SEMINARARBEIT

Complexity Bounds for Cut Elimination

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1 Introduction

Since Gentzen showed that cuts can be eliminated in proofs in the sequent calculus, it has been of interest, how the size of these new proofs can be bound. Various authors published papers regarding this topic and gave different algorithms to eliminate cuts more efficiently. In particular, in [2] Buss gave an efficient algorithm to reduce the alternating quantifier depth of the cut formulas of a given proof π. In [8] Weller focused on eliminating cuts with quantifier-free cut-formulas. In Chapter 3, we will take Weller's approach and adapt it to the proof system used by Buss. In Chapter 4, we will combine the results to obtain a new upper bound for the size of cut-free proofs and the size of expansion-tree-proofs (cf. Chapter 3).

(Non-trivial) lower bounds are in general more difficult to handle than upper bounds since we have to exclude the possibility of certain better algorithms. However, there is an important example first found by Statman in [7] and Orekov in [5], which was later presented in more detail by Pudlák in [6] and by Gerhardy in [3]. In Chapter 5, we will analyse the proof as presented in [3] and see that we can use this example to derive lower bounds for the cut elimination. However, they do not quite match our upper bounds and present a task that can be tackled in the future.

2 Preliminaries

The aim of this chapter is to establish the necessary definitions and a few first results that will later be important for the rest of this work.

One definition, we will need for our complexity estimates is the so-called *tetration*:

Definition 2.1. *Let a, n, d be in $\mathbb{N}$ with $a > 0$. We define:*

$$a_n^d = \begin{cases} d & : n = 0 \\ a^{a_{n-1}^d} & : n \geq 1 \end{cases}$$

We write 2_m for 2_m^0.

In this whole paper, we will only consider formulas in *negation normal form*:

Definition 2.2. *A formula ϕ is in negation normal form if the negation ($\neg$) is only applied to atomic formulas and the only other connectives used are the conjunction ($\wedge$), the disjunction ($\vee$) and the quantifiers ($\forall, \exists$).*

In practice, we will not write $\neg A$, but $\overline{A}$ and mean the formula that we obtain by pushing the negation to the atoms. Therefore, we only need the logical connectives $\wedge, \vee, \exists$ and $\forall$:

Definition 2.3. *For the rest of this paper, we consider an arbitrary but fixed language of constants, function and predicate symbols. Atoms are defined as usual. The set of well-formed formulas is denoted with $\mathcal{F}$. It consists of all the literals (atoms or negated atoms) and is closed under the lawful use of $\wedge, \vee, \exists$ and $\forall$. As a shorthand, we write $\overline{A}$ - this is not a new formula, but rather points to the negation of A in NNF. Therefore, it is inductively defined by the following set of rules:*

- *For atoms A: $\overline{A} = \neg A$ and $\overline{\neg A} = A$.*
- *For propositional connectives: $\overline{A \vee B} = \overline{A} \wedge \overline{B}$ and $\overline{A \wedge B} = \overline{A} \vee \overline{B}$*
- *For quantifiers: $\overline{\exists x A} = \forall x \overline{A}$ and $\overline{\forall x A} = \exists x \overline{A}$.*

We observe that every well formed formula according to our definition is in negation normal form.

A convention that we adapt for this paper: $a, b, c, \dots$ usually denote free variables, $x, y, \dots$ bound variables and $s, t, \dots$ terms.

In general, we will not indicate the free variables in a formula, but work with substitution:

Definition 2.4. *The substitution of a term t in a formula A is defined as usual and denoted by $A[x/t]$. The substitution of several terms $A[x_1/t_1, \dots, x_n/t_n]$ is defined as the simultaneous substitution of the t_i for the x_i. That means that if x_i appears in x_j, then these instances are not replaced by t_i.*

Example 2.5. *Let $A = P(x, y)$. Then $A[x/(x + y), y/(x + y)] = P(x + y, x + y)$, which is not the same as $P[x/(x + y)][y/(x + y)] = P(x + (x + y), x + y)$.*

Another normal form we need later is the following:

Definition 2.6. *A formula φ is in prefix-normal-form (PNF) if there is a quantifier-free formula ψ such that $\varphi = Q_1 \dots Q_n \psi$ and all the Q_i are in $\{\forall, \exists\}$.*

We will later need this fact:

Lemma 2.7. *Let φ be an arbitrary formula in $\mathcal{F}$. Then φ is equivalent to some formula φ' in $\mathcal{F}$ that is in PNF and uses the same number of quantifiers.*

Proof. For atoms this is trivial. For compound formulas, it follows by induction: If $\varphi = Q\psi$ with $Q \in \{\forall, \exists\}$, then consider $Q\psi'$ with ψ being the[1] PNF of ψ. If $\varphi = \psi_1 \circ \psi_2$ with $\circ \in \{\vee, \wedge\}$, then rename the bound variables of ψ_1 and ψ_2 such that they are disjoint and consider $Q_1 \dots Q_n Q'_1 \dots Q'_m(\psi'_1 \circ \psi'_2)$, where $Q_1 \dots Q_n \psi'_1$ and $Q'_1 \dots Q'_m \psi'_2$ are the PNFs of ψ_1 and ψ_2 respectively. ■

In the whole paper, we will use some derivation systems. The most important one will be this Tait-style variant of the sequent calculus, which we take from [2]:

Definition 2.8. *A line in our proof system is called a cedent and consists of a set of formulas; we interpret it as the disjunction of its members. For the sake of readability, A, Γ and Γ_1, Γ_2 are written instead of $\{A\} \cup \Gamma$ and $\Gamma_1 \cup \Gamma_2$. For every atomic formula A, we have the axiom:*

$$A, \overline{A}$$

In the following, let A, B be formulas and $\Gamma, \Gamma_1, \Gamma_2$ sets of formulas. We have the following inference rules:

$$\wedge\text{:}\ \frac{A, \Gamma_1 \qquad B, \Gamma_2}{A \wedge B, \Gamma_1, \Gamma_2} \qquad \vee\text{:}\ \frac{A, B, \Gamma}{A \vee B, \Gamma}$$

$$\exists\text{:}\ \frac{A[x/s], \Gamma}{\exists x : A, \Gamma} \qquad \forall\text{:}\ \frac{A[x/b], \Gamma}{\forall x : A, \Gamma}$$

$$\textit{Weakening:}\ \frac{\Gamma}{\Gamma, \Delta} \qquad \textit{Cut:}\ \frac{A, \Gamma_1 \qquad \overline{A}, \Gamma_2}{\Gamma_1, \Gamma_2}$$

If we introduce a new formula with a rule, it is called the primary formula of that inference. The formulas we eliminate are called the auxiliary formulas. The auxiliary formula of a cut inference is called the cut-formula. The formulas of $\Gamma, \Gamma_1, \Gamma_2$ are called the side-formulas. If an inference has two upper cedents as hypotheses, these are called the left and right upper cedent accordingly. We require the usual eigenvariable-condition for the $\forall$-rule: The free variable b must not appear in the lower cedent. Additionally, the outermost connective of the cut formula in the left upper cedent must not be $\exists$ or $\wedge$. This is without loss of generality since we can set A to $\overline{A}$, for which the connectives are switched. Moreover, we impose the

[1]Note that the PNF of a formula is not unique. However, we will often write *the* PNF of a formula and mean some PNF with the minimum number of quantifiers

*restriction that there cannot be two consecutive weakening inferences. This is again without loss of generality since we allow weakening with sets of formulas. We call this proof system **LK**.*

Later, it will be necessary to associate cedents with formulas. We take the definition from [8] and adapt it to our proof systems:

Definition 2.9. *Let Γ be a cedent. We associate it with the formula $F_\Gamma = \bigvee_{\gamma \in \Gamma} \gamma$, which is the intended meaning of it as mentioned above.*

Similar to normal trees and graphs, we define paths and branches:

Definition 2.10. *Let π be a proof in **LK**. Then a path in π is a sequence of cedents of π such that the (i+1)st cedent is a hypothesis of the ith one for all i. A branch is a path that starts at the conclusion (root) of π and ends at an axiom.*

We also have to define which measures of complexity we use. We do this similarly to [2]:

Definition 2.11. *In general, $|\cdot|$ measures the logical complexity and $\|\cdot\|$ measures the symbol complexity. Given a term t, we define $\|t\|$ as the number of symbols in t. For a formula A, we define $|A|$ as the number of logical symbols (i.e. $\wedge, \vee, \forall, \exists$) and atoms in A and $\|A\|$ as the number of all symbols in A (logical and non-logical). Given a cedent Γ, we define $|\Gamma| = |F_\Gamma|$ and $\|\Gamma\| = \|F_\Gamma\|$. For a proof π in **LK** consisting of the cedents $C_1, \ldots, C_n$, we define $\|\pi\| = \sum_{i=1}^{n} \|C_i\|$ and $|\pi|$ as the number of axioms, $\wedge, \vee, \forall, \exists$ and cut inferences in π[2]. The height of π, $h(\pi)$, is defined as the maximum length of any of the branches of π if is in tree form and as the length of the longest path if it is given as a DAG (again, not counting weakening inferences).*

In our definition, cedents are sets of formulas, which complicates things if we consider cedents like A, Γ, where A is an element of Γ. Therefore, we define the following *auxiliary condition*:

Definition 2.12. *A proof π in **LK** fulfills the auxiliary condition if no auxiliary-formula appears as a side-formula in the same inference. Specifically, this means with regard to Definition 2.8 that for every inference I of π it holds that:*

- *If I is an $\vee$-inference, then neither A nor B occur in Γ.*
- *If I is an $\wedge$-inference, than A nor B occur in Γ_1 or Γ_2.*
- *If I is an $\exists$-inference, then $A[x/s]$ does not occur in Γ.*
- *If I is a cut-inference, then neither A nor $\overline{A}$ occur in Γ_1 or Γ_2*

Since the eigenvariable-condition must be met for the $\forall$-rule, $A[x/b]$ cannot occur in Γ and is therefore no side-formula. By the following lemma, we can assume our proofs to satisfy the auxiliary condition without loss of generality:

[2]Note that we do not count the weakening inferences

Lemma 2.13. *Let π be a proof of Γ in **LK**. Then there is a proof π' of Γ in **LK** with the following properties:*

(i) π' *satisfies the auxiliary condition*

(ii) $|\pi'| \leq |\pi|$ *and* $h(\pi') \leq h(\pi)$

Proof. Let π be a proof of Γ in **LK**. Assume, that I is an inference that violates the auxiliary condition. Case distinction:

I. If I is an $\vee$-inference, then w.l.o.g., A occurs in Γ. We see that we can rewrite this inference by defining $\Gamma^* = \Gamma \setminus \{A\}$:

$$\vee\,\frac{A, B, \Gamma}{A \vee B, \Gamma} \rightsquigarrow \mathrm{w}\,\frac{\vee\,\dfrac{A, B, \Gamma^*}{A \vee B, \Gamma^*}}{A \vee B, \Gamma}$$

Since weakening does not count towards the size or the height of a proof, the transformed proof has the same size and height.

II. If I is an $\wedge$-, $\exists$- or cut-inference, the method from above yields the same result.

With induction, we get that we can transform π into a proof π' of Γ, that satisfies the auxiliary condition and has the same size and height, which proves the claim. ■

We can now find even more names for the structures inside of our proofs:

Definition 2.14. *If the formula A occurs in two cedents C_1 and C_2, then define A_1 and A_2 as the occurrences of A respectively.*

(i) A_1 *is a direct ancestor of* A_2 *if there is a path from* C_1 *to* C_2 *such that* A *occurs in every cedent of that path.*

(ii) A_1 *is a place where* A_2 *is introduced if* A_1 *is a principal formula (in its inference) or an axiom.*

(iii) A_2 *is the place where* A_1 *is eliminated if* A_2 *is an auxiliary formula (in its inference).*

This concept of direct ancestors can be generalised for arbitrary formulas:

Definition 2.15. *Let π be a proof of Γ in **LK** and I be an inference in π. If I is an $\vee-, \wedge-, \exists$ or $\forall-$inference, then the principal formula is called the immediate descendant of each of the auxiliary-formulas of I. The descendant-relation is now the relfexive and transitive closure of the union of the immediate- and direct-descendant-relation. If A' is a descendant of A, then A is an ancestor of A'.*

We can even define a relation on the subformulas:

Definition 2.16. *Let π be a proof in **LK**. Consider the following cases:*

- *If A is a subformula of a side-formula of an upper cedent of an inference, then A corresponds to the same subformula in the lower cedent.*

- *The formula $A[x/s]$ of an $\exists$-inference corresponds to the subformula A in the lower cedent.*
- *The formula $A[x/b]$ of an $\forall$-inference corresponds to the subformula A in the lower cedent.*
- *In an $\vee$- or $\wedge$-inference, the formulas A and B correspond to the subformulas A and B in the lower cedent.*
- *If A corresponds to B, then the ith subformula of A corresponds to the ith subformula of B*[3].

The corresponds-relation is now the reflexive and transitive closure of the relation indicated above.

We now define the *free variable normal form* for proofs:

Definition 2.17. *Let π be a proof in **LK**. Then π is in free variable normal form if the following criteria are met:*

- *Each free variable b is used at most once as an eigenvariable.*
- *If b is used as an eigenvariable for an inference I, then b occurs only above I in π (i.e. Every cedent, where b occurs in π can be reached from the upper cedent of I by a path in π).*

If a free variable c occurs in π, but is not used as an eigenvariable, it is called a parameter variable.

Similarly to above, we can also assume without loss of generality that our proofs are in free variable normal form:

Lemma 2.18. *Let π be a proof of Γ in **LK**. Then there is a proof π' of Γ with the following properties:*

(i) π' satisfies the auxiliary condition and is in free variable normal form.

(ii) $|\pi'| \leq |\pi|$ and $h(\pi') \leq h(\pi)$.

Proof. By Lemma 2.13, we get a proof π'' of Γ that fulfills the auxiliary condition and has the same length and height as π. by renaming the free variables, we get π' in free variable normal form. ■

Lastly, we will also need the classification of formulas regarding their quantifier-complexity:

Definition 2.19. *We define Σ_n and Π_n inductively:*

(i) $\Sigma_0 = \Pi_0 = \{\psi \mid \psi \in \mathcal{F} \wedge \psi$ is equivalent to a quantifer-free formula$\}$

[3]This is well defined since the corresponding subformulas have the same number of subformulas by the previous definitions

(ii) $\psi \in \Sigma_{n+1}$ *if* ψ *is equivalent to* $\exists x_1 \dots \exists x_n \varphi$ *and* $\varphi \in \Pi_n$

(iii) $\psi \in \Pi_{n+1}$ *is* ψ *is equivalent to* $\forall x_1 \dots \forall x_n \varphi$ *and* $\varphi \in \Sigma_n$.

This leads to the following definition:

Definition 2.20. *Let* π *be proof in* ***LK*** *and* δ *a cut inference in* π*:*

$$\delta,\ \mathit{cut}\colon\ \frac{A, \Gamma_1 \qquad \overline{A}, \Gamma_2}{\Gamma_1, \Gamma_2}$$

We define the alternating quantifier depth of δ *as*

$$\mathrm{aqd}(\delta) = \min\{n \in \mathbb{N} \mid A \in \Pi_n\}$$

and

$$\mathrm{aqd}(\pi) = \max\{\mathrm{aqd}(\delta) \mid \delta \textit{ is a cut-inference in } \pi\}.$$

3 Eliminating Quantifier-Free Cuts

The goal of this chapter is to present an algorithm that eliminates quantifier-free cuts from a proof and prove an upper bound for the length of the resulting transformed proof. We will use central notions from [8], but we have to slightly adapt them to fit the proof-system from [2]. Additionally, we sometimes replace or extend the definitions from [8] with the definitions from [1], which better fit to our proof system and are easier to understand.

However, we need a few definitions first:

Definition 3.1. *$\mathcal{F}_\perp$ denotes the set of Formulas obtained by the Definition 2.3 if we add the condition that $\perp$ is contained in the set as well. An Element of $\mathcal{F}_\perp$ is referred to as a formula$_\perp$.*

Definition 3.2. *Let F be a formula. Then the positive subformulas are defined inductively:*

- *If $F = A$ is atomic, then F is the only positive subformula of itself.*
- *If $F = \neg A$ is a negated atom, then F is the only positive subformula of itself.*
- *If $F = G \wedge H$ or $F = G \vee H$, then the positive subformulas of F are F itself and all the positive subformulas of G and H.*

Definition 3.3. *Let F be a formula$_\perp$ and G a formula. We write $F \sqsubseteq G$ if F is obtained by replacing an arbitrary number of positive subformulas of G with $\perp$.*

3.1 Expansion Trees

First we need to introduce the data structure of expansion trees:

Definition 3.4. *Let $\mathcal{E}$ denote the set of expansion trees. Additionally to $\mathcal{E}$, we also define two functions* Sh *and* Dp *from $\mathcal{E}$ to $\mathcal{F}_\perp$, the size functions $|\cdot|$ and $\|\cdot\|$ and the selected variables and expansion terms of a expansion tree.*

(i) $\perp \in \mathcal{E}$, $\mathrm{Sh}(\perp) = \mathrm{Dp}(\perp) = \perp$ *and* $|\perp| = \|\perp\| = 1$

(ii) Assume A is a literal (i.e. an atom or a negated atom). Then $A \in \mathcal{E}$, $\mathrm{Sh}(A) = \mathrm{Dp}(A) = A$, $|A|$ and $\|A\|$ are defined as above for regular formulas.

(iii) Assume that E_1 and E_2 do not share any selected variables. If E_2, E_2 are in $\mathcal{E}$, then $E_1 \vee E_2$ and $E_1 \wedge E_2$ are in $\mathcal{E}$. In any way,

$$\begin{aligned}
\mathrm{Sh}(E_1 \vee E_2) &= \mathrm{Sh}(E_1) \vee \mathrm{Sh}(E_2), \quad \mathrm{Dp}(E_1 \vee E_2) = \mathrm{Dp}(\mathcal{E}_1) \vee \mathrm{Dp}(E_2) \\
|E_1 \vee E_2| &= |E_1| + |E_2| + 1, \quad \|E_1 \vee E_2\| = \|E_1\| + \|E_2\| + 1 \\
\mathrm{Sh}(E_1 \wedge E_2) &= \mathrm{Sh}(E_1) \wedge \mathrm{Sh}(E_2), \quad \mathrm{Dp}(E_1 \wedge E_2) = \mathrm{Dp}(\mathcal{E}_1) \wedge \mathrm{Dp}(E_2) \\
|E_1 \wedge E_2| &= |E_1| + |E_2| + 1, \quad \|E_1 \wedge E_2\| = \|E_1\| + \|E_2\| + 1
\end{aligned}$$

(iv) *Assume $E' \in \mathcal{E}$, $F \in \mathcal{F}$ and there is a variable α not selected in E' such that* $\mathrm{Sh}(E') \sqsubseteq F[x/\alpha]$. *Then $E = \forall x F +^{\alpha} E'$ is in $\mathcal{E}$, α is a selected variable of E and*

$$\mathrm{Sh}(E) = \forall x F, \;\; \mathrm{Dp}(E) = \mathrm{Dp}(E')$$
$$|E| = |E'| + 1, \;\; \|E\| = \left\|E'\right\| + 1$$

(v) *Let $E_1, \ldots E_n$ be in $\mathcal{E}$, $t_1, \ldots, t_n$ be terms and F a formula such that no E_i and E_j share a selected variable if $i \neq j$ and* $\mathrm{Sh}(E_i) \sqsubseteq F[x/t_i]$ *for all i. Then $E = \exists x F +^{t_1} E_1 \cdots +^{t_n} E_n$ is in $\mathcal{E}$ and*

$$\mathrm{Sh}(E) = \exists x F, \;\; \mathrm{Dp}(E) = \bigvee_{i=1}^{n} \mathrm{Dp}(E_i)$$
$$|E| = \sum_{i=1}^{n} (|E_i| + 1), \;\; \|E\| = \sum_{i=1}^{n} (\|E_i\| + 1)$$

We can identify an expansion tree intuitively with an actual tree.

