit. Choose a cofinal, increasing sequence $(\alpha_n)_{n\in\omega}$ in $\beta \cap N$ such that $\alpha = \alpha_0$.

Let $(D_n)_{n\in\omega}$ enumerate a basis of the open dense subsets of P_β that are in N, and $(\underset{\sim}{g}_n)_{n\in\omega}$ all P_β-names in N for elements of $\mathcal{C}$. We may assume that $D_0 = P_\beta$, $D_{n+1} \subseteq D_n$ and that every $p \in D_{n+1}$ decides $\underset{\sim}{g}_m \restriction n$ for $0 \le m \le n$ as well as k and $\underset{\sim}{f}_i \restriction n$ for $0 \le i \le k$.

Let γ_0 and $(p_0^m)_{m\in\omega}$ be P_{α_0}-names for witnesses of the approximation in the assumption. Set $q_{-1} := q$, $k_0 := k$, and $\bar f^{*0} := \bar f^*$. Given k_n, we set $\underset{\sim}{\bar f}^n = (\underset{\sim}{f}_i^n)_{i<k_n} := (\underset{\sim}{f}_0, \dots, \underset{\sim}{f}_{k-1}, \underset{\sim}{g}_0, \dots, \underset{\sim}{g}_{k_n - k})$.

By induction on $n \ge 1$ we can construct the following P_{α_n}-names in N:

1. $(p_n^m)_{m\in\omega}$, a sequence of conditions in P_β / G_{α_n},
2. γ_n, an ordinal,
3. k_n, a natural number $\ge k_{n-1}$,
4. $\bar f^{*n} = (f_i^{*n})_{i<k_n}$, a k_n-sequence of functions from ω to ω,

such that (for $n \ge 1$) P_{α_n} forces that $p_{n-1}^0 \restriction \alpha_n \in G_{\alpha_n}$ implies:[12)]

1. $\bar f^{*n}$ is an honest $(\alpha_n, \gamma_n, \beta)$-approximation of $\underset{\sim}{\bar f}^n$ witnessed by $(p_n^m)_{m\in\omega}$.
2. One of the following cases holds:

   A_n: $\gamma_{n-1} < \alpha_n$. Then there is a maximal $m^* \ge 0$ such that $p_{n-1}^{m^*} \restriction \alpha_n$ is in G_{α_n}. Then we set $k_n := n + k$ and choose $p_n^0 \le p_{n-1}^{m^*} \le p_{n-1}^0$, $p_n^0 \in D_n$.

   B_n: $\gamma_{n-1} = \beta$. (In this case the rest of the iteration is σ-complete and all p_{n-1}^m are identical.) Set $k_n := n + k$ and choose $p_n^0 \le p_{n-1}^0$ in D_n.

   C_n: $\alpha_n \le \gamma_{n-1} < \beta$. (Then all $p_{n-1}^m \restriction \alpha_n$ are identical and therefore in $P_{\alpha_n}/G_{\alpha_n}$.) In this case we "do nothing", i. e. we set $p_n^m := p_{n-1}^m$, $k_n := k_{n-1}$ and $\bar f^{*n} := \bar f^{*n-1}$.

All we need for this construction is Lemma 4.9, 2. Note that in all three cases $p_n^0 \le p_{n-1}^0$; in case A_n or B_n, $p_n^0 \in D_n$ and therefore $p_n^0 \Vdash_{(\alpha_n,\beta)} \underset{\sim}{f}_i^n \restriction n = f^{*n} \restriction n$ for $i < n$. In case B_n, γ_n is again β, in case C_n, $\gamma_n = \gamma_{n-1}$. In all three cases, f^{*n} is an honest $(\alpha_n, \gamma_n, \alpha_{n+1})$-approximation witnessed by $(p_n^m \restriction \alpha_{n+1})_{m\in\omega}$.

To see this, we just have to show that $p_n^m \restriction \alpha_{n+1} \Vdash_{(\alpha_n, \alpha_{n+1})} f_i^{*n+1} \restriction m = f_i^{*n} \restriction m$. Assume $G_{\alpha_{n+1}}$ contains $p_n^m \restriction \alpha_{n+1}$. Then in $V_{\alpha+2}$, case A_{n+1}, B_{n+1} or C_{n+1} holds. In each case we can extend $G_{\alpha_{n+1}}$ to a P_β-generic filter G_β containing p_{n+1}^m. Then (by case distinction) G_β contains p_n^m as well, i. e.

$$f_i^{*n} \restriction m = \underset{\sim}{f}_i \restriction m = f_i^{*n+1} \restriction m.$$

12) Or: $\Vdash_{\alpha_{n-1}} p_{n-1}^0 \restriction \alpha_n \Vdash_{(\alpha_{n-1}, \alpha_n)}$.

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Next we construct (by induction on $n \geq 0$) $q_n \in P_{\alpha_{n+1}}$ such that $q_n \restriction \alpha_n = q_{n-1}$ and q_n forces:

1. $G_{\alpha_{n+1}}$ is N-generic and η covers $N[G_{\alpha_{n+1}}]$;
2. $p_n^0 \restriction \alpha_{n+1} \in G_{\alpha_{n+1}}$;
3. $f_i^{*n} R_j \eta$ implies $f_i^{*n+1} R_j \eta$ for $i \in k_n$, $j \in \omega$;
4. $(f_i^{*n+1})_{i<k_{n+1}}$ approximates $(f_i^{*n+2})_{i<k_{n+1}}$ witnessed by $(p_{n+1}^m \restriction \alpha_{n+2})_{m\in\omega}$.

