Bachelor Thesis

# The Tutte Polynomial

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# Contents

# 1 Introduction

This thesis treats aspects of matroid theory — an area of study introduced by Hassler Whitney in 1935. Matroids are mathematical structures that generalize the concept of independence from linear algebra and have applications in numerous fields of combinatorics, particularly in combinatorial optimization and graph theory.

The primary focus of this thesis lies on the Tutte polynomial, a two-variable polynomial defined on matroids. To ensure a thorough exploration of the Tutte polynomial and also increase the accessibility of this thesis, we begin by discussing the fundamentals of matroid theory. This will be explored in Section 2. After going over the foundational definition of matroids as well as other important terminology, it will be introduced how new matroids can be created out of already existing ones through operations defined on matroids. Furthermore, the dynamic interplay between matroids and graphs will be explored, as well as the concept of duality for graphs and matroids.

In Section 3, our focus shifts to the central aspect of this thesis, the Tutte polynomial. After proving that it is well-defined through the corank-nullity polynomial, we proceed to showcase its diverse applications. The Tutte polynomial not only serves as a versatile tool for deriving graph theoretical results but also uncovers its role in various contexts, including Tutte-Grothendieck invariants. In this context, we also introduce the chromatic polynomial, initially proposed by Birkhoff. The Tutte polynomial, alongside the chromatic polynomial have emerged as crucial mathematical constructs, with applications extending far beyond their initial definitions. These polynomials capture essential characteristics of matroids and graphs and shed light on their intricate properties. Furthermore the flow polynomial will be introduced when talking about acyclic orientations and nowhere-zero flows.

This thesis mainly builds upon the foundational work presented in Gary Gordon and Jennifer McNulty's book *Matroids: A Geometric Introduction* [GM12], utilizing their terminology and notation for consistency and clarity. Additionally, it is worth noting that this thesis includes proofs for certain theorems that are not covered in [GM12] and even corrections. Throughout, the provided information and newly acquired knowledge are reinforced through the utilization of illustrative examples. While the material presented herein is approachable for those without prior knowledge of matroids, a basic understanding of linear algebra and graph theory concepts is recommended to fully appreciate the insights and connections that this paper provides.

# 2 Matroids

This chapter introduces the fundamentals of matroid theory, which are essential for understanding the central component of this thesis. Additionally, it explores the connection between graph theory and matroids. For a more in-depth understanding of the basics of matroids, refer to [GM12, Chapter 1 - 4] or the paper *What is a Matroid?* by James Oxley [Oxl03].

## 2.1 Defining a Matroid

There are many different definitions for a matroid, however, this thesis will be using Whitney's initial definition in terms of a generalized notion of independence. Consider the matrix $A$ with entries in an arbitrary field. Let $E$ be the set of the column vectors of $A$ and let $\mathcal{I}$ represent the collection of all subsets of $E$ that are linearly independent. The sets $E$ and $\mathcal{I}$ generate the pair $(E, \mathcal{I})$ which is a particular example of a matroid. The term "matroid" already suggests a structure which indicates a connection with matrices and indeed matroids were introduced by Hassler Whitney in 1935 as an abstract generalization of a matrix, where the Greek-derived suffix "-oid" means similar in shape or form.

**Definition 2.1.** Let $E$ be a finite set and $\mathcal{I}$ a family of subsets of $E$. The pair $M = (E, \mathcal{I})$ is called a *matroid* if $\mathcal{I}$ has the following properties:

(I1) $\mathcal{I} \neq \emptyset$,

(I2) if $J \in \mathcal{I} \wedge I \subseteq J$, then $I \in \mathcal{I}$,

(I3) if $I, J \in \mathcal{I}$ with $|I| < |J|$, then $\exists x \in J \setminus I : I \cup \{x\} \in \mathcal{I}$.

The collection of sets $\mathcal{I}$ forms the independent sets of the matroid $M$.

The third characteristic (I3) is also called augmentation. Note that since $\mathcal{I}$ is non-trivial and is closed under subsets the empty set $\emptyset$ is contained in $\mathcal{I}$.