Definition 3.5. *Let E be an expansion tree. We define the associated tree* $\mathrm{tr}(E)$ *inductively:*

- *If $E = \bot$ or $E = A$ with A being a literal,* $\mathrm{tr}(E)$ *has exactly one node which is labeled with $\bot$ or A respectively.*
- *If $E = E_1 \circ E_2$ with $\circ \in \{\wedge, \vee\}$, we take the trees* $\mathrm{tr}(E_1)$ *and* $\mathrm{tr}(E_2)$ *and add a root node v, which is labeled with $\circ$ and has the roots of E_1 and E_2 as children.*
- *If $E = \forall x F +^{\alpha} E'$, then we take* $\mathrm{tr}(E')$ *and add a root node v, which is labeled $\forall x F$ and has the root of* $\mathrm{tr}(E')$ *as its only child. This edge is then labeled with α*
- *If $E = \exists x F +^{t_1} E_1 \cdots +^{t_n} E_n$, we take all the* $\mathrm{tr}(E_i)$ *and add the root node v, which is labeled with $\exists x F$ and has all the root nodes of the E_i as children. These edges are labeled with the t_i respectively.*

An expansion tree defines a relation on its nodes:

Definition 3.6. *We now define the expansions* $\exp(E^*)$ *of an expansion tree E^* inductively:*

- *If E^* is constructed with propositional connectives only, then* $\exp(E^*) = \emptyset$.
- *If $E^* = \exists x F +^{t_1} E_1 +^{t_2} \cdots +^{t_n} E_n$, then* $\exp(E^*) = \{t_1, \ldots, t_n\} \cup \bigcup_{i=1}^{n} \exp(E_i)$. *Moreover, for each i we say that t_i dominates all the expansions of E_i.*
- *If $E^* = \forall x F +^{\alpha} E$, then* $\exp(E^*) = \{\alpha\} \cup \exp(E)$. *Again, we say that α dominates all the expansions of E.*

We could also say an element t_1 of $\exp(E^*)$ *dominates another element t_2 of* $\exp(E^*)$ *if there is path in* $\mathrm{tr}(E^*)$, *in which an edge labelled with t_1 occurs prior to t_2. We now define a relation $<^0_E$ on the expansions of E^* with $t_1 <^0_E t_2$ if one of the following holds:*

(i) t_1 *dominates* t_2

(ii) $t_1 = \alpha$ *is the expansion of a* $\forall$*-expansion while* t_2 *is the term of an* $\exists$*-expansion and contains* α*.*

Now, define $<_E$ *as the transitive closure of* $<_E^0$*.*

Having defined expansion trees, we can now define expansion tree proofs:

Definition 3.7. *Let* E *be an expansion tree. We say* E *is tautologous if* $\mathrm{Dp}(E) \in \mathcal{F}_\perp$ *is a tautology.* E *is an expansion tree for a formula* F *if the following holds:*

(i) $\mathrm{Sh}(E) \sqsubseteq F$

(ii) The free variables of F *are not selected in* E

(iii) $<_E$ *is acyclic.*

We then write $E \succ F$*. If* E *is tautologous and* $E \succ F$*, then* E *is an expansion tree proof of* F *and we write* $\vdash_E F$*.*

$<_E$ being acyclic will in some way be the eigenvariable condition in expansion trees.

Example 3.8. *Consider the expansion tree*

$$E = \exists x \forall y(\overline{P(x)} \lor P(y)) +^{t_1} [\forall y(\overline{P(t_1)} \lor P(y)) +^{\alpha} (\overline{P(t_1)} \lor P(\alpha))]$$

with α *not occurring in* t_1*. We have that* $\mathrm{Sh}(E) = \exists x \forall y(\overline{P(x)} \lor P(y))$ *and* $\mathrm{Dp}(E) = \overline{P(t_1)} \lor P(\alpha)$*. For* $\mathrm{tr}(E)$ *we can draw the following graph:*

$$\begin{tikzcd}
\overline{P(t_1)} \arrow[dr, dash] & & P(\alpha) \arrow[dl, dash] \\
& \lor \arrow[d, dash, "\alpha"'] & \\
& \forall y(\overline{P(t_1)} \lor P(y)) \arrow[d, dash, "t_1"'] & \\
& \exists x \forall y(\overline{P(x)} \lor P(y) &
\end{tikzcd}$$

We see that $<_E$ *is acyclic, that* $\mathrm{Sh}(E)$ *contains no free variables (therefore none of them are selected in E) and* $\mathrm{Sh}(E) \sqsubseteq \mathrm{Sh}(E)$*. Therefore,* $E \succ Sh(E) = \forall y \exists x(\overline{P(x)} \lor P(y))$*. Note that* $Dp(E) = \overline{P(t_1)} \lor P(\alpha)$ *is not tautologous since* α *may not occur in* t_1 *by Definition 3.4 and therefore* E *is not an expansion tree proof of* $\mathrm{Sh}(E)$*.*

Now we can define the first lemma regarding expansion trees:

Lemma 3.9. *Let* $E, E_A, E_B, E_1, \ldots, E_n \in \mathcal{E}$ *and* $A, B, F \in \mathcal{F}$*. Then:*

(i) If $E_A \succ A$ *and* $E_B \succ B$*, then* $E_{A \lor B} = E_A \lor E_B \succ A \lor B$

(ii) *If* $E_A \succ A$ *and* $E_B \succ B$*, then* $E_{A\wedge B} = E_A \wedge E_B \succ A \wedge B$

(iii) *If* $E \succ F[x/\alpha]$*, then* $E' = \forall x F +^{\alpha} E \succ \forall x F$

(iv) *If for all* i*,* $E_i \succ F[x/t_i]$*, then* $E'' = \exists x F +^{t_1} E_1 \cdots +^{t_n} E_n \succ \exists x F$

Proof. For all these cases, we simply have to prove the three conditions:

(i) Clearly, $<_{E_{A\vee B}}$ is acyclic if E_A and E_B are. Moreover, $\mathrm{Sh}(E_{A\vee B}) = \mathrm{Sh}(E_A) \vee \mathrm{Sh}(E_B) \sqsubseteq A \vee B$. By renaming the variables in A, B, E_A and E_B, we also have that the free variables of $A \vee B$ are not selected in $E_{A\vee B}$.

(ii) The case of $E_{A\wedge B}$ is identical to the one of $E_{A\vee B}$

(iii) By assumption we have that $\mathrm{Sh}(E) \sqsubseteq F[x/\alpha]$. Also, note that selected variables do not occur in shallow formulas. Therefore, we can assume that α is not selected in E by renaming. Thus, E' is a well formed expansion tree. Moreover, $\mathrm{Sh}(E') = \forall x F \sqsubseteq \forall x F$. Since $E \succ F[x/\alpha]$, no free variable of $F[x/\alpha]$ is selected in E. The free variables of $\forall x F$ are a subset and therefore, no free variable of $\forall x F$ is selected in E', in which all the selected variables of E and α are selected. Again, since the expansion terms do not occur in shallow formulas, we get by renaming that α is not dominated by any term in E. Thus, the relation $<_{E'}$ is acyclic. In summary, this means that $E' \succ \forall x F$.

(iv) By renaming the variables, we can assume that the selected variables in all the E_i are disjoint. By assumption $\mathrm{Sh}(E_i) \sqsubseteq F[x/t_i]$. Therefore, E'' is a well formed expansion tree and clearly $\mathrm{Sh}(E'') = \exists x F \sqsubseteq \exists x F$. Since the selected variables of E'' are the selected variables of all the E_i, no free variable of $F[x/t_i]$ is selected in E_i and x is not free in $\exists x F$, we have that no free variable of $\exists x F$ is selected in E''. Moreover, we obtain that $<_{E''}$ is acyclic by renaming the selected variables such that no selected α occurs in any t_i and that a cycle in E'' would imply a cycle in some E_i. Therefore, $<_{E''}$ is acyclic and $E'' \succ \exists x F$.

■

For the next theorem, we need to be able to merge expansion trees in some way:

Lemma 3.10. *Let* E_1, E_2 *be in* $\mathcal{E}$ *with* $E_i \succ F$*. Then, there exists* E_3 *in* $\mathcal{E}$ *such that* $E_3 \succ F$*,* $|E_3| \leq |E_1| + |E_2|$*,* $\|E_3\| \leq \|E_1\| + \|E_2\|$*. Also* $\mathrm{Dp}(E_1) \vee \mathrm{Dp}(E_2) \to \mathrm{Dp}(E_3)$ *is a tautology.*

Proof. We prove this via induction and case distinction. For the base cases:

(i) $E_1 = \bot$, then set $E_3 = E_2$ (analogously for $E_2 = \bot$). All the desired restrictions hold.

(ii) $E_1 = A$ with A being an atom. Then $F = \mathrm{Sh}(E_1) = A$ and therefore $E_2 = A$. Again, take $E_3 = E_i$ and again, E_3 has the properties, we want.

Now, we can consider the induction step:

(iii) Let $\circ \in \{\wedge, \vee\}$. If $E_1 = E_1' \circ E_1''$, then $\mathrm{Sh}(E_1') \circ \mathrm{Sh}(E_1'') \sqsubseteq F = F' \circ F'' \sqsubseteq \mathrm{Sh}(E_2)$. Thus, $E_2 = E_2' \circ E_2''$ and also $\mathrm{Sh}(E_1') \sqsubseteq \mathrm{Sh}(E_2')$ and $\mathrm{Sh}(E_1'') \sqsubseteq Sh(E_2'')$. Therefore, $E_1', E_2' \succ F'$ and $E_1'', E_2'' \succ F''$. By the induction hypothesis, we get $E_3 \succ F'$ and $E_3'' \succ F''$ with the desired properties. By Lemma 3.9, we have that $E_3 = E_3' \circ E_3'' \succ F$. Furthermore, by the induction hypothesis, we have that

$$\begin{aligned}
\mathrm{Dp}(E_1) \vee \mathrm{Dp}(E_2) &= (\mathrm{Dp}(E_1') \circ \mathrm{Dp}(E_1'')) \vee (\mathrm{Dp}(E_2') \circ \mathrm{Dp}(E_2'')) \\
&= ((\mathrm{Dp}(E_1') \circ \mathrm{Dp}(E_1'')) \vee \mathrm{Dp}(E_2')) \circ ((\mathrm{Dp}(E_1') \circ \mathrm{Dp}(E_1'')) \vee \mathrm{Dp}(E_2'')) \\
&= ((\mathrm{Dp}(E_1') \vee \mathrm{Dp}(E_2')) \circ \dots) \circ (\dots \circ (\mathrm{Dp}(E_1'') \vee \mathrm{Dp}(E_2''))) \\
&\to \mathrm{Dp}(E_3') \circ \mathrm{Dp}(E_3'') \\
&= \mathrm{Dp}(E_3)
\end{aligned}$$

is a tautology. Moreover, $|E_3| = |E_3'| + |E_3''| \leq |E_1'| + |E_2'| + |E_1''| + |E_2''| = |E_1| + |E_2|$. The same holds for $\|\cdot\|$.

(iv) If $E_1 = \forall x G +^{\alpha} E$, then $\mathrm{Sh}(E_1) = \forall x G$. Therefore, E_2 has to be of the form $\forall x G +^{\beta} E'$. Now, $\mathrm{Sh}(E) \sqsubseteq G[x/\alpha]$ and $\mathrm{Sh}(E') \sqsubseteq G[x/\beta]$. Since the other requirements are inherited from E_1 and E_2, we have that $E, E' \succ G$. By the induction hypothesis, we get $E_3' \succ G$ with the desired properties. Now $E_3 = \forall x G +^{\alpha} E_3' \succ \forall x G \sqsubseteq F$ by Lemma 3.9. Furthermore, $|E_3| = 1 + |E_3'| \leq 1 + |E| + |E'| \leq 1 + 1 + |E| + |E'| = |E_1| + |E_2|$. Similarly for $\|\cdot\|$. Lastly, since $\mathrm{Dp}(E) \vee \mathrm{Dp}(E') \to \mathrm{Dp}(E_3')$ is a tautology, so is $\mathrm{Dp}(E_1) \vee \mathrm{Dp}(E_2) \to \mathrm{Dp}(E_3)$.

(v) If $E_1 = \exists x G +^{t_1} E_1' \cdots +^{t_n} E_n'$, then $\mathrm{Sh}(E_1) = \exists x G \sqsubseteq F$. Therefore, E_2 must have the form $\exists x G +^{s_1} E_1'' \cdots +^{s_m} E_m''$. By renaming the variables, we can assume that E_1 and E_2 share no selected variables. Define $E_3 = \exists x G +^{t_1} E_1' \cdots +^{t_n} E_n' +^{s_1} E_1'' \cdots +^{s_m} E_m''$. Again, since $<_{E_1}$ and $<_{E_2}$ are acyclic, so is $<_{E_3}$. Clearly, the free variables of F are not selected in E_3 and $\mathrm{Sh}(E_3) \sqsubseteq F$. Thus $E_3 \succ F$. Also, the size estimation of E_3 follows directly. Lastly, by the definition of Dp it also holds that $\mathrm{Dp}(E_1) \vee \mathrm{Dp}(E_2) \to \mathrm{Dp}(E_3)$.

∎

Now, we can formulate the first connection between our proof system **LK** and the expansion trees:

Theorem 3.11. *Let π be a proof of S in **LK** with only quantifier-free cuts. Then there exists an expansion tree E such that $\vdash_E S, |E| \leq \frac{5}{2}|\pi|$ and $\|E\| \leq \frac{5}{2}\|\pi\|$.*

Proof. Let ρ be an inference in π with conclusion Δ_s, Δ_c, where Δ_s are the ancestors of the endcedent S and Δ_c are the ancestors of a cut. Let $h(\rho)$ be the maximal number of inferences between ρ and an axiom of π and π_ρ the proof that ends with ρ as its last inference. We perform induction on $h(\rho)$ and construct an expansion tree E that such that $\mathrm{Dp}(E) \vee \Delta_c$ is tautologous, $E \succ \Delta_s, |E| \leq c|\pi_\rho|$ and $\|E\| \leq d\,\|\pi_\rho\|$ for some $c, d \in \mathbb{N}$. Furthermore, no free variable of Δ_s, Δ_c is selected in E. Then, by setting ρ to the last inference, we obtain the desired result.

For the base case, there is only one option:

(i) ρ is an axiom $A, \overline{A}$. We have three options:

(a) If both A and $\overline{A}$ are cut ancestors, then set $E = \bot$. Obviously, $E \vee (A \vee \overline{A})$ is tautologous, $E \succ \bot = \Delta_s$ and the size- and free-variables-constraints hold.

(b) If both A and $\overline{A}$ are ancestors of the endcedent, then set $E = A \vee \overline{A}$. Again, all the constraints hold.

(c) If w.l.o.g. A is an ancestor of the endcedent and $\overline{A}$ is an ancestor of a cut, then set $E = A$. Thus, $E \vee \overline{A}$ is tautologous and $E \succ A$. Also, the other constraints hold.

From the cases above, we see that $c, d \geq 2$.

Now, for the induction step, we refine the length condition and require that the expansion tree E from the induction hypothesis fulfills that $|E| \leq c_{h(\rho)}|\pi_\rho|$ and $\|E\| \leq c_{h(\rho)} \|\pi_\rho\|$ with $c_1 = 2$ from the base case:

(ii) ρ is a $\forall$-inference.

$$\forall\colon \ \frac{A[x/b], \Delta_s, \Delta_c}{\forall x : A, \Delta_s, \Delta_c}$$

Since all our cuts are quantifier-free, $A[x/b]$ and $\forall x : A$ are ancestors of the endcedent. By the induction hypothesis, we have an expansion tree E such that $E \succ \Delta_s \vee A[x/b]$, $\mathrm{Dp}(E) \vee \Delta_c$ is tautologous and $|E| \leq c_{h(\rho)-1}(|(\pi_\rho| - 1)$ and $\|E\| \leq d_{h(\rho)}(\|\pi_\rho\| - 1)$. Therefore, $E = E' \vee E''$ with $\mathrm{Sh}(E'') \sqsubseteq A[x/b]$. If b is selected in E, then b is not free in $A[x/b]$ by induction hypothesis. Therefore, $\mathrm{Sh}(E'') \sqsubseteq A[x/b] = A[x/c]$ for any c which is not selected in E and not free in $\forall x : A, \Delta_s, \Delta_c$. Thus, we can set $E^* = \forall x A +^c E''$. If on the other hand, b is not selected in E, we can set $E^* = \forall x A +^b E''$. Either way, if we define $E^+ = E' \vee E^*$, we have that $\mathrm{Sh}(E^+) = \mathrm{Sh}(E') \vee \forall x A \sqsubseteq \Delta_s, \forall x A$ and $\mathrm{Dp}(E^+) = \mathrm{Dp}(E') \vee \mathrm{Dp}(E'')$, which means that $\mathrm{Dp}(E^+) \vee \Delta_c$ is tautologous. Moreover, the free-variable-condition holds and we have $|E^+| = |E| + 1 \leq c_{h(\rho)-1}(|\pi_\rho| - 1) + 1 = c_{h(\rho)-1}|\pi_\rho| - c_{h(\rho)-1} + 1 = (c_{h(\rho)-1} - \frac{c_{h(\rho)-1} - 1}{|\pi_\rho|})|\pi_\rho|$ and $\|E^+\| = \|E\| + 1 \leq (c_{h(\rho)-1} - \frac{c_{h(\rho)-1} - 1}{\|\pi_\rho\|}) \|\pi_\rho\|$.

(iii) ρ is an $\exists$-inference

$$\exists\colon \ \frac{A[x/s], \Delta_s, \Delta_c}{\exists x : A, \Delta_s, \Delta_c}$$

Since all our cuts are quantifier-free, we have that $A[x/s]$ and $\exists x A$ have to be ancestors of the endcedent. By the induction hypothesis, there is an expansion tree E such that $E \succ A[x/s] \vee \Delta_s$, $E \vee \Delta_c$ is tautologous and $|E| \leq c_{h\rho)}|\lambda|$ and $\|E\| \leq 2 \|\lambda\|$. Since $\mathrm{Sh}(E) \sqsubseteq \Delta_s \vee A[x/s]$, it holds that $E = E' \vee E''$ with $\mathrm{Sh}(E'') \sqsubseteq A[x/s]$. Now, set $E^* = \exists x A +^s E''$. By Lemma 3.9, we have that $E^* \succ \exists x A$ and thus $E^+ = E' \vee E^* \succ \exists x A, \Delta_s$. Also, $\mathrm{Dp}(E^+) = \mathrm{Dp}(E') \vee \mathrm{Dp}(E'')$ and therefore, $E^+ \vee \Delta_c$ is tautologous. Furthermore, the free-variable-condition still holds and we have that $|E^+| = |E| + 1 \leq c_{h(\rho)-1}(|\pi_\rho| - 1) + 1 = (c_{h(\rho)-1} - \frac{c_{h(\rho)-1} - 1}{|\pi_\rho|})|\pi_\rho|$ and $\|E^+\| = \|E\| + 1 \leq (c_{h(\rho)-1} - \frac{c_{h(\rho)-1} - 1}{\|\pi_\rho\|}) \|\pi_\rho\|$.