We can do this simply by applying the induction lemma iteratively: Given q_{n-1}, we choose q_n using the Induction Lemma 4.10, setting $\alpha := \alpha_n$, $\beta := \alpha_{n+1}$, $q := q_{n-1}$, $q^+ := q_n$, $p := p_n^0$, $k := k_n$, $\bar{f}^* := \bar{f}^{*n}$, $\bar{f} := \bar{f}^{*n+1}$.

Now $q_\beta := \bigcup q_{\alpha_n}$ is as required: Assume G_β is a P_β-generic filter over V containing q_β. We write p_n^m for $p_n^m[G_\beta] = p_n^m[G_{\alpha_n}]$ etc. Then we have:

1. $p_n^0 \in G_\beta$ for all n: $q_m \Vdash p_{m-1}^0 \restriction \alpha_m \in G_{\alpha_m}$ for all m. Therefore $p_m^0 \leq p_{m-1}^0$ for all m. So for $m > n$, $q_m \Vdash p_n^0 \restriction \alpha_m \in G_{\alpha_m}$. Therefore $p_n^0 \restriction \alpha_m \in G_{\alpha_m}$ for all m, i. e. $p_n^0 \in G_\beta$.

2. $\gamma_n = \gamma_{n-1}$ unless $\gamma_{n-1} < \alpha_n$ (i. e. case A_n holds).

3. $\bigcup_{n\in\omega} k_n = \omega$, and infinitely often case A_n or case B_n holds: If $\gamma_m = \beta$ for some m, then case B_n holds (and $k_n = n$) for all $n > m$. Whenever $\alpha_{m+1} \leq \gamma_m < \beta$ (i. e. case C_{m+1} holds), then for some $n > m$ (the smallest n such that $\alpha_n > \gamma_m$) case A_n holds and therefore $k_n = n$.

4. G_β is N-generic: Let $D \in N$ be dense. Then $D \supseteq D_m \in N$, and for some $n \geq m$, case A_n or case B_n holds. Therefore $p_n^0 \in N \cap D_n \cap G_\beta$, and $D_n \subseteq D_m$.

5. We set $f_i^\infty := f_i^l[G_\beta]$ for some l sufficiently large (i. e. l such that $k_l > i$). So

$$(f_0^\infty, f_1^\infty, \ldots) = (f_0, \ldots, f_{k-1}, g_0, g_1, \ldots).$$

6. If $k_n > i$ and $l > n$, then $f_i^{*n} R_j \eta$ implies $f_i^{*l} R_j \eta$.

7. If $k_n > i$, then $f^{*n} R_j \eta$ implies $f_i^\infty R_j \eta$: Recall that $\{f : f R_j \eta\}$ is closed. For every m there is $l > m$ such that case A_l or B_l holds, i. e. $f_i^{*l} \restriction l = f_i^\infty \restriction l$, and by the last item $f_i^{*l} R_j \eta$.

8. η covers $N[G_\beta]$: Let $g \in N[G_\beta] \cap \mathcal{C}$. Then for some i, $g = f_i^\infty$. Pick an n such that $k_n > i$. Since η covers $N[G_{\alpha_n}]$ and $f_i^{*n} \in N[G_{\alpha_n}]$, $f_i^{*n} R_j \eta$ for some $j \in \omega$.

This ends the proof of the limit case. □

Applying the induction lemma to $\alpha = k = 0$, we get that the limit P_ε is weakly preserving. However, the lemma applied to $k > 0$ does not immediately give the preservation theorem (Theorem 2.4), since we only get preservation for honest approximations. This turns out to be no problem, however: We can find a dense $P' \subseteq P_\varepsilon$ consisting only of "honest" conditions. Then any P'-approximation is an honest P-approximation, so we can apply the induction lemma, which shows that P' is preserving, i.e. P_ε (and therefore P_ε^0) is densely preserving. In more detail: Set

$$\begin{aligned} P' := \{1_{P_\varepsilon}\} \cup \{p \in P_\varepsilon \,:\, & (\exists \gamma \leq \varepsilon)\,(\gamma = \varepsilon \vee p \restriction \gamma \Vdash_\gamma p(\gamma) \in Q^{\neg\sigma}) \\ & \& \,(\forall \alpha < \gamma)\,(p \restriction \alpha \Vdash_\alpha p(\alpha) \in Q^\sigma)\}. \end{aligned}$$

P' is a dense subforcing of P_ε (and therefore a dense subforcing of the original P_ε^0 of Theorem 2.4). We assign to every $p \in P' \setminus \{1_{P'}\}$ the (unique) corresponding $\gamma(p)$. If $q \leq p$, then $\gamma(q) = \gamma(p)$.

We claim that P' is preserving (this finishes the proof of the iteration theorem). Assume that (in P') $\bar{f}^*$ interprets $\bar{f}$ witnessed by $(p^m)_{m\in\omega}$. We have to show that there is an honest witness $(p_1^m)_{m\in\omega}$ such that $p_1^0 \leq p^0$.

(a) If all p^m are 1_P, then $\bar{f}$ is the standard name for $\bar{f}^*$ and there is nothing to do. So let m^* be the smallest m such that $p^{m^*} \neq 1_P$. Set $\gamma = \gamma(p^{m^*})$.

(b) There exists p^ω in P_γ such that $p^\omega \leq p^m \restriction \gamma$ for all m. Set $p_1^m := p^\omega \wedge p^m$. (So if $\gamma = \varepsilon$, then $p_1^m = p^\omega$ for all m.)

(c) If $\gamma < \varepsilon$, we can assume that p^ω decides whether the set $\{p^m(\gamma) \,:\, m \in \omega\}$ is consistent. If it decides positively, then we redefine $p_1^m(\gamma)$ to be any inconsistent sequence in Q_γ stronger than all $p^m(\gamma)$.

(d) The resulting sequence $(p_1^m)_{m\in\omega}$ witnesses that $\bar{f}^*$ is an honest approximation of $\bar{f}$.

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