(iv) ρ is a cut inference

$$\text{Cut:}\ \frac{\overset{(\lambda_1)}{A, \Delta_s, \Delta_c} \qquad \overset{(\lambda_2)}{\overline{A}, \Delta'_s, \Delta'_c}}{\Delta_s, \Delta'_s, \Delta_c, \Delta'_c}$$

By the induction hypothesis, we have two expansion trees E_1 and E_2 such that $E_1 \succ \Delta_s, E_2 \succ \Delta'_s$, $E_1 \vee A \vee \Delta_c$ and $E_2 \vee \overline{A} \vee \Delta'_c$ are both tautologous and all the other conditions hold. Since the the selected variables of E_1 and E_2 do not occur in $\Delta_s, \Delta'_s, \Delta_c, \Delta'_c$, we can rename them so they are disjoint and construct $E = E_1 \vee E_2$. By Lemma 3.9, we have that $E \succ \Delta_s \vee \Delta'_s$. Also, $\mathrm{Dp}(E \vee \Delta_c \vee \Delta'_c) = \mathrm{Dp}(E_1) \vee \mathrm{Dp}(\Delta_c) \vee \mathrm{Dp}(E_2) \vee \mathrm{Dp}(\Delta'_c)$, which is a tautology by the induction hypothesis. The free-variable-condition holds by construction and we have that $|E| = |E_1| + |E_2| + 1 \leq c_{h(\rho)-1}|\lambda_1| + c_{h(\rho)-1}|\lambda_2| + 1 = c_{h(\rho)-1}(|\lambda_1| + |\lambda_2|) + 1 = c_{h(\rho)-1}(|\pi_\rho| - 1) + 1 = (c_{h(\rho)-1} - \frac{c_{h(\rho)-1}-1}{|\pi_\rho|})|\pi_\rho|$ and analogously for $\|\cdot\|$.

(v) ρ is an $\wedge$-inference

$$\wedge:\ \frac{\overset{(\lambda_1)}{A, \Delta_s, \Delta_c} \qquad \overset{(\lambda_2)}{B, \Delta'_s, \Delta'_c}}{A \wedge B, \Delta_s, \Delta'_s, \Delta_c, \Delta'_c}$$

By the induction hypothesis, we have two expansion trees E_1 and E_2. We distinguish two cases:

(a) $A \wedge B$ is a cut ancestor. Then A and B are cut ancestors and we have that $E_1 \succ \Delta_s, E_2 \succ \Delta'_s$ and $E_1 \vee \Delta_c, A$ and $E_2 \vee \Delta'_c, B$ are both tautologous. Thus, $E = E_1 \vee E_2 \succ \Delta_s \vee \Delta'_s$ by Lemma 3.9. Also $\mathrm{Dp}(E \vee (A \wedge B) \vee \Delta_c \vee \Delta'_c) = \mathrm{Dp}(E_1) \vee \mathrm{Dp}(E_2) \vee (A \wedge B) \vee \Delta_c \vee \Delta'_c$ is tautologous. Again, for the size we have $|E| = |E_1| + |E_2| \leq c_{h(\rho)-1}(|\lambda_1| + |\lambda_2|) + 1 = c_{h(\rho)-1}(|\pi_\rho| - 1) + 1 = (c_{h(\rho)-1} - \frac{c_{h(\rho)-1}-1}{|\pi_\rho|})|\pi_\rho|$ and similarly for $\|\cdot\|$.

(b) $A \wedge B$ is an ancestor of the endcedent. Then, we have that $E_1 = E_A \vee E'_1$ and $E_2 = E_B \vee E'_2$ with $E_A \succ A, E_B \succ B, E'_1 \succ \Delta_s, E'_2 \succ \Delta'_s$. We can now define $E_3 = (E_A \wedge E_B) \vee E'_1 \vee E'_2$. Now by iterating Lemma 3.9, we have that $E_3 \succ A \wedge B \vee \Delta_s \vee \Delta'_s$. Also $E_3 \vee \Delta_c \vee \Delta'_c$ is tautologous by construction. Again, $|E_3| = |E_1| + |E_2| + 1 \leq (c_{h(\rho)-1} - \frac{c_{h(\rho)-1}-1}{|\pi_\rho|})|\pi_\rho|$.

(vi) ρ is an $\vee$-inference

$$\vee:\ \frac{\overset{(\lambda_1)}{A, B, \Delta_s, \Delta_c}}{A \vee B, \Delta_s, \Delta_c}$$

Let E be the extension tree from the induction hypothesis. Since the cedents A, B and $A \vee B$ are associated with the same formula, E is the expansion tree with the desired properties. Also, we have that $|E| \leq 2|\lambda| = c_{h(\rho)-1}(|\pi_\rho| - 1) \leq (c_{h(\rho)-1} - \frac{c_{h(\rho)-1}-1}{|\pi_\rho|})|\pi_\rho|$

(vii) ρ is a weakening-inference

$$\text{w}, \rho\colon \dfrac{\delta\colon \dfrac{\overset{(\lambda)}{\Delta}}{\Delta_s, \Delta_c}}{\Delta_s, \Delta_c, \Gamma_s, \Gamma_c}$$

We separated the new formulas of Γ in two subsets Γ_s and Γ_c with the endcedents ancestors and cut ancestors respectively. By the definition of our proof system, the inference δ cannot be a weakening inference. Let E be the expansion tree from the induction hypothesis. We have to distinguish two cases:

(a) If Γ_s is empty, we set $E^* = E$. All the conditions hold.

(b) If Γ_s is not empty, we set $E^* = E \vee \bot$. We observe that $E^* \vee \Delta_c \vee \Gamma_c$ is still tautologous and that the free variable condition still holds. Moreover, $\mathrm{Sh}(E^*) = \mathrm{Sh}(E) \vee \bot \sqsubseteq \Delta_s \vee \Gamma_s$ also holds since we obtain $\mathrm{Sh}(E^*)$ be replacing $\bigvee_{\gamma \in \Gamma} \gamma$ with $\bot$ and the whatever we need to replace in Δ_s.

Now, since δ is not a weakening inference, we can use, that in all the cases before we obtained the bound $|E| \leq c_{h(\rho)-1}|\pi_\delta| - 1$. Therefore, $|E^*| \leq |E| + 2 \leq c_{h(\rho)-1}|\pi_\delta| + 1 = c_{h(\rho)-1}|\pi_\rho| + 1 = (c_{h(\rho)-1} + \frac{1}{|\pi_\rho|})|\pi_\rho|$

Since only every second inference can be a weakening inference and $h(\rho) \leq |\pi_\rho| \leq \|\pi_\rho\|$, we see that $c_{h(\rho)}$ is dominated by x_n for $n = h(\rho)$ with

$$x_n = \begin{cases} 2 & : n = 1 \\ x_{n-1} - \frac{x_{n-1}-1}{n} & : n > 1 \wedge n \equiv 1 \ (2) \\ x_{n-1} + \frac{1}{n} & : n > 1 \wedge n \equiv 0 \ (2) \end{cases}$$

Now, we see that $x_1 = 2, x_2 = \frac{5}{2}$. Our induction hypothesis is now that for odd n, we have that $x_n = 2$. It follows that

$$\begin{aligned} x_{n+2} &= x_{n+1} - \frac{x_{n+1} - 1}{n+2} \\ &= x_n + \frac{1}{n+1} - \frac{x_n + \frac{1}{n+1} - 1}{n+2} \\ &= 2 + \frac{n + 2 - (n+1)2 - 1 + n + 1}{n+2} \\ &= 2 + \frac{2(n+1) - 2(n+1)}{n+2} \\ &= 2 \end{aligned}$$

It therefore follows that $x_n = 2 + \frac{1}{n}$ if n is even and thus that $c = d = \max_{n \in \mathbb{N}} x_n = \frac{5}{2}$. ∎

Note that these constants are optimal. For this consider the proof π

$$\dfrac{A, \overline{A}}{A, \overline{A}, B}$$

Since, we count axioms towards the proof-length, it follows that $|\pi| = 2$. Moreover, for any expansion tree E with a tautologous deep formula and $\mathrm{Sh}(E) \sqsubseteq F_C = A \vee \overline{A} \vee B$ it holds that $E \geq 5 = \frac{5}{2}|\pi|$.

We can now define a sequent calculus for sets of expansion trees:

Definition 3.12. *We now consider sets of expansion trees. For every atom A, we have the axiom*

$$A, \overline{A}$$

Then, we have the rules $\vee, \wedge$ and weakening from ***LK***. *Additionally, we have*

$$\forall\text{:}\ \frac{\Gamma, E}{\Gamma, \forall x F +^{\alpha} E} \qquad \exists'\text{:}\ \frac{\Delta, E_1}{\Delta, \exists x F +^{t_1} E_1}$$

$$\exists\text{:}\ \frac{\Delta, \exists x F +^{t_1} E_1 \cdots +^{t_{i-1}} E_{i-1} +^{t_{i+1}} E_{i+1} \cdots +^{t_n} E_n, E_i}{\Delta, \exists x F +^{t_1} E_1 \cdots +^{t_n} E_n}$$

For the $\exists$ and $\exists'$ rule, the terms t_1 or t_i respectively have to be admissible. A term t is admissible in an expansion tree E if no free variable of t is selected in E. We call the derivation system obtained by this $\boldsymbol{LK}_E$.

Since we use a one-sided sequent calculus, this is a bit different from [8]. We still associate formulas with cedents:

Definition 3.13. *Let Γ be a cedent of expansion trees. We associate Γ with the expansion tree $E_\Gamma = \bigvee_{\gamma \in \Gamma} \gamma$. Now, define, $\mathrm{Sh}(\Gamma) = \mathrm{Sh}(E_\Gamma)$ and $\mathrm{Dp}(\Gamma) = \mathrm{Dp}(E_\Gamma)$. We say C is tautologous if $\mathrm{Dp}(C)$ is a tautology.*

Definition 3.14. *Let C be a cedent of expansion trees and π an $\boldsymbol{LK}_E$-proof. We define $|C| = |\Gamma_C|$, $\|C\| = |\Gamma_C|$, $\|\pi\| = \sum_{S \in \pi} \|S\|$ and $|\pi|$ as the number of inferences except for weakening.*

Definition 3.15. *Let $C = E_1, \ldots, E_n$ be a cedent of expansion trees. We define $<_C = <_{\Gamma_C}$.*

Lemma 3.16. *Let $E = E_1^d, \ldots E_n^d$ be a cedent of expansion trees such that $<_E$ is acyclic and all the E_i are existential expansion trees. Then there exists an E_i such that an expansion term t of E_i is admissible.*

Proof. Since all of the E_i are existential, they have the form $E_i = \exists x F_i +_{i,1}^{t} E_{i,1} \cdots +^{t_{i,n_i}} E_{i,n_i}$. Now assume that none of the expansion terms of E_i are admissible. Then for some $t_{i,j}$ there is a variable α free in $t_{i,j}$ with α being selected in E and therefore in some $E_k = \exists x F_k +^{t_{k,1}} E_{k,1} \cdots +^{t_{k,n_k}} E_{k,n_k}$. Thus, there is a $t_{k,l} <_E t_{i,j}$. Since this holds for all expansion terms by assumption and E is finite, we have that $<_E$ is cyclic. ∎

Lemma 3.17. *Let C be derived from C' by a rule of $\boldsymbol{LK}_E$ except for weakening. Then if C is tautologous so is C' and if $<_C$ is acyclic, then so is $<_{C'}$. Moreover, if C is derived with the (arbitrary) inference ρ, then C is tautologous if all the upper cedents of ρ are.*

Proof. For this proof, let us call the inference in question ρ. We now have to distinguish the different cases:

(i) Let ρ be an $\vee$-inference

$$\vee\colon\ \frac{C' : \Gamma, A, B}{C : \Gamma, A \vee B}$$

Clearly, the associated formulas F_C and $F_{C'}$ are logically equivalent. By definition of $<_{C'}$ and $<_C$, they are the same.

(ii) Let ρ be an $\wedge$-inference

$$\wedge\colon\ \frac{C_1 : \Gamma_1, A \qquad C_2 : \Gamma_2, B}{C : \Gamma_1, \Gamma_2, A \wedge B}$$

Then both C_1 and C_2 are tautologous by definition of the associated formula. Moreover, since $<_{C_1} \subseteq <_C$ and $<_{C_2} \subseteq <_C$, they are both acyclic. On the other hand, if both upper cedents are tautologous, so is C by constructing $\mathrm{Dp}(C)$.

(iii) Let ρ be a weakening-inference:

$$\mathrm{w}\colon\ \frac{\Gamma}{\Gamma, \Delta}$$

Clearly, if the upper cedent is tautologous, so is the lower one.

(iv) Let ρ be an $\exists$-inference:

$$\exists\colon\ \frac{C' : \Delta, \exists x F +^{t_1} E_1 \cdots +^{t_{i-1}} E_{i-1} +^{t_{i+1}} E_{i+1} \cdots +^{t_n} E_n, E_i}{C : \Delta, \exists x F +^{t_1} E_1 \cdots +^{t_n} E_n}$$

We have that $<_{C'} \subseteq <_C$ and therefore, $<_{C'}$ is acyclic. Also, the associated deep formulas are logically equivalent.

(v) Let ρ be a $\forall$- or an $\exists'$-inference:

$$\forall\colon\ \frac{C' : \Gamma, E}{C : \forall x F +^{\alpha} E} \qquad \exists'\colon\ \frac{C' : \Delta, E_1}{C : \Delta, \exists x F +^{t_1} E_1}$$

Then again, the associated deep formulas are logically equivalent. Again, $<_{C'} \subseteq <_C$ and therefore, the acyclicity is preserved.

■

Lemma 3.18. *Let $E = E_1, \ldots, E_n$ be a tautologous expansion cedent such that $<_E$ is acyclic. Then there is a a proof π of E in $\boldsymbol{LK}_E$ such that $|\pi| \leq 2^{|E|}$ and $\|\pi\| \leq 2^{4\|E\|}$.*

Proof. Let $E = E_1, \ldots, E_n$. Let $m(E) = \sum_{i=1}^{n} \mathrm{lc}(E_i)$ with lc being the number of logical connectives in E_i (without the atoms). We proceed via induction on $m(E)$ and construct π such that $\|\pi\| \leq |E| \cdot \|E\| \cdot 2^{2m(E)}$ and $h(\pi) \leq |E| - (n-1)$. The desired result then follows since $m(E) \leq |E| \leq \|E\|$ and thus $|\pi| \leq 2^{h(\pi)} \leq 2^{|E|}$ and $\|\pi\| \leq |E|\, \|E\|\, 2^{2\|E\|} \leq 2^{2\log(\|E\|)+2\|E\|} \leq 2^{4\|E\|}$. If $m(E) = 0$, then all of the E_i are literals. Since, E is tautologous, there are i, j such that $E_i = A$ and $E_j = \overline{A}$. Therefore, we can derive E in $\mathbf{LK}_{\mathrm{E}}$ with weakening only and receive a proof π with $h(\pi) = 1 \leq |E| - (n-1)$ and $\|\pi\| \leq |E|\, \|E\|$. Now let $m(E) > 0$. We can now distinguish two cases:

(i) Let $E = E_1, \dots E_n$ be such that all of the E_i are existential expansion trees. By Lemma 3.16 there is an admissible expansion term t_j of some $E_i = \exists x F +^{t_1} E'_1 \cdots +^{t_m} E'_m$. Now define $\Gamma = E_1 \dots E_{i-1}, E_{i+1}, \dots E_n$ and

$$E' = \exists x F +^{t_1} E'_1 \cdots +^{t_{j-1}} E'_{j-1} +^{t_{j+1}} E'_{j+1} \cdots +^{t_m} E'_m, E'_j, \Gamma.$$

We now have that $|E'| = |E|, \|E'\| = \|E\|$ and $m(E') = m(E)-1$ since the associated formulas are the same, but we reduce $m(E)$ by eliminating the $+^{t_j}$. Moreover, it holds that

$$\exists\text{:}\ \frac{E'}{E}$$

By the induction hypothesis, we now have a proof π' of E' such that $\|\pi'\| \leq |E'|\,\|E'\|\,2^{2m(E')}$ and $h(\pi') \leq |E'|$. This now gives us a proof π of E with

$$\begin{aligned}
\|\pi\| &= \left\|\pi'\right\| + \|E\| \\
&\leq |E'|\,\left\|E'\right\|\,2^{2m(E')} + \|E\| \\
&= |E|\,\|E\|\,2^{2m(E)-2} + \|E\| \\
&= |E|\,\|E\|\,2^{2m(E)}(\frac{1}{4} + \frac{1}{|E|2^{2m(E)}}) \\
&\leq |E|\,\|E\|\,2^{2m(E)}
\end{aligned}$$

since $m(E), E \geq 1$. Moreover, $h(\pi) = h(\pi') + 1 \leq |E'| - (n-1) + 1 = |E| - (n-2)$.

(ii) Let $E = E_1, \dots, E_n$ such that one of the E_i is not an existential expansion tree. Without loss of generality all the E_i are critical for E being tautologous and also without loss of generality none of the E_i are $\bot$, otherwise we could just consider $E' = E \setminus E_i$ and get E by weakening from E'. Therefore, there is an E_i that has a root (outermost connective) different from $\bot$. We fix this E_i and consider different cases:

(a) If $E_i = A$ is atomic. We group all the atomic E_j and consider $E' = E \setminus \{E_j \mid E_j \text{ is atomic}\}$. Now, if all the E_l in E' are existential, there has to be an admissible term t in some E_l. This term is still admissible in E since no variables are selected in the atomic expansion trees. We can now use the strategy from above. If there is an E_l which is not existential, then we can just consider this E_l, which is not atomic and use one of the strategies below.

(b) If $E_i = A \vee B$ and $E = \Gamma, A \vee B$. Then, we define $E' = A, B, \Gamma$ and see that $m(E') = m(E) - 1, |E'| = |E|, \|E'\| = \|E\|$ and

$$\vee\text{:}\ \frac{A, B, \Gamma}{A \vee B, \Gamma}$$

By the induction hypothesis, we get a proof of E' and all the size-estimations hold for the same reasons as above.

(c) Let $E_i = A \wedge B$ and $E = A \wedge B, \Gamma$. Then, define $E_1 = A, \Gamma$ and $E_2 = B, \Gamma$. We see that $m(E_2), m(E_1) \leq m(E) - 1$. By the induction hypothesis, we have two proofs π_1, π_2 such that[1]:

$$\wedge\colon \dfrac{\begin{matrix}(\pi_1) & & (\pi_2)\\ A, \Gamma & & B, \Gamma\end{matrix}}{A \wedge B, \Gamma}$$

This constitutes our proof π with $h(\pi) = \max\{h(\pi_1), h(\pi_2)\} + 1 \leq \max\{|A, \Gamma| - (n-1), |B, \Gamma| - (n-1)\} + 1 = \max\{|A, \Gamma|, |B, \Gamma|\} - (n-2) \leq |A \wedge B, \Gamma| - (n-2)$ and

$$\begin{aligned}
\|\pi\| &= \|\pi_1\| + \|\pi_2\| + \|E\| \\
&\leq |E_1|\,\|E_1\|\,2^{2m(E_1)} + |E_2|\,\|E_2\|\,2^{2m(E_2)} + \|E_2\| \\
&\leq 2(|E| - 1)(\|E\| - 1)2^{m(E)-4} + \|E\| \\
&\leq |E|\,\|E\|\,2^{m(E)-2} + \|E\| \\
&\leq |E|\,\|E\|\,2^{2m(E)}
\end{aligned}$$

(d) Let $E_i = \forall x F +^{\alpha} E^*$. Now, define $E' = \Gamma, E^*$. We observe that $m(E') = m(E) - 1, |E'| < |E|$ and $\|E'\| < \|E\|$. By the induction hypothesis, we have a proof π' of E' and we obtain π by

$$\forall\colon \dfrac{\begin{matrix}(\pi')\\ \Gamma, E^*\end{matrix}}{\Gamma, \forall x +^{\alpha} E^*}$$

Note that we do not impose any restrictions on α other than not being selected in E^*. Again, we have that $h(\pi) = h(\pi') + 1 \leq |\Gamma, E^*| - (n-1) + 1 \leq |\Gamma, \forall +^{\alpha} E^*| - (n-2)$ and $\|\pi\| = \|\pi'\| + \|E\| = |E'|\,\|E'\|\,2^{2m(E')} + \|E\| \leq |E|\,\|E\|\,2^{2m(E)}$.

■

Lemma 3.19. *Let π be an $\boldsymbol{LK}_E$-proof of S_E such that $S_E \succ F$ and let S be an expansion cedent occurring in π. Then $FV(\mathrm{Sh}(S)) \cap SV(S) = \emptyset$ with $FV(G)$ being the free variables of a formula G and $SV(S)$ the selected variables in an expansion cedent.*

Proof. We proceed via induction on the number of inferences between S and S_E. For the base case consider $S = S_E$. We have that $\vdash_S F$ and therefore $\mathrm{Sh}(S) \sqsubseteq F$ and $FV(F) \cap SV(S) = \emptyset$. Since $FV(\mathrm{Sh}(S)) \subseteq FV(F)$, we have $FV(\mathrm{Sh}(S)) \cap SV(S) = \emptyset$. For the induction step, we distinguish the next inference applied to S to obtain the next cedent S'.

(i) If S is subject to an $\vee-, \wedge-$ or weakening-inference, we have that $FV(\mathrm{Sh}(S)) \subseteq FV(\mathrm{Sh}(S'))$ and $SV(S) \subseteq SV(S')$. By the induction hypothesis, we have that $FV(\mathrm{Sh}(S)) \cap SV(S) = \emptyset$.

(ii) If S' is derived from S by an $\forall$-inference

$$, \forall\colon \dfrac{S : \Gamma, E}{S' : \Gamma, \forall x G +^{\alpha} E}$$

[1]Note that we do not need a contraction since we deal with sets

we have that $FV(\mathrm{Sh}(S)) \subseteq FV(\mathrm{Sh}(S')) \cup \{\alpha\}$ and $SV(S) = SV(S') \setminus \{\alpha\}$ since α cannot be selected in S' twice. Therefore, by the induction hypothesis, we have that $FV(\mathrm{Sh}(S)) \cap SV(S) = \emptyset$.

(iii) Let S' be obtained by an $\exists$-inference

$$\exists\colon \ \frac{S : \Gamma, \exists x G +^{t_1} E_1 \cdots +^{t_{i-1}} E_{i-1} +^{t+1} E_{i+1} \cdots +^{t_n} E_n, E_i}{S' : \Gamma, \forall x G +^{t_1} E_1 \cdots +^{t_n} E_n}$$

or an $\exists'$-inference

$$\exists'\colon \ \frac{S : \Gamma, E}{S' : \Gamma, \forall x G +^{t_i} E}$$

then t_i is admissible in S'. Therefore, $FV(t_i) \cap SV(S') = \emptyset$. Also $SV(S) = SV(S')$ and $FV(\mathrm{Sh}(S)) = FV(\mathrm{Sh}(S')) \cup FV(t_i)$. Thus and by the induction hypothesis, it follows that $FV(\mathrm{Sh}(S)) \cap SV(S) = \emptyset$.

■

3.2 Eliminating Cuts

Definition 3.20. *Let π a proof in any of our proof systems. We define $C(\pi)$ to be the number of cedents in π.*

We can now return to the proofs in our original system and formulate a Lemma:

Lemma 3.21. *Let π be a cut-free **LK**-proof of S and let $b \in \mathbb{N}$ such that for all $\exists$-inferences in π*

$$\exists\colon \ \frac{F[x/t], \Delta}{\exists x F, \Delta}$$

in π it holds that $\|t\| \leq b$. Then $\|\pi\| \leq C(\pi)\,\|S\|\,(\|S\| + C(\pi))(b+1)(C(\pi)+1)$.

Proof. Let ρ be an inference of π. We proceed with an induction on the number n of inferences between ρ and S, the root of π. We show that for every formula F in the conclusion of ρ it holds that $\|F\| \leq \|S\|\,(b+1)(n+1)$ and that the conclusion of ρ contains at most $\|S\| + n$ subformulas. We can then bound n by $C(\pi)$ and get that $\|\pi\| \leq C(\pi)\,\|S\|\,(\|S\| + C(\pi))(b+1)(C(\pi)+1)$. For the base case, consider that ρ is the last inference. Then clearly for all formulas F of S, it holds that $\|F\|$ is bounded by $\|S\|$. Also, there are at most $\|S\|$ many formulas in S. Now for the induction step:

(i) If ρ is an axiom, then there are only two formulas in the conclusion of ρ, namely A and $\overline{A}$ both of which have $\|F\| = 1 \leq \|S\|$. Moreover, S contains at least one formula and $n \geq 1$. Therefore, the number of formulas is bounded by $\|S\| + n$

(ii) Let δ be the next derivation after ρ and let it be of the kind $\vee, \wedge, \forall$ or weakening. Then for all formulas F in the conclusion of ρ, there is a formula F' in the conclusion of δ such that $\|F\| \leq \|F'\|$ and by the induction hypothesis, $\|F\| \leq \|F'\| \leq \|S\|\,(b+1)(n+1) \leq \|S\|\,(b+1)(n+2)$. Moreover, at most one formula is added (going upwards). Therefore, the number of formulas is still bounded by $\|S\| + n$.

(iii) If the next inference δ is a $\exists$-inference

$$\exists\colon\ \frac{F[x/t], \Delta}{\exists x F, \Delta}$$

Then by the induction hypothesis, we have that $\|\exists x F\| \leq \|S\| (b+1)(n+1)$. Since our cuts are quantifier-free, all occurrences of x in F have to be in S as well. Therefore, $\|F[x/t]\| \leq \|F\| + \|S\| b \leq \|S\| (b+1)(n+1) + \|S\| b \leq \|S\| (b+1)(n+2)$. By the induction hypothesis, the inequality holds for all formulas in Δ as well. Since no formula is added (going upwards), the number of formulas is still bounded by $\|S\| + n$.

■

Lemma 3.22. *Let π be an $\boldsymbol{LK}_E$-proof of an expansion cedent S_E such that $S_E \succ S$. Then there exists a cut-free proof π' in $\boldsymbol{LK}$ of S such that $|\pi'| \leq |\pi|$ and $\|\pi'\| \leq 32 \|\pi\|^4 \|S\|^2$.*

Proof. We proceed via induction and construct an **LK**-proof π' of $\mathrm{Sh}(S_E)$ such that every axiom of π' is also an axiom of π (modulo renaming of variables). Note that we can restrict our focus to $\mathrm{Sh}(S_E)$ since $\overline{\bot}$ is not in $\mathcal{E}$ and can therefore not occur in our axioms. We will also have that $C(\pi') \leq 2C(\pi)$ and $|\pi'| \leq |\pi|$. For the base case, let $\pi = A, \overline{A}$ be an axiom. Then, define π' as π. Now, π' proves $A, \overline{A} = \mathrm{Sh}(\pi)$. Also, all the bounds hold. For the induction step, consider these cases:

(i) If the last inference of π is an $\vee$-inference:

$$\vee\colon\ \frac{\begin{array}{c}(\pi_1)\\ A, B, \Gamma\end{array}}{A \vee B, \Gamma}$$

Then, by the induction hypothesis we have a proof π_1' of $\mathrm{Sh}(A), \mathrm{Sh}(B), \mathrm{Sh}(\Gamma)$ with the desired bounds. We can construct π' by adding an $\vee$-inference:

$$\vee\colon\ \frac{\begin{array}{c}\pi_1'\\ \mathrm{Sh}(A), \mathrm{Sh}(B), \mathrm{Sh}(\Gamma)\end{array}}{\mathrm{Sh}(A) \vee \mathrm{Sh}(B), \mathrm{Sh}(\Gamma)}$$

Moreover, $|\pi'| = |\pi_1'| + 1 \leq |\pi_1| + 1 = |\pi|$ and $C(\pi') = C(\pi_1') + 1 \leq 2C(\pi_1) + 1 = 2C(\pi) - 1$

(ii) If the last inference of π is an $\wedge$, we repeat the construction from the prior point, but add an $\wedge$-inference instead.

(iii) If the last inference of π is a weakening inference, we repeat the construction from above one more time, but with a weakening inference. However, it is noteworthy that the weakening neither counts towards $|\pi|$ nor to $|\pi'|$. The bounds still hold.

(iv) If the last inference in π is a $\forall$-inference.

$$\forall\colon\ \frac{\begin{array}{c}(\pi_1)\\ \Gamma, E\end{array}}{\Gamma, \forall x F +^{\alpha} E}$$

By the induction hypothesis, we have a cut-free proof π_1' of $\mathrm{Sh}(\Gamma), \mathrm{Sh}(E)$. We also now have that $\mathrm{Sh}(E) \sqsubseteq F[x/\alpha]$ and that $\perp$ does not occur in an axiom of π_1'. Thus, we can replace the necessary occurrences of $\perp$ in our proof with subformulas of $F[x/\alpha]$ to get a proof π^* of $\mathrm{Sh}(\Gamma), F[x/\alpha]$ (some of the eigenvariables of π_1' might need renaming). Now, define π' by

$$\forall\colon \frac{\begin{matrix}(\pi^*)\\ \mathrm{Sh}(\Gamma), F[x/\alpha]\end{matrix}}{\mathrm{Sh}(\Gamma), \forall x F}$$

Lemma 3.19 ensures that the eigenvariable condition is met. The bounds are as obvious as before.

(v) Let the last inference of π is an $\exists$-inference

$$\exists\colon \frac{\begin{matrix}(\pi_1)\\ \Delta, \exists x F +^{t_1} E_1 \cdots +^{t_{i-1}} E_{i-1} +^{t_{i+1}} E_{i+1} \cdots +^{t_n} E_n, E_i\end{matrix}}{\Delta, \exists x F +^{t_1} E_1 \cdots +^{t_n} E_n}$$

Then, by the induction hypothesis, we have a proof π_1' of $\mathrm{Sh}(\Delta), \exists x F, \mathrm{Sh}(E_i)$ with $\mathrm{Sh}(E_i) \sqsubseteq F[x/t_i]$. Since $\perp$ does not occur in our axioms, we can replace some of the occurrences, to obtain a proof π^* of $\mathrm{Sh}(\Delta), \exists x F, F[x/t_i]$. Since we deal with sets of formulas, we can define π' in one step by

$$\exists\colon \frac{\begin{matrix}(\pi^*)\\ \exists x F, F[x/t_i], \mathrm{Sh}(\Delta)\end{matrix}}{\exists x F, \mathrm{Sh}(\Delta)}$$

The bounds hold again.

(vi) If the last inference is an $\exists'$-inference:

$$\exists'\colon \frac{\Delta, E_i}{\Delta, \exists x F +^{t_1} E_1}$$

We repeat the construction from the prior point.

Since $\perp$ does not occur in our axioms and we have a proof of $\mathrm{Sh}(S_E) \sqsubseteq S$, we can replace some of the occurrences of $\perp$ by subformulas of S and obtain a cut-free proof ψ' of S. It still holds that $C(\psi) \leq 2C(\pi)$ and $|\psi| \leq |\pi|$. We can now obtain ψ' by the following construction: Consider every $\exists$-inference

$$\exists : \frac{\begin{matrix}(\lambda)\\ F[x/t]\end{matrix}}{\exists x F, \Delta}$$

such that no occurrence of t introduced by the substitution of x by t has an ancestor in an axiom. Now replace this inference with the following:

$$\exists : \frac{\begin{matrix}(\lambda')\\ F[x/\alpha]\end{matrix}}{\exists x F, \Delta}$$

with α being a fresh variable and λ' being obtained from λ by replacing all ancestors of the occurrences of t by α. Since we do not alter the axioms, it still holds that every axiom of ψ' is an axiom of π. Also, it holds that for all t occurring in an $\exists$-inference, $t = \alpha$ or t occurs in an axiom of π. Therefore, for such t, we have that $\|t\| \leq \|\pi\|$. By setting $b = \|\pi\|$, we can apply Lemma 3.21 and receive

$$\begin{aligned}
\left\|\psi'\right\| &\leq C(\psi') \|S\| (\|S\| + C(\psi'))(\|\pi\| + 1)(C(\psi') + 1) \\
&\leq 2C(\pi) \|S\| (\|S\| + 2C(\pi))(\|\pi\| + 1)(2C(\pi) + 1) \\
&\leq 2 \|\pi\| \|S\| (\|S\| + 2 \|\pi\|)(\|\pi\| + 1)(2 \|\pi\| + 1) \\
&= 2 \|\pi\| \|S\| (\|\pi\| \|S\| + 2 \|\pi\|^2 + \|S\| + 2 \|\pi\|)(2 \|\pi\| + 1) \\
&= 2 \|\pi\| \|S\| (2 \|\pi\|^2 \|S\| + 2 \|\pi\|^3 + 2 \|\pi\| \|S\| + 4 \|\pi\|^2 + \|\pi\| \|S\| + 2 \|\pi\|^2 + \|S\| + 2 \|\pi\|) \\
&\leq 2 \|\pi\| \|S\| 4 \cdot 4 \|\pi\|^3 \|S\| \\
&= 32 \|\pi\|^4 \|S\|^2
\end{aligned}$$

■

Theorem 3.23. *Let π be a **LK**-proof of C with only quantifier-free cuts. Then there exists a cut-free proof π' of C in **LK** such that $|\pi'| \leq 2^{\frac{5}{2}|\pi|}$ and $\|\pi'\| \leq 2^{45\|\pi\|}$.*

Proof. By Theorem 3.11 we have an expansion tree E such that $\vdash_E C$, $|E| \leq \frac{5}{2}|\pi|$ and $\|E\| \leq \frac{5}{2} \|\pi\|$. By Lemma 3.18, we have an expansion-tree proof ψ such that $|\psi| \leq 2^{|E|} \leq 2^{\frac{5}{2}|\pi|}$ and $\|\psi\| \leq 2^{4\|E\|} \leq 2^{10\|\pi\|}$. By Lemma 3.22, we obtain a proof π' such that $|\pi'| \leq |\psi| \leq 2^{\frac{5}{2}|\pi|}$ and $\|\pi'\| \leq 32 \|\psi\|^4 \|C\|^2 \leq 2^5 2^{40\|\pi\|} \|C\|^2 = 2^{5+40\|\pi\|+2\log(\|C\|)}$. Since $x - 1 \geq \log(x)$ for $x \geq 1$ and $\|\pi\| \leq \|C\|$, we have that $\|\pi'\| \leq 2^{5+40\|\pi\|+2(\|\pi\|-1)} \leq 2^{45\|\pi\|}$. ■

4 Eliminating general cuts

4.1 Transforming proofs to PNF

We will later need to consider proofs that have the property that all their cut formulas are in PNF. Since this is generally not the case, we first have to transform a given proof into a proof of that kind.

Definition 4.1. *Let π be a proof in $\boldsymbol{LK}$. We say, π is in PNF if all its cut formulas are in PNF. The set containing all the $\boldsymbol{LK}$-proofs in PNF is denoted by $\boldsymbol{LK}_{PNF}$.*

Lemma 4.2. *Let B be a formula and B' be its prenex normal form. Then $B \in \Sigma_n$ iff $B' \in \Sigma_n$ for all $n \in \mathbb{N}$. The same holds for Π_n*

Proof. We have defined the membership of Σ_n and Π_n by equivalence to PNF formulas. Since the equivalence relation is transitive, the claim follows. ■

Lemma 4.3. *Let B be a propositional formula. Then there is proof π of $B, \overline{B}$ in $\boldsymbol{LK}$ (and in $\boldsymbol{LK}_{PNF}$) with $|\pi| \leq |B, \overline{B}| = |B \vee \overline{B}|$.*

Proof. For atoms, the case is clear, since we can derive $A, \overline{A}$ with one inference for atoms A. For the induction step, consider two cases:

- If $B = B_1 \wedge B_2$, we can construct the following proof:

$$\dfrac{\dfrac{\begin{matrix}(\lambda_0) \\ B_0, \overline{B_0}\end{matrix} \qquad \begin{matrix}(\lambda_1) \\ B_1, \overline{B_1}\end{matrix}}{\overline{B_0}, \overline{B_1}, B_0 \wedge B_1}}{\overline{B_0} \vee \overline{B_1}, B_0 \wedge B_1}$$

  with $\overline{B_0} \vee \overline{B_1}, B_0 \wedge B_1 = \overline{B}, B$. If we define λ as the whole proof, we have that $|\lambda| = |\lambda_0| + |\lambda_1| + 2 \leq |B_0 \vee \overline{B_0}| + |B_1 \vee \overline{B_1}| + 2 = |B \vee \overline{B}|$.

- If $B = B_0 \vee B_1$, we can construct the proof of $\overline{B}, \overline{\overline{B}}$ with the means above for $\overline{B} = \overline{B_0} \wedge \overline{B_1}$. This is then clearly also a proof of $B, \overline{B}$.

■

Lemma 4.4. *Let π be a proof of Γ in $\boldsymbol{LK}$. Then there is a proof π' of Γ such that $|\pi'| = |\pi|$ and for every formula F in π', it holds that $|F| \leq |\pi| + |\Gamma|$. If F is a cut formula, it even holds that $|F| \leq |\pi|$.*

Proof. We start with a case distinction: If F corresponds to a subformula in the endcedent, then clearly, the size of F is bounded by the size of the formulas in the endcedent. If F does not correspond to a subformula in the endcedent, then it has a cut-formula as a descendant. The other simple case: If F has no ancestors that are the primary formula of a weakening inference, then the size of F is clearly bound by the number of inferences. The only problem we can have is if F is derived by weakening. Since F has a cut formula F' as a descendant, we can look at this cut formula directly since $|F| \leq |F'|$. Consider all the subformulas of F' that stem from weakening on both sides of the cut (the other ones are trivially bounded). We can now replace the corresponding formulas in the weakening with an atom $P(c)$ (c being a constant) and adapt the inferences below accordingly. This leads to a new cut formula F'', which is bounded by the size of $\vee, \wedge, \exists$ and $\forall$ inferences. Note that we have to adapt the other cut-formula of the same cut accordingly, but the size bound holds for the same reason. ∎

Lemma 4.5. *Let B and C be two formulas and for any formula F let F' be the corresponding formula in PNF. Then there are proofs π of $\overline{B'}, \overline{C'}, (B \wedge C)'$, π' of $\overline{B'}, (B \vee C)'$ and π'' of $\overline{C'}, (B \vee C)'$ such that $|\pi|, |\pi'|, |\pi''| \leq |S|$ with S being the respective endcedent.*

Proof. For the rest of the proof let $B' = Q_1 B_0$ and $C' = Q_2 C_0$ with Q_i being blocks of quantifiers and B_0 and C_0 being quantifier-free. Similarly $(B \wedge C)' = Q(B_0 \wedge C_0)$ and $(B \vee C)' = Q(B_0 \vee C_0)$ with Q being the result of arbitrarily interleaving Q_1 and Q_2 (while preserving their initial ordering). We now proceed via induction and claim that there is a tree-like proof of the cedents $\overline{C}, \overline{B}, (B \wedge C)'$, $\overline{B}, (B \vee C)'$ and $\overline{C}, (B \vee C)'$ such that their length is at most the number of the logical connectives in the respective cedent. For the base case, we consider $Q = \emptyset$ and B, C being propositional formulas. In this case, we have that $(B \vee C)' = B' \vee C' = B \vee C$ and $(B \wedge C)' = B' \wedge C' = B \wedge C$. It follows that we can derive the cedents in the following way:

$$
\delta_1, \wedge\colon \dfrac{\begin{matrix}(\lambda_0)\\ \overline{B}, B\end{matrix} \qquad \begin{matrix}(\lambda_1)\\ \overline{C}, C\end{matrix}}{\overline{B}, \overline{C}, B \wedge C}
\quad
\delta_2, \vee\colon \dfrac{\text{w:}\ \dfrac{\begin{matrix}(\lambda_0)\\ B, \overline{B}\end{matrix}}{\overline{B}, B, C}}{\overline{B}, B \vee C}
\quad
\delta_3, \vee\colon \dfrac{\text{w:}\ \dfrac{\begin{matrix}(\lambda_1)\\ C, \overline{C}\end{matrix}}{\overline{C}, C, B}}{\overline{C}, B \vee C}
$$

In Lemma 4.3, we have seen that we can prove $B, \overline{B}$ and $C, \overline{C}$ with λ_0 and λ_2 of size at most $|B \vee \overline{B}|$ and $|C \vee \overline{C}|$ respectively. Therefore, the proof of $\overline{B}, \overline{C}, B \wedge C$ has length $|\lambda_0| + |\lambda_1| + 1 \leq |B \vee \overline{B}| + |C \vee \overline{C}| + 1 = |\overline{B} \vee \overline{C} \vee (B \wedge C)| = |\overline{B}, \overline{C}, B \wedge C|$. The bounds for the proofs of $|\overline{B}, B \vee C|$ hold for similar reasons if we remember that we do not count weakening inferences.

For the induction step, assume that Q contains at least one quantifier. We need a case distinction:

- Let the outermost quantifier of Q be $\exists x$. Since this quantifier has to be the outermost quantifier of either Q_1 or Q_2, we can assume without loss of generality that it is the outermost quantifier of Q_1. Now consider Q' and Q'_1 defined as Q and Q_1 respectively but without the outermost quantifier. Now, by the induction hypothesis, we can write the following derivations:

$$
\begin{array}{cc}
(\lambda) & (\lambda') \\
\exists\text{:}\ \dfrac{\overline{Q'_1 B_0[x/b]}, \overline{Q_2 B_1}, Q'(B_0[x/b] \wedge B_1)}{\forall\text{:}\ \dfrac{\overline{Q'_1 B_0[x/b]}, \overline{Q_2 B_1}, \exists x Q'(B_0 \wedge B_1)}{\forall x \overline{Q'_1 B_0}, \overline{Q_2 B_1}, Q(B_0 \wedge B_1)}} &
\exists\text{:}\ \dfrac{\overline{Q'_1 B_0[x/b]}, Q'(B_0[x/b] \vee B_1)}{\forall\text{:}\ \dfrac{\overline{Q'_1 B_0[x/b]}, \exists x Q'(B_0 \vee B_1)}{\forall x \overline{Q'_1 B_0}, Q(B_0 \vee B_1)}}
\end{array}
$$

$$
\begin{array}{c}
(\lambda'') \\
\exists\text{:}\ \dfrac{\overline{Q_2 B_1}, Q'(B_0[x/b] \vee B_1)}{\overline{Q_2 B_1}, \exists x Q'(B_0 \vee B_1)}
\end{array}
$$

Note that all the $\forall$ inferences are legal (eigenvariable-condition) since we can assume without loss of generality that the variables of B_0 and B_1 are disjoint by renaming them. Moreover, all of our inferences add one connective to the endcedent. Thus, the size restrictions are met.

- Now assume that outermost quantifier of Q is a $\forall$. The same construction from above holds if we switch the quantifier. However, we still need to introduce the $\exists$-quantifier first to fulfill the eigenvariable condition in the second step.

■

Lemma 4.6. *Let π be a proof of S in $\boldsymbol{LK}$. Then there is a proof π' of S in $\boldsymbol{LK}_{PNF}$ with $|\pi'| \leq 5|\pi|^2$.*

Proof. For every cedent Δ in π, we define the cedent Δ' in the following way:

- If B is a formula of Δ and B has no cut formula as a descendant, add B to Δ'
- If B is a formula of Δ and has a cut-formula as a descendant, then add B', the PNF of B, to Δ. Without loss of generality choose B such that it has the minimum number of blocks of alternating quantifiers.

Now, we construct the (invalid) proof π'' by replacing every cedent Δ with Δ' and using the same inferences. This proof is not correct since there are illegal inferences: If the formula $A \wedge B$ is the primary formula of an $\wedge-$inference and has a cut formula as a descendant, then the new inference is not valid anymore. Similarly, if the formula $A \vee B$ is the primary formula of an $\vee-$inference and has a cut formula as descendant, then the new inference is not valid either. We now deal with these two cases:

- Consider the inference

$$
\dfrac{\begin{array}{cc}(\lambda_0) & (\lambda_1) \\ A', \Gamma_1 & B', \Gamma_2\end{array}}{(A \wedge B)', \Gamma_1, \Gamma_2}
$$

By Lemma 4.5 we obtain a proof λ_2 of $\overline{A'}, \overline{B'}, (A \wedge B)'$. We can now adapt π'':

$$
\text{cut:}\ \dfrac{\text{cut:}\ \dfrac{\begin{array}{cc}(\lambda_2) & (\lambda_0) \\ \overline{A'}, \overline{B'}, (A \wedge B)' & A', \Gamma_1\end{array}}{\overline{B'}, (A \wedge B)', \Gamma_1} \quad \begin{array}{c}(\lambda_1) \\ B', \Gamma_2\end{array}}{(A \wedge B)', \Gamma_1, \Gamma_2}
$$

By Lemma 4.5, we have that $|\lambda_2| \leq |\overline{A} \vee \overline{B} \vee (A \wedge B)'| = 2|A \wedge B| + 1 \leq 3|A \wedge B|$. Since $A \wedge B$ has a cut-formula as a descendant, we can bound this by $3|\pi|$.

- Consider the inference

$$\dfrac{\begin{matrix}(\lambda_0)\\ A', B', \Gamma\end{matrix}}{(A \vee B)', \Gamma}$$

Again, by Lemma 4.5, we have two proofs λ_1 and λ_2 of $\overline{A'}, (A \vee B)'$ and $\overline{B'}, (A \vee B)'$ respectively. We can adapt π'' again:

$$\text{cut: } \dfrac{\text{cut: } \dfrac{\begin{matrix}(\lambda_1)\\ \overline{A'}, (A \vee B)'\end{matrix} \qquad \begin{matrix}(\lambda_0)\\ A', B', \Gamma\end{matrix}}{\overline{B'}, (A \vee B)', \Gamma} \qquad \begin{matrix}(\lambda_2)\\ B', (A \vee B)'\end{matrix}}{(A \vee B)', \Gamma}$$

By Lemma 4.5, we have that $|\lambda_1| + |\lambda_2| \leq |\overline{A'} \vee (A \vee B)'| + |\overline{B'} \vee (A \vee B)'| \leq 3|\pi| + 1$ since $(A \vee B)'$ has a cut formula as a descendant and can therefore be bounded by $|\pi|$.

If we replace all these inferences by the ones constructed above, we obtain a valid proof. Furthermore, by the size bounds from above, we see that each of these inferences have size at most $5|\pi|$ (we replace one inference with two cut inferences and branches of cumulated size at most $4|\pi|$). Therefore, the new proof π' has size at most $4|\pi|^2$. ■

4.2 Reducing the Quantifier Depth of Cuts

This section aims to give an algorithm, on how to eliminate the outermost quantifiers from cut formulas of a given proof and prove an upper bound of the logical complexity of the resulting proof. The main result will be Corollary 4.23. Applying this algorithm repeatedly yields a proof, of which all the cuts are quantifier-free and therefore permits us to apply Theorem 3.23. For this section, we will assume all the formulas in our proofs are in PNF and thus in $\mathbf{LK}_{PNF}$. Moreover, we will assume that the proofs are in variable normal form and fulfill the auxiliary condition.

First, we need a few definitions:

Definition 4.7. *Let A be a formula. An $\exists$-subformula of A is a subformula that is occurring only in the scope of $\exists$-quantifiers. An $\exists$-component is a minimal $\exists$-subformula. The $\forall$-subformulas and $\forall$-components are defined accordingly. Let π be a proof in **LK**. A $\forall/\exists$-component of a cut-formula in π is a $\forall$-component of the left cut-formula or an $\exists$-component of the right cut-formula.*

Example 4.8. *Consider the Formula $\varphi = \forall x \forall y \exists z (A \wedge \exists w B)$ with A and B being quantifier-free and x, y, z not occurring in B. Then the $\forall$-subformulas of φ are $\forall y \exists z (A \wedge \exists w B)$ and $\exists z (A \wedge \exists w B)$ and there is no $\exists$-subformula. The only $\forall$-component is $\exists z A$ and again, there is no $\exists$-component.*

Definition 4.9. *Let B be a formula occurring in π. Then B is of one of the following kinds:*

α) *B has a left cut formula A as a descendant and corresponds to a $\forall$-subformula of A.*

β) *B has a right cut formula A as a descendant and corresponds to an $\exists$-subformula of A.*

γ) *Neither α) nor β) hold.*

Definition 4.10. *Let D be an $\exists$-inference and $A[x/s]$ the auxiliary formula. We say:*

(i) D is critical if the outermost connective of $A[x/s]$ is not an $\exists$. In this case $A[x/s]$ is called $\exists$-critical.

(ii) If (i) holds and $A[x/s]$ is of category β) from Definition 4.9, then the $\exists$-jump-target of $A[x/s]$ is defined as the cut inference which has a descendant of $A[x/s]$ as a right cut formula. The $\exists$-jump-target-cedent of $A[x/s]$ is defined as the upper left cedent of the jump-target of $A[x/s]$. This cedent is also referred to as the jump-target-cedent of the cedent containing $A[x/s]$.

Lemma 4.11. *The $\exists$-jump-target is well-defined.*

Proof. Since cut-formulas are eliminated in their respective cut, one formula can have at most one cut-formula as a descendant. By definition, every formula of type β) has at least one cut-formula as a descendant and thus exactly one, which proves the claim. ■

The next definition is probably the most crucial one to eliminate like quantifiers from cut inferences:

Definition 4.12. *Let D a cut inference in π. D is called to-be-eliminated if the outermost connective of its cut formula is a quantifier. A sequence of cedents of π $(\Delta_1, \ldots, \Delta_m)$ is called an $\exists$-path if Δ_1 is the endcedent of π and for each $i < m$ one of the following holds:*

- *Δ_i is the lower cedent of a to-be-eliminated cut inference and Δ_{i+1} is the right upper cedent of the same inference.*
- *Δ_i is the lower cedent of any inference but a to-be-eliminated cut inference and Δ_{i+1} is an upper cedent of the same inference.*
- *Δ_i is the upper cedent of an $\exists$-critical inference of type β) and Δ_{i+1} is the jump-target-cedent of Δ_i.*

This $\exists$-path is said to lead to Δ_m.

The intuition behind an $\exists$-path is that starting from the endcedent, one goes upwards through the proof, always choosing the right upper cedent at a to-be-eliminated cut inference and jumping back down to the $\exists$-jump-target in the next step.

Lemma 4.13. *Let φ be an $\exists$-path in π and Δ_i in φ. Then φ contains every cedent below Δ_i.*

Proof. We proceed via induction. Let φ be any $\exists$-path to Δ_m. All the cedents below Δ_1 are trivially contained in φ. Now, we note that every cedent has exactly one cedent directly beneath it and consider two cases for the induction step:

- If Δ_i is not the $\exists$-jump-target of another cedent in φ, then Δ_{i-1} is by definition of the $\exists$-path the child cedent of Δ_i. From the induction hypothesis, it follows that all the cedents below Δ_i are in φ.
- If Δ_i is the jump-target of Δ_{i-1}, which is the upper cedent of an $\exists$-critical inference, we have that all the cedents below Δ_i are also below Δ_{i-1} and therefore in φ by the induction hypothesis.

■

Lemma 4.14. *Let $\varphi = (\Delta_1, \ldots, \Delta_m)$ be an $\exists$-path and $\varphi' = (\Delta_{i_1}, \ldots, \Delta_{i_k})$ the subsequence consisting of the $\exists$-critical cedents of φ. Then the subsequence φ' and knowledge of Δ_m allow to uniquely reconstruct φ.*

Proof. We proceed via induction on the number of jump-targets. For the base case assume that φ contains no jump target. Then φ only goes *upwards* in the graph and the uniqueness follows since Δ_m is given. For the induction step consider an $\exists$-path φ and the sequence of jump-targets $(\Delta_{i_1}, \ldots, \Delta_{i_k})$. By the induction hypothesis, we can now uniquely construct the $\exists$-path φ' containing the jump-targets $(\Delta_{i_1}, \ldots, \Delta_{i_{k-1}})$ and leading to the lower cedent of the cut inference of Δ_{i_k}, which we call Δ'. There is only one option to extend φ' to Δ_{i_k-1} since this path contains no extra jump-targets. Note that φ' is just a cut version of φ. From Δ_{i_k-1} there is again only one possibility to extend the path and that is Δ_{i_k}. Again, since there are no more jump targets, there is only one possibility to extend this path to Δ_m, which is the original path φ. ■

We can also associate a substitution σ_φ with an $\exists$-path φ. This substitution will be defined on all free variables occurring in or below Δ_m and all the outermost universally quantified variables occurring in type α) formulas in Δ_m. For that, we need the following observation: Let $B = \forall x_i \ldots \forall x_l A$ of type α) with $l > i > 1$ and A not having a $\forall$-quantifier as the outermost connective. Then, B has a descendant cut-formula $B' = \forall x_1 \ldots \forall x_l A$ in the cedent Δ'. By Lemma 4.13 we have that Δ' is also in φ and since it is the left cut formula of a to-be-eliminated cut, φ also contains the cedent Δ'', of which Δ' is the jump-target. By definition, Δ'' contains the formula $\overline{A[x_1/s_1, \ldots, x_l/s_l]}$ which is an ancestor of the corresponding right cut-formula. This cedent (and thus the s_i) is unique because it has to be the upper cedent of an $\exists$-critical inference.

$$
\dfrac{\begin{array}{c} \forall x_i \ldots \forall x_l A, \Gamma \\ \ddots\vdots\cdot^{\cdot^{\cdot}} \\ \Delta' : \forall x_1 \ldots \forall x_l A, \Gamma_1 \end{array} \qquad \begin{array}{c} \dfrac{\Delta'' : \overline{A[x_1/s_1, \ldots, x_l/s_l]}, \Gamma'}{\exists x_l \overline{A[x_1/s_1, \ldots, x_{l-1}/s_{l-1}]}, \Gamma'} \\ \ddots\vdots\cdot^{\cdot^{\cdot}} \\ \exists x_1 \ldots \exists x_l \overline{A}, \Gamma_2 \end{array}}{\Gamma_1, \Gamma_2}
$$

This leads to the following definition:

Definition 4.15. *Let $\varphi = (\Delta_1, \dots, \Delta_m)$ be an $\exists$-path and B a type α) formula of Δ_m of the form $\forall x_i \forall x_{i+1} \dots \forall x_l B'(b_1, \dots, b_{i-1}, x_i, \dots, x_l)$ with at least one outermost universally quantified variable. We construct the formula $\overline{A[x_1/s_1, \dots x_l/s_l]}$ as above and define σ_φ inductively by $x_k\sigma_\varphi = s_k\sigma_{\varphi'}$ with $i \leq k \leq l$ and φ' being φ truncated to end at $\overline{A[x_1/s_1, \dots, x_k/s_k]}, \Gamma'$ (cf. graphic above). If b is a free variable occurring in Δ_m or below, we distinguish two cases:*

- *If there is a $\forall$-inference*
$$\forall : \frac{A[x/b], \Gamma}{\forall x A, \Gamma}$$
below Δ_m, which uses b as an eigenvariable and if $\forall x A$ is a type α) formula, we define $b\sigma_\varphi = x\sigma_{\varphi'}$ with φ' being φ truncated to end at the lower cedent of the $\forall$-inference.

- *If there is no such inference, we define $b\sigma_\varphi = b$*

Definition 4.16. *Let A be a formula occuring in a cedent Δ of π and φ be an $\exists$-path leading to Δ. We define $*_\varphi(A)$:*

- *If A is of kind α) and has the form $\forall x_1 \forall x_2 \dots \forall x_l B$ with $l > 0$ and B having a different outermost connective than $\forall$, then we define $*_\varphi(A) = B\sigma_\varphi$.*

- *If A is of kind β) and has an $\exists$ as the outermost connective, then we define $*_\varphi(A)$ as the empty cedent.*

- *Otherwise, we define $*_\varphi(A)$ as $A\sigma_\varphi$.*

If A appears below Δ, we define $*_\varphi(A)$ as $*_{\varphi'}(A)$ with φ' being φ truncated to end at the cedent Δ' containing A. The $*_\varphi$-translation of Δ, $*_\varphi(\Delta)$ is the cedent containing exactly the formulas $*_\varphi(A)$ for A occurring in or below Δ in π.

Example 4.17. *Consider the following proof*

$$
\begin{prooftree}
\AxiomC{(λ_1)}
\LeftLabel{δ_6:}
\noLine
\UnaryInfC{$A[x/b,y/c],\Gamma_1$}
\LeftLabel{$\delta_5,\forall$:}
\UnaryInfC{$\forall y A[x/b],\Gamma_1$}
\LeftLabel{$\delta_4,\forall$:}
\UnaryInfC{$\forall x \forall y A,\Gamma_1$}
\AxiomC{(λ_2)}
\LeftLabel{δ_3:}
\noLine
\UnaryInfC{$\overline{A[x/s_1,y/s_2]},\Gamma_2$}
\LeftLabel{$\delta_2,\exists$:}
\UnaryInfC{$\exists y \overline{A[x/s_1]},\Gamma_2$}
\LeftLabel{$\delta_1,\exists$:}
\UnaryInfC{$\exists x \exists y \overline{A},\Gamma_2$}
\LeftLabel{δ_0, cut:}
\BinaryInfC{Γ_1,Γ_2}
\end{prooftree}
$$

with λ_1, λ_2 being valid proofs in $\boldsymbol{LK}_{PNF}$ and the outermost connective of A not being an $\exists$-connective. All the inferences are labeled with δ_i. The corresponding lower cedent of the inference δ_i is called Δ_i. Then, we have the following structure: The inference δ_0 is to-be-eliminated, the inference δ_2 is $\exists$-critical and Δ_4 is the $\exists$-jump-target of δ_2. Now, we can construct seven $\exists$-paths from the graphic: $\varphi_n = (\Delta_0, \cdots, \Delta_n)$ for $0 \leq n \leq 6$ (note that every φ_n leads to Δ_n) and consider the induced substitutions σ_{φ_n} and the mapping $*_{\varphi_n}$:

(i) For $n = 0$, we have $\sigma_{\varphi_0} = \mathrm{id}$ and $*_{\varphi_0}(\Delta_0) = \Delta_0$.

(ii) *For $n = 1$, we have that σ_{φ_1} is still the identity. Note that the descendant relation is reflexive and therefore $\overline{\exists x \exists y A}$ has a right cut formula as a descendant (itself). Therefore, $*_{\varphi_1}(\Delta_1) = \emptyset, \Gamma_2\sigma_{\varphi_1}, \Gamma_1\sigma_{\varphi_0} = \Gamma_1, \Gamma_2$*

(iii) *For $n = 2$, we have the same result as for $n = 1$ and $\sigma_{\varphi_2} = \mathrm{id}$ and $*_{\varphi_2}(\Delta_2) = \Gamma_1, \Gamma_2$.*

(iv) *For $n = 3$, it still holds that $\sigma_{\varphi_3} = \mathrm{id}$. However, since the outermost connective of $\overline{A[x/s_1, y/s_2]}$ is not an $\exists$-connective, we have that $*_{\varphi_3}(\Delta_3) = \Delta_3\sigma_{\varphi_3}, \Gamma_1\sigma_{\varphi_0} = \Delta_3, \Gamma_1 = \overline{A[x/s_1, y/s_2]}, \Gamma_1, \Gamma_2$.*

(v) *For $n = 4$, it follows that $x\sigma_{\varphi_4} = s_1\sigma_{\varphi_3} = s_1$ and $y\sigma_{\varphi_4} = s_2\sigma_{\varphi_3} = s_2$. Moreover, we have that $*_{\varphi_4}(\forall x \forall y A) = A\sigma_{\varphi_4} = A[x/s_1, y/s_2]$. Since, we imposed the eigenvariable condition, it follows that x, y do not occur in Γ_1 and thus $*_{\varphi_4}(\Gamma_1) = \Gamma_1\sigma_{\varphi_4} = \Gamma_1$. Therefore, $*_{\varphi_4}(\Delta_4) = A[x/s_1, y/s_2], \Gamma_1, *_{\varphi_0}(\Gamma_2) = A[x/s_1, y/s_2], \Gamma_1, \Gamma_2$.*

(vi) *For $n = 5$, we have that $y\sigma_{\varphi_5} = s_2$ and that $b\sigma_5 = x\sigma_{\varphi_4} = s_1$. And again, $*_{\varphi_5}(\Delta_5) = A[x/s_1, y/s_2], \Gamma_1, \Gamma_2$*

(vii) *Lastly, for $n = 6$, we obtain the same result for different reasons and have that $b\sigma_{\varphi_6} = s_1, c\sigma_{\varphi_6} = s_2$ and $*_{\varphi_6}(\Delta_6) = A[x/s_1, y/s_2], \Gamma_1, \Gamma_2$.*

We consider the set

$$\{*_{\varphi_n}(\Delta_n) \mid 0 \leq n \leq 5\} = \{(\Gamma_1, \Gamma_2); (\overline{A[x/s_1, y/s_2]}, \Gamma_1, \Gamma_2); (A[x/s_1, y/s_2], \Gamma_1, \Gamma_2)\}$$

and observe that, we could rewrite the proof from above in the following way

$$\text{cut: } \frac{\begin{array}{c}(\lambda_1') \\ A[x/s_1, y/s_2], \Gamma_1, \Gamma_2\end{array} \qquad \begin{array}{c}(\lambda_2') \\ \overline{A[x/s_1, y/s_2]}, \Gamma_1, \Gamma_2\end{array}}{\Gamma_1, \Gamma_2}$$

if we had the subproofs λ_1', λ_2'. Note that the proof λ_2' can be constructed naturally from λ_2. Furthermore, $A[x/s_1, y/s_2], \Gamma_1, \Gamma_2$ is a valid formula since $A[x/b, y/c], \Gamma_1$ is and therefore, there is a proof of $A[x/s_1, y/s_2], \Gamma_1, \Gamma_2$. Moreover, we observe that we eliminated the outermost layer of like quantifiers in the cut formulas (if we only consider the cut depicted).

We will generalize the construction from the example above inductively, in order to eliminate the outermost layer of like quantifiers in all the cut formulas at once. Moreover, we observed that $*_{\varphi_0}(\Delta_0) = \Delta_0$. This generalizes to every proof tree:

Lemma 4.18. *For the endcedent Δ of π it holds that $*_\varphi(\Delta) = \Delta$ for the (only) $\exists$-path φ leading to Δ*

Proof. Obviously, there is only one $\exists$-path $\varphi = (\Delta)$ leading to Δ. All of the formulas of Δ are of type γ). Therefore, $\sigma_\varphi = \mathrm{id}$. Moreover, there are no formulas below Δ. Therefore, $*_\varphi(\Delta) = \Delta\sigma_\varphi = \Delta$. ∎

We can now formulate (and prove) the most important theorem of this section:

Theorem 4.19. *Let π be a tree-like proof of Γ in $\boldsymbol{LK}_{PNF}$. Then there is a tree-like proof π' of Γ in $\boldsymbol{LK}_{PNF}$ with the following properties:*

(i) All cut-formulas of π' are $\forall/\exists$-components of cut-formulas in π.

(ii) $|\pi'| \leq 2^{|\pi|}$ and $h(\pi') \leq 2^{h(\pi')}$

Proof. For the proof, we will construct a new proof π' with the $*\varphi$ translations from Definition 4.16 that has all the properties we wish for. First, consider the set of possible cedents

$$S = \{*_\varphi(\Delta) \mid \Delta \text{ is a cedent in } \pi, \varphi \text{ is an } \exists\text{-path in } \pi \text{ leading to } \Delta\}.$$

Note that by Lemma 4.18, we have that Γ is in S. We now proceed via induction and show that every sub-proof of π can be reconstructed in the sense that if π^* is a subproof of π of Γ^*, we can construct a proof of $*_\varphi(\Gamma^*)$ for any $\exists$-path φ in π. After showing that, it immediately follows that we can construct such a proof for $\Gamma = *_\varphi(\Gamma)$. We now consider the cedent Δ and construct a DAG-like proof, which we can later transform into a tree-like proof. For the base case, assume that Δ is an axiom. We have that Δ has the form $A, \overline{A}$ with A being atomic. From the definition of $*_\varphi$ it follows that $*_\varphi(\Delta) = A\sigma_\varphi, \overline{A}\sigma_\varphi, \Lambda = A\sigma_\varphi\overline{A\sigma_\varphi}, \Lambda$ with Λ being the images of the formulas occurring below Δ. We can therefore derive $*_\varphi(\Delta)$ in the following way:

$$\text{w:}\ \frac{A\sigma_\varphi, \overline{A\sigma_\varphi}}{A\sigma_\varphi, \overline{A\sigma_\varphi}, \Lambda}$$

with $A\sigma_\varphi, \overline{A\sigma_\varphi}$ still being an axiom since $A\sigma_\varphi$ is still atomic. The length of the proof is the same because we do not count weakening. For the induction step, we need a case distinction:

(i) Let Δ be derived from Δ' by an $\vee$-inference

$$\begin{array}{c}(\lambda)\\ \vee\text{:}\ \dfrac{A, B, \Gamma'}{A \vee B, \Gamma'}\end{array}$$

and assume that we already have a proof λ'' of $*_\varphi(A, B, \Gamma')$. By construction, we have that

$$*_\varphi(\Delta') = *_\varphi(A), *_\varphi(B), *_\varphi(\Gamma'), \Lambda \text{ and } *_\varphi\ (\Delta) = *_\varphi(A \vee B), *_\varphi(\Gamma'), \Lambda \text{ for some } \Lambda.$$

Since the outermost connective of $A \vee B$ is an $\vee$-connective, we have that $*_\varphi(A \vee B) = (A \vee B)\sigma_\varphi = A\sigma_\varphi \vee B\sigma_\varphi$. Moreover, we have that A, B do not correspond to $\forall-$ or $\exists$-subformulas and therefore, they are of type γ), which means that $*_\varphi(A) = A\sigma_\varphi$ and $*_\varphi(B) = B\sigma_\varphi$. Thus, by the induction hypothesis, we have the following proof

$$\begin{array}{c}(\lambda')\\ \vee :\ \dfrac{A\sigma_\varphi, B\sigma_\varphi, *_\varphi(\Gamma), \Lambda}{A\sigma_\varphi \vee B\sigma_\varphi, *_\varphi(\Gamma), \Lambda}\end{array}$$

(ii) If Δ is derived from Δ' with an $\wedge$-inference, the same thought as in the case above yields the desired result.

(iii) If Δ is derived from Δ' with weakening, the construction is self-evident.

(iv) Let Δ be derived with a cut inference:

$$\delta, \text{ cut: } \frac{A, \Gamma_1 \qquad \overline{A}, \Gamma_2}{\Gamma_1, \Gamma_2}$$

Now, we distinguish two cases:

- Let δ is not a to-be-eliminated cut and let φ_1, φ_2 be the two $\exists$-paths that extend φ to the left and right upper cedent of the cut respectively (i.e. A, Γ_1 and $\overline{A}, \Gamma_2$). By definition of $*_{\varphi_1}$ and $*_{\varphi_2}$, we have that $*_{\varphi_1}(A) = A\sigma_{\varphi_1}$ and $*_{\varphi_2}(\overline{A}) = \overline{A}\sigma_{\varphi_2} = \overline{A\sigma_{\varphi_2}}$. Also since A and $\overline{A}$ contain the same free variables and the variables below are only one cut inference away, we have that $A\sigma_{\varphi_1} = A\sigma_{\varphi_2}$ and thus, $A\sigma_{\varphi_1} = A\sigma_{\varphi_2}$. Therefore, we can derive $*_\varphi(\Delta)$ with a cut inference as well:

$$\text{cut: } \frac{*_{\varphi_1}(A), *_{\varphi_1}(\Gamma_1), \Lambda \qquad \overline{*_{\varphi_1}(A)}, *_{\varphi_2}(\Gamma_2), \Lambda}{*_\varphi(\Gamma_1), *_\varphi(\Gamma_1), \Lambda}$$

- Let δ be a to-be-eliminated cut and let φ_2 be the $\exists$-path that extends φ to the upper right cut formula. Then the outermost connective of $\overline{A}$ is an $\exists$-quantifier and it is clearly of type β since it is its own descendant. Therefore, by definition of $*_{\varphi_2}$, we have that $*_{\varphi_2}(A)$ is the empty cedent. Thus, we have $*_{\varphi_2}(A, \Gamma_2) = *_\varphi(\Gamma_1, \Gamma_2)$ and we can omit the inference.

(v) Let Δ be derived with a $\forall$-inference:

$$\forall\text{: } \frac{A[x/b], \Gamma}{\forall x : A, \Gamma}$$

We define φ' to be the $\exists$-path that extends φ by one step to the cedent $A[x/b], \Gamma$. Now, we distinguish two cases:

- Let $A[x/b]$ (and therefore $\forall x : A$) not be in category α). Then by definition $b\sigma_{\varphi'} = b$ and $*_{\varphi'}(A[x/b]) = A[x/b]\sigma_{\varphi'} = C[x/b]$. Moreover, since $\forall x : A$ is not of type α), we have that x is not in the scope of σ_φ and therefore $*_\varphi(\forall x : A) = (\forall x : A)\sigma_\varphi = (\forall x : A)\sigma_{\varphi'} = \forall x : C$. Thus, we can derive $*_\varphi(\Delta)$ by a $\forall$-inference (note that $*_\varphi(\Gamma) = *_{\varphi'}(\Gamma)$):

$$\forall\text{: } \frac{C[x/b], *_\varphi(\Gamma), \Lambda}{\forall x : C, *_\varphi(\Gamma), \Lambda}$$

- Let $A[x/b]$ (and therefore $\forall x : A$) be in category α and let C be the $\forall$-component of A. Then by definition of $*_\varphi$ it holds that $*_{\varphi'}(A[x/b]) = *_\varphi(\forall x : A) = C\sigma_\varphi$. Therefore, we can omit this inference in the transformed proof.

(vi) For the last case, let Δ be derived by an $\exists$-inference:

$$\exists\text{: } \frac{A[x/s], \Gamma}{\exists x : A, \Gamma}$$

We define φ' as in the previous case to be the $\exists$-path that extends φ to the cedent $A[x/s], \Gamma$. Now, we again have to distinguish two cases:

- Let $A[x/s]$ (and therefore $\exists x : A$) not be of kind β). Then have that $*_{\varphi'}(A[x/s]) = A[x/s]\sigma_{\varphi'} = C[x/t]$. Moreover, it follows that $*\varphi(\exists x : A) = (\exists x : A)\sigma_\varphi = \exists x : C$. Therefore, we can derive $*_\varphi(\Delta)$ with the following inference (note that $*_{\varphi'}(\Gamma) = *_\varphi(\Gamma)$):

$$\exists\colon \ \frac{C[x/t], *_\varphi(\Gamma)}{\exists x : C, *_\varphi(\Gamma)}$$

- Let $A[x/s]$ (and therefore $\exists x : A$) be of type β). If the outermost connective of $A[x/s]$ is an $\exists$-quantifier, then $*_{\varphi'}(A[x/s]) = *_\varphi(\exists x : A) = \emptyset$. The inference can therefore be omitted since the the upper and lower cedent are identical. Therefore, without loss of generality, we assume that the outermost connective of $A[x/s]$ is not an $\exists$-quantifier. Then, we have that $*_{\varphi'}(A[x/s], \Gamma) = A[x/s]\sigma_{\varphi'}, \Lambda$ and $*_\varphi(\exists x : A, \Gamma) = \Lambda$. Now, consider the $\exists$-jump-target $\forall x_1 \dots \forall x_n : \overline{A}, \Gamma'$ of $A[x/s] = A[x_1/s_1, \dots, x_n/s_n]$ and the $\exists$-path φ'' that extends φ' to this cedent. By definition, it holds that $*_{\varphi''}(\forall x_1 \dots \forall x_n : A) = A[x_1/s_1, \dots, x_n/s_n]\sigma_{\varphi''} = A[x_1/s_1, \dots, x_n/s_n]\sigma_{\varphi'} = \overline{*_{\varphi'}(A[x_1/s_1, \dots, x_n/s_n])}$. Therefore, we have that $*_{\varphi''}(\forall x_1, \dots, \forall x_n : A) = \overline{*_{\varphi'}(A[x_1/s_1, \dots, x_n/s_n])}$. Moreover, since $*_\varphi(\Gamma)$ also contains the $*_\varphi$ translations of all the formulas occurring below Γ and Γ' appears in the lower cedent of the cut, which is also a lower cedent of Γ, we have that $\Lambda' = *_{\varphi'}(\Gamma') \subseteq *_\varphi(\Gamma) = \Lambda$. Therefore, we can infer $\Lambda = *_\varphi(\Delta)$ with a cut:

$$\text{cut:} \ \frac{*_{\varphi''}(\forall x_1, \dots, \forall x_n : A), \Lambda' \qquad *_{\varphi'}(A[x_1/s_1, \dots, x_n/s_n]), \Lambda}{\Lambda', \Lambda}$$

  with $\Lambda', \Lambda = \Lambda$ since $\Lambda' \subseteq \Lambda$. The crucial detail here is that $A[x_1/s_1, \dots, x_n/s_n]$ is an $\exists$-component of $\exists x_1, \dots, \exists x_n : A$. Therefore, the cuts in the transformed proof have lower complexity than the ones in our original proof.

For the size bounds, we remark the following: In the construction of the proof π' from π, for every inference δ of π and every $\exists$-path to an axiom, we added at most one inference to π', with all inferences belonging to the same $\exists$-path being on the same branch. Therefore, $h(\pi') \leq |\pi|$. We can now canonically transform π' into a tree-like proof π''. Since in proof-trees (in our system) it holds that for every node v the degree $d(v)$ is ≤ 2, we have that $|\pi''| \leq 2^{h(\pi')} - 1 \leq 2^{|\pi|}$. Moreover, we have that $h(\pi'') = h(\pi') \leq |\pi|$. Again, since π is a tree with degree ≤ 2, it follows that $|\pi| \leq 2^{h(\pi)} - 1$ and therefore $h(\pi'') \leq 2^{h(\pi)}$. ■

We can now apply this theorem iteratively the following corollary:

Corollary 4.20. *Let π be a proof of the cedent Γ in $\boldsymbol{LK}_{PNF}$ with $\operatorname{aqd}(\pi) > 0$. Then there is a proof π' of Γ such that all the cut formulas of π' are quantifier-free and $|\pi'| \leq 2^{|\pi|}_{\operatorname{aqd}(\pi)}$ and $h(\pi') \leq 2^{h(\pi)}_{\operatorname{aqd}(\pi)}$.*

For the next few corollaries, we need a Lemma:

Lemma 4.21. *Let $n, k \in \mathbb{N}$ with $n, k \geq 1$. Then $\frac{5}{2} \cdot 2^k_n \leq 2^{3k}_n$*

Proof. By induction: For $n = 1$, we have that $\frac{5}{2} \cdot 2^k = 2^{\log(5/2)+k} \leq 2^{3k}$ since $k \geq 1$ and $\log(5) \leq 2$. For the induction step, we have that $\frac{5}{2} \cdot 2^k_{n+1} = 2^{\log(5/2)+2^k_n} \leq 2^{2+2^k_n} \leq 2^{2 \cdot 2^k_n} \leq 2^{2^{3k}_n} = 2^{3k}_{n+1}$. ■

We can now combine this corollary with Theorem 3.23:

Corollary 4.22. *Let π be a proof of Γ in $\boldsymbol{LK}_{PNF}$. Then there is a cut-free proof π' of Γ with $|\pi'| \leq 2^{3|\pi|}_{\mathrm{aqd}(\pi)+1}$.*

Proof. For $\mathrm{aqd}(\pi) = 0$, the claim follows directly from Theorem 3.23. For $\mathrm{aqd}(\pi) > 0$, we can use Theorem 4.19 and transform π into a proof π'' such that $\mathrm{aqd}(\pi'') = 0$ and $|\pi''| \leq 2^{|\pi|}_{\mathrm{aqd}(\pi)}$. Now, by Theorem 3.23 we receive a proof π' such that $|\pi'| \leq 2^{\frac{5}{2}|\pi''|}$. By Lemma 4.21 we have that $\frac{5}{2}|\pi''| = \frac{5}{2} \cdot 2^{|\pi|}_{\mathrm{aqd}(\pi)} \leq 2^{3|\pi|}_{\mathrm{aqd}(\pi)}$ and therefore $|\pi'| \leq 2^{3|\pi|}_{\mathrm{aqd}(\pi)+1}$. ■

The main result takes into account that we can transform proofs from **LK** to proofs of $\mathbf{LK}_{PNF}$:

Corollary 4.23. *Let π by any proof of Γ in **LK**. Then there is a cut-free proof π' of Γ in **LK** with $|\pi'| \leq 2^{15|\pi|^2}_{\mathrm{aqd}(\pi)+1}$*

Proof. By Lemma 4.6, we obtain a proof π'' of Γ in PNF with $|\pi''| \leq 5|\pi|^2$ and by Corollary 4.22, we now get a proof π' with $|\pi'| \leq 2^{3|\pi''|}_{aqd(\pi')+1}$. In Lemma 4.2, we have established that Σ_i membership is invariant under PNF transformation and therefore $\mathrm{aqd}(\pi') = \mathrm{aqd}(\pi)$. Thus $|\pi'| \leq 2^{15|\pi|^2}_{aqd(\pi)+1}$. ■

We can also obtain an upper bound for the size of expansion trees:

Corollary 4.24. *Let π be a proof in **LK** of Γ. Then there is an expansion tree E with $\vdash_E \Gamma$ and $|E| \leq 2^{7|\pi|^2}_{\mathrm{aqd}(\pi)}$.*

Proof. By Lemma 4.6, we obtain a proof π' of Γ in $\mathbf{LK}_{PNF}$ with $|\pi'| \leq 5|\pi|^2$. By Corollary 4.20, we have a proof π'' with only quantifier-free cuts and $|\pi''| \leq 2^{|\pi'|}_{\mathrm{aqd}(\pi')} = 2^{5|\pi|^2}_{\mathrm{aqd}(\pi)}$. Lastly by Theorem 3.11, we obtain an expansion tree proof E of Γ with $|E| \leq \frac{5}{2}|\pi''| = \frac{5}{2} 2^{5|\pi|^2}_{\mathrm{aqd}(\pi)} \leq 2^{5|\pi|^2+2}_{\mathrm{aqd}(\pi)} \leq 2^{7|\pi|^2}_{\mathrm{aqd}(\pi)}$. ■

5 Lower Bounds

Having established upper bounds for the complexity of cut elimination, the natural next step is to inspect whether those upper bounds are tight or if there is space for improvement. Usually lower bounds are more difficult to prove since we cannot consider a single algorithm, but have to show that there is no better algorithm (at least for some instance). Therefore, we will heavily rely on results that have already been proven in the past. The most important sources for this are [6] and [3].

First we specifiy a language and an axiom-system

Definition 5.1. *We define a language consisting of the function symbols* $+, 2^{(\cdot)}$, *the constants* $0, 1$ *and the predicates* $I, =$. *For the axioms*[1], *we start with the* $=$, *which is interpreted as the regular equality:*

$$\forall x : x = x \tag{5.1}$$
$$\forall x \forall y : x = y \to y = x \tag{5.2}$$
$$\forall x \forall y \forall z : x = y \land y = z \to x = z \tag{5.3}$$
$$\forall x \forall y : x = y \to 2^x = 2^y \tag{5.4}$$
$$\forall x \forall y \forall x' \forall y' : x = y \land x' = y' \to x + x' = y + y' \tag{5.5}$$
$$\forall x \forall y : x = y \to (I(x) \to I(y)) \tag{5.6}$$

For the function symbols we add the following axioms:

$$\forall x : x + 0 = x \tag{5.7}$$
$$\forall x \forall y \forall z : x + (y + z) = (x + y) + z \tag{5.8}$$
$$\forall x \forall y : x + y = y + x \tag{5.9}$$
$$2^0 = 1 \tag{5.10}$$
$$\forall x : 2^{x+1} = 2^x + 2^x \tag{5.11}$$

Finally, for the predicate symbol I, *we add the following axioms:*

$$I(0) \tag{5.12}$$
$$\forall x : I(x) \to I(x + 1) \tag{5.13}$$

$\mathcal{A}$ *now describes the set of our axioms and* $\overline{\mathcal{A}}$ *is the set* $\{\overline{A} \mid A \in \mathcal{A}\}$.

We will write terms like 2_i, but this is not a new function-symbol in our language, but will only be a shorthand for the numeral $2_i = 2_i^0$. Moreover, we will write sequents like $\Gamma \vdash \Delta$

[1] We use symbols like $\to$ for the sake of readability but have in mind that these are just shorthands and have to be translated to our language.

as a better readable shorthand for cedents of the form $\{\overline{\gamma} \mid \gamma \in \Gamma\}, \Delta$. The inference-rules are adapted naturally. Now, we define some formulas with the free variable x:

$$\varphi_0(x) = I(x) \tag{5.14}$$

$$\varphi_{i+1}(x) = \forall y(\varphi_i(y) \to \varphi_i(2^x + y)) \tag{5.15}$$

Note that $\varphi_0(\underline{m})$ is valid for any numeral $\underline{m}$. Moreover, we define the following parameterized sentence:

$$\xi_n = \bigwedge_{A \in \mathcal{A}} A \to \varphi_0(2_n) \tag{5.16}$$

We start with the first property of φ_0 as in [6]:

Lemma 5.2. *For any numeral $\underline{m}$ and any cut-free proof π of $\mathcal{A} \vdash I(\underline{m})$, it holds that $|\pi| \geq m$.*

Proof. Take any cut-free proof the sequent $\mathcal{A} \vdash I(\underline{m})$. Since every formula is in PNF, there is a midsequent $S : \Gamma \vdash \Delta$ such that all the formulas Γ and Δ are quantifier-free and that all the inferences in π underneath S are quantifier-introductions and the inferences above S are propositional or weakening. Seeing that we only have $I(\underline{m})$ on the right side of the endsequent, it follows that $\Delta = I(\underline{m})$. Moreover, Γ consists of instances of axioms of $\mathcal{A}$. Our claim now is that there are at least m instances of the axiom $\forall x(I(x) \to I(x+1))$. To prove this, assume there are fewer instances of this axiom. Since we associate each numeral $\underline{n}$ with the number n, there has to be a numeral $\underline{n}$ such that $n < m$ and $I(\underline{n}) \to I(\underline{n}+1)$ is not in Γ. Then we construct the Model $\mathcal{M} = (\mathbb{N}, \Phi)$ with the usual interpretation $0, 1, +, 2^{(\cdot)}$ and we define $I^{\mathcal{M}} = \{0, \ldots, n\}$. Now, in this model, all the axioms that do not contain I (and therefore their instances) are true. Moreover, every instance of $\forall x(I(x) \to I(x+1))$ except for $I(\underline{n}) \to I(\underline{n}+1)$ is true. Therefore, all formulas of Γ are true. However, $I(\underline{m})$ is false since $m > n$. Thus, the sequent $\Gamma \vdash \Delta$ is not valid, which is a contradiction to the assumption that is appears in our (valid) proof. ∎

Corollary 5.3. *There is no $n \in \mathbb{N}$ and cut-free proof π of ξ_n with $|\pi| < 2_n$.*

Similarly to the Lemma above, we obtain the following result for expansion trees:

Lemma 5.4. *Let E be an expansion tree such that $\vdash_E \xi_n$. Then $|E| \geq 2_n$.*

Proof. Let E be an expansion tree with $\vdash_E \xi_n$ and assume that $|E| < 2_n$. By definition, it holds that $\mathrm{Dp}(E)$ is tautologous and that $E \succ \xi_n$ and thus $\mathrm{Sh}(E) \sqsubseteq \xi_n$. In particular, that means that of $\mathrm{Sh}(E)$ has the form $\bigvee_{A \in \mathcal{A}} \overline{B_A} \vee B$ with $B_A \sqsubseteq A$ and $B \sqsubseteq I(2_n)$. Since $\mathrm{Dp}(E) = \bigvee_{A \in \mathcal{A}} \mathrm{Dp}(\overline{B_A}) \vee \mathrm{Dp}(B)$ is tautologous and negated instances of the axioms of $\mathcal{A}$ alone are not tautologous, it follows that $B = I(2_n)$. By assumption, we have that $|E| < 2_n$ and therefore, there cannot be 2_n instances of the axiom $\forall x(I(x) \to I(x+1))$ and there is some $m < 2_n$ such that $I(\underline{m}) \to I(\underline{m}+1)$ does not appear in the respective $\mathrm{Dp}(B_A)$. Thus, we can construct the model $\mathcal{M} = (\mathbb{N}, \Phi)$ with the usual interpretation $0, 1, +, 2^{(\cdot)}$ and we define $I^{\mathcal{M}} = \{0, \ldots, m\}$. Now, in this model, all the axioms that do not contain I (and therefore their instances) are true. Moreover, every instance of $\forall x(I(x) \to I(x+1))$

except for $I(\underline{m}) \to I(\underline{m}+1)$ is true. Therefore, all the formulas B_A are true and $\overline{B_A}$ are false. Moreover, $B = I(2_n)$ is false in this model and thus, the whole formula $\mathrm{Dp}(E)$ is false, which contradicts the assumption of it being tautologous. ■

The next goal is to give a short proof of ξ_n with cuts. For this, we need the following Lemma:

Lemma 5.5. *For every $i \in \mathbb{N}$, it holds that $\varphi_i(x)$ is in Π_i*

Proof. We proceed via induction. Since $I(x)$ is atomic, it follows that $\varphi_0(x)$ is in Π_0. For the induction step, we observe that $\overline{\varphi_i(a)} \vee \varphi_i(2^x + a)$ is in Π_{i+1} and thus $\varphi_{i+1}(x)$ is in Π_{i+1}. ■

Lemma 5.6. *For any $i \in \mathbb{N}$ and any term t it holds that $|\varphi_i(t)| \leq 2^{i+1}$.*

Proof. For $i = 0$, we have that $|\varphi_0(t)| = |I(t)| = 1 = 2^0$. For the induction step, we have $|\varphi_{i+1}(t) = |\forall y(\varphi_i(y) \to \varphi_{i+1}(2^t + y))| = 1 + |\varphi_i(y)| + 1 + |\varphi_i(y + 2^t)| \leq 2 + 2 \cdot 2^i = 2(2^i + 1) \leq 2^{i+2}$. ■

Lemma 5.7. *For any formula ψ there is a cut-free proof of $\overline{\psi}, \psi$ with length $\mathcal{O}(|\psi|)$. If $\psi = \chi_1 \vee \chi_2$, then there is also a cut-free proof of $\overline{\psi}, \chi_1, \chi_2$ with length $\mathcal{O}(|\psi|)$.*

Proof. For literals L, we obtain trivial proofs with length $1 = 2|L|$. For the induction step, we need a case distinction. Consider the formula ψ with $|\psi| = n + 1$:

i If $\psi = \forall x \chi$, then we have a proof π of $\overline{\chi}, \chi$ with $|\pi| \leq 2 \cdot n$. With two more inferences, we obtain $\exists x \overline{\chi}, \forall x \chi$, which is exactly the cedent $\overline{\psi}, \psi$ and therefore, we have a proof of $\overline{\psi}, \psi$ with length in $2 \cdot (n + 1)$.

ii If the outermost connective is an $\exists$-quantifier, the case is identical to the $\forall$-quantifier.

iii If $\psi = \chi_1 \wedge \chi_2$, we have two proofs π_1 and π_2 of $\overline{\chi_1}, \chi_1$ and $\overline{\chi_2}, \chi_2$ with $|\pi_i| \leq 2|\chi_i|$. Now, we can write the following proof:

$$\dfrac{\dfrac{\begin{matrix}(\pi_1) \\ \overline{\chi_1}, \chi_1\end{matrix} \qquad \begin{matrix}(\pi_2) \\ \overline{\chi_2}, \chi_2\end{matrix}}{\overline{\chi_1}, \overline{\chi_2}, \chi_1 \wedge \chi_2}\ \wedge\text{:}}{\overline{\chi_1} \vee \overline{\chi_2}, \chi_1 \wedge \chi_2}\ \vee\text{:}$$

However, this is exactly the proof π of $\overline{\psi}, \psi$ with $|\pi| \leq 2|\chi_1| + 2|\chi_2| + 2 = 2(|\chi_1| + |\chi_2| + 1) = 2|\psi|$.

iv The case that $\psi = \chi_1 \vee \chi_2$ is analogous to the case of $\wedge$.

Therefore, we have a proof of any ψ with length $\leq 2|\psi| = \mathcal{O}(|\psi|)$. If $\psi = \chi_1 \vee \chi_2$, we obtain the proof from the fourth induction step and omitting the last inference. ■

Corollary 5.8. *For any $i \in \mathbb{N}$ and any term t there is a cut-free proof of $\varphi_i(t), \overline{\varphi_i(t)}$ with length $\mathcal{O}(2^i)$.*

In our proof we will need the sequent $\mathcal{A} \vdash \varphi_i(2^{b+1} + a), \overline{\varphi_i(2^b + 2^b + a)}$. This sequent is not only provable in our theory, but we can also derive it with a relatively short proof:

Lemma 5.9. *For any $i \in \mathbb{N}$ there is a proof π_i of $\mathcal{A} \vdash \varphi_i(2^{b+2^0} + a), \overline{\varphi_i(2^b + 2^b + a)}$ with $|\pi_i| = \mathcal{O}(2^i)$. Furthermore, it holds that* $\operatorname{aqd}(\pi_i) = 0$.

Proof. First, we fix some $i \in \mathbb{N}$. Now we take any $k \leq i$ and define the terms t_k and t'_k inductively:

$$t_0 = 2^{b+2^0} + a \tag{5.17}$$
$$t'_0 = 2^b + 2^b + a \tag{5.18}$$
$$t_{k+1} = 2^{t_k} + a_{k+1} \tag{5.19}$$
$$t'_{k+1} = 2^{t'_k} + a_{k+1} \tag{5.20}$$

Note that we can give proofs of $\mathcal{A} \vdash t_k = t'_k$ with size linear in k. This is done with induction. Since they are instances of axioms, we have proofs in constant size of the following sequents:

$$\mathcal{A} \vdash 2^0 = 1 \tag{5.21}$$
$$\mathcal{A} \vdash b = b \tag{5.22}$$
$$\mathcal{A} \vdash \overline{2^0 = 1 \land b = b}, b + 2^0 = b + 1 \tag{5.23}$$
$$\mathcal{A} \vdash \overline{b + 2^0 = b + 1}, 2^{b+1} = 2^{b+2^0} \tag{5.24}$$
$$\mathcal{A} \vdash 2^b + 2^b = 2^{b+1} \tag{5.25}$$
$$\mathcal{A} \vdash \overline{2^b + 2^b = 2^{b+1} \land 2^{b+1} = 2^{b+2^0}}, 2^b + 2^b = 2^{b+2^0} \tag{5.26}$$
$$\mathcal{A} \vdash a = a \tag{5.27}$$
$$\mathcal{A} \vdash \overline{2^{b+2^0} = 2^b + 2^b \land a = a}, 2^{b+2^0} + a = 2^b + 2^b + a \tag{5.28}$$

By some cuts and $\land$-inferences, we obtain the proof of $\mathcal{A} \vdash t_0 = t'_0$. Now the induction step: Since they are again instances of axioms, we have proofs of the sequents

$$\mathcal{A} \vdash \overline{t_k = t'_k}, 2^{t_k} = 2^{t'_k} \tag{5.29}$$
$$\mathcal{A} \vdash a_{k+1} = a_{k+1} \tag{5.30}$$
$$\mathcal{A} \vdash \overline{2^{t_k} = 2^{t'_k} \land a_{k+1} = a_{k+1}}, 2^{t_k} + a_{t+1} = 2^{t'_k} + a_{t+1} \tag{5.31}$$

By the induction hypothesis, one $\land$ inference and two cuts, we obtain the desired proofs. Since we only add a constant number of inferences, the length of the proof is linear in k. Moreover, since we can prove $t_k = t'_k$ and we have a constant-size proof of the following instance of an axiom

$$\mathcal{A} \vdash \overline{t_k = t'_k}, \overline{I(t'_k)}, I(t_k), \tag{5.32}$$

we can cut once again and obtain a proof of

$$\mathcal{A} \vdash \overline{I(t'_k)}, I(t_k) \tag{5.33}$$

These proofs are still linear in k. Our induction hypothesis now is that we have a proof

$\pi_{n,k}$ of

$$\mathcal{A} \vdash \overline{\varphi_n(t'_k)}, \varphi_n(t_k) \text{ for some } n < i \text{ and all } k \leq i - n \tag{5.34}$$

with $|\pi_{n,k}| = \mathcal{O}(2^n + k)$, which we have proven above for $n = 0$. For the induction step, we consider the following proof for any $k \leq i - n - 1$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}(\lambda) \\ \mathcal{A} \vdash \overline{\varphi_n(a_{k+1})}, \varphi_n(a_{k+1})\end{matrix} \qquad \begin{matrix}(\pi_{n,k+1}) \\ \mathcal{A} \vdash \overline{\varphi_n(2^{t'_k} + a_{k+1})}, \varphi_n(2^{t_k} + a_{k+1})\end{matrix}}{\mathcal{A} \vdash \varphi_n(a_{k+1}) \wedge \overline{\varphi_n(2^{t'_k} + a_{k+1})}, \overline{\varphi_n(a_{k+1})}, \varphi_n(2^{t_k} + a_{k+1})}\ \wedge}{\mathcal{A} \vdash \varphi_n(a_{k+1}) \wedge \overline{\varphi_n(2^{t'_k} + a_{k+1})}, \overline{\varphi_n(a_{k+1})} \vee \varphi_n(2^{t_k} + a_{k+1})}\ \vee}{\mathcal{A} \vdash \overline{\varphi_{n+1}(t'_k)}, \overline{\varphi_n(2^{t_k})} \vee \varphi_n(2^{t_k} + a_{k+1})}\ \exists}{\mathcal{A} \vdash \overline{\varphi_{n+1}(t'_k)}, \varphi_{n+1}(t_k)}\ \forall$$

Note that λ is the proof from Corollary 5.8 with size $\mathcal{O}(2^n)$. Since we add a constant number of inferences underneath λ and $\pi_{n,k}$, we have that $|\pi_{n+1,k}| = \mathcal{O}(2^n + 2^n + k) = \mathcal{O}(2^n + k) \leq \mathcal{O}(2^i)$ since $n, k \leq i$. If we set $n = i$ and $k = 0$, we have provided a proof of

$$\mathcal{A} \vdash \varphi_i(2^{b+2^0} + a), \overline{\varphi_i(2^b + 2^b + a)}.$$

Moreover, all the cut formulas were quantifier-free and therefore, the aqd of the proof in question is 0. This concludes the proof. ■

Similarly, we obtain the following result:

Lemma 5.10. *For any $i \in \mathbb{N}$ and any term t there is a proof $\pi_{i,t}$ of the sequent*

$$\mathcal{A} \vdash \overline{\varphi_i(t+0)}, \varphi_i(t)$$

with $|\pi_{i,t}| = \mathcal{O}(2^i)$ and $\mathrm{aqd}(\pi_{i,t}) = 0$.

Proof. We define the following sequence of terms t_k for all $k \leq i$:

$$t_0 = t \tag{5.35}$$
$$t'_0 = t + 0 \tag{5.36}$$
$$t_{k+1} = 2^{t_k} + a_{k+1} \tag{5.37}$$
$$t'_{k+1} = 2^{t'_k} + a_{k+1} \tag{5.38}$$

Since they are instances of axioms, we obtain proofs of the following sequents:

$$\mathcal{A} \vdash t_0 = t'_0 \tag{5.39}$$
$$\mathcal{A} \vdash a_k = a_k \text{ for any } k \tag{5.40}$$
$$\mathcal{A} \vdash \overline{t_k = t'_k \wedge a_k = a_k}, 2^{t_k} + a_k = 2^{t'_k} + a_k \text{ for any } k \tag{5.41}$$

By an $\wedge$-inference and some cuts, we obtain a proof of $\mathcal{A} \vdash t_k = t'_k$ for any k. Again, we have the axiom-instance $\mathcal{A} \vdash \overline{t_i = t'_i}, \overline{I(t'_i)}, I(t_i)$ and by cutting, we obtain a proof of $\mathcal{A} \vdash$

$\overline{I(t'_i)}, I(t_i)$. The induction hypothesis now is that we have a proof of $\mathcal{A} \vdash \overline{\varphi_k(t'_{i-k})}, \varphi_k(t_{i-k})$. For the induction step, we consider the following:

$$
\forall\colon\ \dfrac{\vee\colon\ \dfrac{\exists\colon\ \dfrac{\wedge\colon\ \dfrac{\overset{(\lambda)}{\mathcal{A} \vdash \overline{\varphi_k(a_{i-(k+1)})}\varphi_k(a_{i-(k+1)})} \qquad \overset{(\lambda')}{\mathcal{A} \vdash \overline{\varphi_k(2^{t'_{i-(k+1)}} + a_{i-(k+1)})}, \varphi_k(2^{t_{i-(k+1)}} + a_{i-(k+1)})}}{\mathcal{A} \vdash \varphi_k(a_{i-(k+1)}) \wedge \overline{\varphi_k(2^{t'_{i-(k+1)}} + a_{i-(k+1)})}, \varphi_k(2^{t_{i-(k+1)}} + a_{i-(k+1)}), \overline{\varphi_k(a_{i-(k+1)})}}}{\mathcal{A} \vdash \overline{\varphi_{k+1}(t'_{i-(k+1)})}, \varphi_k(2^{t_{i-(k+1)}} + a_{i-(k+1)}), \overline{\varphi_k(a_{i-(k+1)})}}}{\mathcal{A} \vdash \overline{\varphi_{k+1}(t'_{i-(k+1)})}, \varphi_k(2^{t_{i-(k+1)}} + a_{i-(k+1)}) \vee \overline{\varphi_k(a_{i-(k+1)})}}}{\mathcal{A} \vdash \overline{\varphi_{k+1}(t'_{i-(k+1)})}, \varphi_{k+1}(t_{i-(k+1)})}
$$

For $k = i$ this yields a proof of $\mathcal{A} \vdash \overline{\varphi_i(t'_0}, \varphi_i(t_0)$. The bounds of size of the proof follow from the same reasons as in the Lemmata above. ■

Lemma 5.11. *For any $i \in \mathbb{N}$, there is proof π_i of the sequent*

$$\mathcal{A} \vdash \varphi_i(0)$$

with $|\pi_i| = \mathcal{O}(2^i)$ and $\mathrm{aqd}(\pi_i) = 0$.

Proof. First, note that $\varphi_0(0) = I(0)$ is an axiom. Moreover, we can construct a proof of $\varphi_1(0) = \forall y(I(y) \to I(2^0 + y))$. However, we have to do this in several steps. Since they are only instances of axioms and as a consequence of Lemma 5.7, there are cut-free proofs of the following sequents:

$$\mathcal{A} \vdash 2^0 = 1 \tag{5.42}$$
$$\mathcal{A} \vdash b = b \tag{5.43}$$
$$\mathcal{A} \vdash \overline{b = b \wedge 2^0 = 1}, 2^0 + b = 1 + b \tag{5.44}$$
$$\mathcal{A} \vdash \overline{1 + b = 2^0 + b}, b + 1 = 2^0 + b \tag{5.45}$$
$$\mathcal{A} \vdash \overline{b + 1 = 2^0}, \overline{I(b+1)}, I(2^0 + b) \tag{5.46}$$

By using one $\wedge$-inference and several atomic cuts, we obtain a proof λ of $\mathcal{A} \vdash \overline{I(b+1)}, I(2^0 + b)$. By Lemma 5.7 there is a proof λ' of $\overline{I(b) \to I(b+1)}, \overline{I(b)}, I(b+1)$. This yields:

$$
\forall\colon\ \dfrac{\vee\colon\ \dfrac{\text{cut}\colon\ \dfrac{\text{w}\colon\ \dfrac{\exists\colon\ \dfrac{\overset{(\lambda')}{\overline{I(b) \to I(b+1)}, \overline{I(b)}, I(b+1)}}{\overline{\exists x I(x) \to I(x+1)}, \overline{I(b)}, I(b+1)}}{\mathcal{A} \vdash \overline{I(b)}, I(b+1)} \qquad \overset{(\lambda)}{\mathcal{A} \vdash \overline{I(b+1)}, I(2^0 + b)}}{\mathcal{A} \vdash \overline{I(b)}, I(2^0 + b)}}{\mathcal{A} \vdash I(b) \to I(2^0 + b)}}{\mathcal{A} \vdash \forall y(I(y) \to I(2^0 + y))}
$$

Therefore, we have a proof of $\varphi_1(0)$ with atomic cuts only.

Now, assume that $i \geq 2$. Then, by Lemma 5.7 and Lemma 5.9 we have the following proofs with size $\mathcal{O}(2^{i-2}) = \mathcal{O}(2^i)$ and aqd ≤ 0:

- λ_0 of $\overline{\varphi_{i-2}(2^b + a)}, \varphi_{i-2}(2^b + a)$
- λ_1 of $\mathcal{A} \vdash \overline{\varphi_{i-2}(2^{b+2^0} + a)}, \varphi_{i-2}(2^{b+2^0} + a)$
- λ_2 of $\overline{\varphi_{i-2}(a)}, \varphi_{i-2}(a)$

We can now construct the following proof:

$$
\begin{prooftree}
\AxiomC{(λ_2)}
\noLine
\UnaryInfC{$\varphi_{i-2}(a), \overline{\varphi_{i-2}(a)}$}
\AxiomC{(λ_0)}
\noLine
\UnaryInfC{$\varphi_{i-2}(2^b + a), \overline{\varphi_{i-2}(2^b + a)}$}
\LeftLabel{$\wedge$:}
\BinaryInfC{$\varphi_{i-2}(a) \wedge \overline{\varphi_{i-1}(2^b + a)}, \overline{\varphi_{i-2}(a)}, \varphi_{i-2}(2^b + a)$}
\LeftLabel{$\exists$:}
\UnaryInfC{$\overline{\varphi_{i-1}(b)}, \overline{\varphi_{i-2}(a)}, \varphi_{i-2}(2^b + a)$}
\AxiomC{(λ_1)}
\noLine
\UnaryInfC{$\overline{\mathcal{A}}, \varphi_{i-2}(2^{b+2^0} + a), \overline{\varphi_{i-2}(2^b + 2^b + a)}$}
\LeftLabel{$\wedge$:}
\BinaryInfC{$\overline{\mathcal{A}}, \overline{\varphi_{i-1}(b)}, \overline{\varphi_{i-2}(a)}, \varphi_{i-2}(2^b + a) \wedge \overline{\varphi_{i-2}(2^b + 2^b + a)}, \varphi_{i-2}(2^{b+2^0} + a)$}
\LeftLabel{$\exists$:}
\UnaryInfC{$\overline{\mathcal{A}}, \overline{\varphi_{i-1}(b)}, \overline{\varphi_{i-2}(a)}, \varphi_{i-2}(2^{b+2^0} + a)$}
\LeftLabel{$\vee$:}
\UnaryInfC{$\overline{\mathcal{A}}, \overline{\varphi_{i-1}(b)}, \overline{\varphi_{i-2}(a)} \vee \varphi_{i-2}(2^{b+2^0} + a)$}
\LeftLabel{$\forall$:}
\UnaryInfC{$\overline{\mathcal{A}}, \overline{\varphi_{i-1}(b)}, \varphi_{i-1}(b + 2^0)$}
\LeftLabel{$\vee$:}
\UnaryInfC{$\overline{\mathcal{A}}, \overline{\varphi_{i-1}(b)} \vee \varphi_{i-1}(b + 2^0)$}
\LeftLabel{$\forall$:}
\UnaryInfC{$\overline{\mathcal{A}}, \varphi_i(0)$}
\end{prooftree}
$$

Note that we only add a constant number of inferences underneath $\lambda_0, \lambda_1, \lambda_2$ (independent of i) and obtain therefore a proof with size $\mathcal{O}(2^i + 2^i + 2^i) = \mathcal{O}(2^i)$. Since there is no cut in the new inferences and the sub-proofs λ_i all are cut free or have aqd 0, the new proof also has aqd 0. ■

Lemma 5.12. *For every $n \geq 1$ there is a proof π^* of the sequent $\mathcal{A} \vdash \overline{\varphi_{n-1}(0)}, \varphi_{n-1}(2_1)$ with $|\pi^*| = \mathcal{O}(2^n)$ and $\mathrm{aqd}(\pi^*) = n - 1$.*

Proof. For $n = 1$ we accept the fact that there is some proof by the completeness theorem. Without loss of generality, we therefore assume that $n \geq 2$ and recycle the strategy from Lemma 5.11: By Lemma 5.7 and Lemma 5.9 we have the following proofs with size $\mathcal{O}(2^i)$ and aqd ≤ 0:

- λ_0 of $\overline{\varphi_{n-2}(2^b + a)}, \varphi_{n-2}(2^b + a)$
- λ_1 of $\mathcal{A} \vdash \overline{\varphi_{n-2}(2^{b+2^0} + a)}, \varphi_{n-2}(2^{b+2^0} + a)$
- λ_2 of $\overline{\varphi_{n-2}(a)}, \varphi_{n-2}(a)$

This leads to the following proof:

$$
\begin{array}{l}
\wedge: \dfrac{\overset{(\lambda_2)}{\varphi_{n-2}(a), \overline{\varphi_{n-2}(a)}} \qquad \overset{(\lambda_0)}{\varphi_{n-2}(2^b+a), \overline{\varphi_{n-2}(2^b+a)}}}{\varphi_{n-2}(a) \wedge \overline{\varphi_{n-1}(2^b+a)}, \overline{\varphi_{n-2}(a)}, \varphi_{n-2}(2^b+a)} \\
\exists: \dfrac{\varphi_{n-2}(a) \wedge \overline{\varphi_{n-1}(2^b+a)}, \overline{\varphi_{n-2}(a)}, \varphi_{n-2}(2^b+a)}{\overline{\varphi_{n-1}(b)}, \overline{\varphi_{n-2}(a)}, \varphi_{n-2}(2^b+a)} \\
\wedge: \dfrac{\overline{\varphi_{n-1}(b)}, \overline{\varphi_{n-2}(a)}, \varphi_{n-2}(2^b+a) \qquad \overset{(\lambda_1)}{\overline{\mathcal{A}}, \varphi_{n-2}(2^{b+2^0}+a), \overline{\varphi_{n-2}(2^b+2^b+a)}}}{\overline{\mathcal{A}}, \overline{\varphi_{n-1}(b)}, \overline{\varphi_{n-2}(a)}, \varphi_{n-2}(2^b+a) \wedge \overline{\varphi_{n-2}(2^b+2^b+a)}, \varphi_{n-2}(2^{b+2^0}+a)} \\
\exists: \dfrac{\overline{\mathcal{A}}, \overline{\varphi_{n-1}(b)}, \overline{\varphi_{n-2}(a)}, \varphi_{n-2}(2^b+a) \wedge \overline{\varphi_{n-2}(2^b+2^b+a)}, \varphi_{n-2}(2^{b+2^0}+a)}{\overline{\mathcal{A}}, \overline{\varphi_{n-1}(b)}, \overline{\varphi_{n-2}(a)}, \varphi_{n-2}(2^{b+2^0}+a)} \\
\vee: \dfrac{\overline{\mathcal{A}}, \overline{\varphi_{n-1}(b)}, \overline{\varphi_{n-2}(a)}, \varphi_{n-2}(2^{b+2^0}+a)}{\overline{\mathcal{A}}, \overline{\varphi_{n-1}(b)}, \overline{\varphi_{n-2}(a)} \vee \varphi_{n-2}(2^{b+2^0}+a)} \\
\forall: \dfrac{\overline{\mathcal{A}}, \overline{\varphi_{n-1}(b)}, \overline{\varphi_{n-2}(a)} \vee \varphi_{n-2}(2^{b+2^0}+a)}{\overline{\mathcal{A}}, \overline{\varphi_{n-1}(b)}, \varphi_{n-1}(b+2^0)}
\end{array}
$$

Note that by Definition 2.1 2^0 is exactly the term for 2_1. Moreover, this proof holds even if we replace b with a specific value. If we set b to 0, we have a proof of $\mathcal{A} \vdash \overline{\varphi_{n-1}(0)}, \varphi_{n-1}(0+2_1)$. By Lemma 5.10, we also have a proof of $\mathcal{A} \vdash \overline{\varphi_{n-1}(0+2^0)}, \varphi_{n-1}(2^0)$. Cutting yields the proof of $\mathcal{A} \vdash \overline{\varphi_{n-1}(0)}, \varphi_{n-1}(2_1)$. ■

Lemma 5.13. *For every $i \in \mathbb{N}$ and $n \geq i$ there is a proof π_i' of the sequent*

$$\mathcal{A} \vdash \varphi_{i+1}(2_{n-(i+1)}), \varphi_i(0) \vdash \varphi_i(2_{n-i})$$

with $|\pi_i'| = \mathcal{O}(2^i)$ and $\mathrm{aqd}(\pi_i') = 0$.

Proof. By Corollary 5.8 and by Lemma 5.10 we have the following proofs with size $\mathcal{O}(2^i)$ with aqd ≤ 0:

- λ of $\overline{\varphi_i(0)}, \varphi_i(0)$
- λ' of $\mathcal{A} \vdash \overline{\varphi_i(2^{2_{n-i+1}}+0)}, \varphi_i(2_{n-i})$

We construct the following proof:

$$
\begin{array}{l}
\wedge: \dfrac{\overset{(\lambda)}{\overline{\varphi_i(0)}, \varphi_i(0)} \qquad \overset{(\lambda')}{\overline{\varphi_i(2^{2_{n-i+1}}+0)}, \varphi_i(2_{n-i})}}{\varphi_i(0) \wedge \overline{\varphi_i(2^{2_{n-i+1}})}, \overline{\varphi_i(0)}, \varphi_i(2_{n-i})} \\
\exists: \dfrac{\varphi_i(0) \wedge \overline{\varphi_i(2^{2_{n-i+1}})}, \overline{\varphi_i(0)}, \varphi_i(2_{n-i})}{\overline{\varphi_{i+1}(2_{n-(i+1)})}, \overline{\varphi_i(0)}, \varphi_i(2_{n-i})}
\end{array}
$$

■

These lemmata together now lead to the following theorem:

Theorem 5.14. *For every $n \in \mathbb{N}$ there is a proof π of $\varphi_0(2_n)$ with $|\pi| = \mathcal{O}(2^{2n})$ and $\mathrm{aqd}(\pi) = n - 1$.*

Proof. We proceed via induction and let i run from $n-1$ to 0. We will construct proofs of $\varphi_i(2_{n-i})$ with length $\mathcal{O}((n-i)2^n)$. By setting i to 0, we then have a proof of $\varphi_0(2_n)$ with length $\mathcal{O}(n \cdot 2^n) \leq \mathcal{O}(2^n \cdot 2^n) = \mathcal{O}(2^{2n})$. From the Lemmata 5.11,5.12 and 5.13 we obtain the following proofs in length $\mathcal{O}(2^i)$ and aqd ≤ 0:

- π_i of $\mathcal{A} \vdash \varphi_i(0)$

- π_i' of $\varphi_{i+1}(2_{n-(i+1)}), \varphi_i(0) \vdash \varphi_i(2_{n-i})$
- π^* of $\mathcal{A} \vdash \overline{\varphi_{n-1}(0)}, \varphi_{n-1}(2_1)$

For the base case, we consider the following proof:

$$\text{cut: } \dfrac{\begin{matrix}(\pi_{n-1}) \\ \mathcal{A} \vdash \varphi_{n-1}(0)\end{matrix} \qquad \begin{matrix}(\pi^*) \\ \mathcal{A} \vdash \overline{\varphi_{n-1}(0)}, \varphi_{n-1}(2_1)\end{matrix}}{\mathcal{A} \vdash \varphi_{n-1}(2_1)}$$

The size restriction clearly holds. We call the proof from the induction hypothesis λ and consider the following proof for the induction step:

$$\text{cut: } \dfrac{\begin{matrix}(\pi_i) \\ \mathcal{A} \vdash \varphi_i(0)\end{matrix} \qquad \text{cut: } \dfrac{\begin{matrix}(\lambda) \\ \mathcal{A} \vdash \varphi_{i+1}(2_{n-(i+1)})\end{matrix} \qquad \begin{matrix}(\pi_i') \\ \varphi_{i+1}(2_{n-(i+1)}), \varphi_i(0) \vdash \varphi_i(2_{n-i})\end{matrix}}{\mathcal{A}, \varphi_i(0) \vdash \varphi_i(2_{n-i})}}{\mathcal{A} \vdash \varphi_i(2_{n-i})}$$

By the induction hypothesis, we have that $|\lambda| = \mathcal{O}((n-(i+1))2^n)$ and therefore, the new proof has size in $\mathcal{O}((n-(i+1))2^n + 2^n + 2^n) = \mathcal{O}((n-i)2^n + 2^n) = \mathcal{O}((n-i)2^n)$. Moreover, note that all the cut-inferences used some $\varphi_i(m)$ with $i \leq n-1$ as cut formulas. Therefore, the aqd is $n-1$. ∎

If we compare this to Corollary 4.24, we see that our upper bound for the expansion-proofs of ξ_n is $2_{n-1}^{7 \cdot 2^{4n}} \leq 2_n^{7n}$. Summarising this chapter leads to the following result:

Corollary 5.15. *For the sequence ξ_n the following holds:*

- *There are proofs π_n of ξ_n with $|\pi_n| = \mathcal{O}(2^{2n})$ and $\mathrm{aqd}(\pi_n) = n-1$*
- *There are expansion-trees E_n with $\vdash_{E_n} \xi_n$ and $|E_n| = \mathcal{O}(2_n^{7n})$*
- *Any expansion-tree E_n' with $\vdash_{E_n'} \xi_n$ has length at least 2_n*

Unfortunately, $2_n = 2_n^0 = 2_{n-1}^1 = 2_{n-1}^{n^0}$. That means that our number of exponents is not yet optimal. Even worse, we have the following result:

Lemma 5.16. *For any $i \leq n$ it holds that $2_n = \mathcal{O}(2_{n-i}^n)$.*

Proof. We have that $2_n = 2_n^0 = 2_{n-i}^{2_i}$. Note that i is fixed and independent of n. Therefore, there is an $N \in \mathbb{N}$ such that for all $n \geq N$ it holds that $n \geq 2_i$. It follows that $2_{n-i}^n \geq 2_{n-i}^{2_i} = 2_n$ for all $n \geq N$. ∎

However, we have the following Theorem:

Theorem 5.17. *Let F be any function that maps a proof to a cut-free proof of the same end-sequent and define $F' : \mathbb{N} \to \mathbb{N} : n \mapsto |F(\xi_n)|$. Then the following holds: If any cut-free proof of ξ_n has length at least 2_n^n, then $F' = \Omega(2_n^n)$ and $F' > \mathcal{O}(2_{n-1}^{n^k})$ for any $k \in \mathbb{N}$.*

Proof. If every proof of ξ_n has length at least 2_n^n, then $F'(n) \geq 2_n^n$ and $F' = \Omega(2_n^n)$. For the fact that $F' > \mathcal{O}(2_{n-1}^{n^k})$, we show that there is an $N \in \mathbb{N}$ such that for all $n \geq N$ it holds that $2_n^n > 2_{n-1}^{n^k}$. We observe that $2_n^n = 2_{n-1}^{2^n}$. Since there is an $N \in \mathbb{N}$ such that for all $n \geq N$ it holds that $2^n > n^k$ and consequently $2_n^n = 2_{n-1}^{2^n} > 2_{n-1}^{n^k}$. ∎

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