**Definition 2.2.** Let $M = (E, \mathcal{I})$ be a matroid. Every $B \in \mathcal{I}$ which does not have an independent superset, meaning $\nexists A \in \mathcal{I} : B \subset A$, is called *basis* of the matroid $M$.

Considering two arbitrary bases $B_1$ and $B_2$ of the matroid $M = (E, \mathcal{I})$ with $|B_1| < |B_2|$, then according to the augmentation of the independent sets of $M$ there exists an

$$x \in B_2 \setminus B_1 : \{x\} \cup B_1 \in \mathcal{I}.$$

Since this contradicts the assumption that $B_1$ is a basis, it holds $|B_1| = |B_2|$. This shows that all bases of a matroid have the same cardinality.

**Definition 2.3.** Let $M = (E,\mathcal{I})$ be a matroid. The *rank* of a subset $A$ of $E$, written as $r(A)$, can be defined through the *rank function*:

$$r(A) = \begin{cases} \mathcal{P}(E) \to \mathbb{N} \cup \{0\} \\ A \mapsto \max_{I \subseteq A}\{|I| : I \in \mathcal{I}\} \end{cases}$$

The rank of a matroid $M = (E,\mathcal{I})$ is $r(E)$, representing the rank of the ground set $E$. For simplicity we will write $r(M)$ to refer to the rank of a matroid $M$. Through Definition 2.2 it becomes clear that the bases of a matroid are the maximal independent sets of the matroid. Therefore, the rank of a matroid is exactly the cardinality of its bases.

The previously defined matroid which arose from a matrix $A$, indeed satisfies the properties (I1) - (I3). Thus, the linear independence of columns of a matrix can always be captured by a matroid. Furthermore, the rank of the matrix $A$ is equal to the rank of its corresponding matroid. For an outline of the proof and additional reading, consult [GM12, Chapter 6.1].

Let $E = \{a, b\}$ be the ground set. Then there are exactly five matroids on $E$ with $\mathcal{I}$ being either $\{\emptyset\}, \{\emptyset, \{a\}\}, \{\emptyset, \{b\}\}, \{\emptyset, \{a\}, \{b\}\}$ or $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. By observing the matroids $M_2 = (E, \{\emptyset, \{a\}\})$ and $M_3 = (E, \{\emptyset, \{b\}\})$ one might see that both matroids not only share the same rank, but also have the same number of bases and independent sets. Both of these numbers are *matroid invariants*, functions that yield identical values on isomorphic matroids. These will be further discussed in Section 3, see Definition 3.4. In general one can say that the matroids $M_2$ and $M_3$ have the same structure. If that is the case these matroids are called *isomorphic*, denoted by $M_2 \cong M_3$. More specific, two matroids $M_1$ and $M_2$ with the ground sets $E_1$ and $E_2$ are isomorphic, if a bijection from $E_1$ to $E_2$ exists such that a set is independent in $M_1$ if and only if its image is independent in $M_2$.

Although Definition 2.1 is the most common way to define a matroid, there are many other equivalent ways to describe one.

**Definition 2.4.** Let $M$ be a matroid with the ground set $E$. If the subset $C$ of $E$ is dependent but every proper subset of $C$ is independent, then $C$ is called a *circuit* in $M$.

The family of subsets $\mathcal{C}$ is the *set of circuits* of a matroid if and only if it satisfies

(C1) $\emptyset \in C$,

(C2) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$ then $C_1 = C_2$,

(C3) If $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq C_2$, and $x \in C_1 \cap C_2$, then $C_3 \subseteq (C_1 \cup C_2) \setminus \{x\}$ for some $C_3 \in \mathcal{C}$.

Matroids can be characterized by their *circuits* $\mathcal{C}$, see Definition 2.4, especially because through $\mathcal{I}$ one can define $\mathcal{C}$,

$$\mathcal{C} = \{C \subseteq E \mid C \notin \mathcal{I} \wedge \text{ if } I \subset C \text{ then } I \in \mathcal{I}\} \subseteq \mathcal{P}(E),$$

as well as define $\mathcal{I}$ through $\mathcal{C}$,

$$\mathcal{I} = \{I \subseteq E \mid \text{if } C \in \mathcal{C} \text{ then } C \nsubseteq I\} \subseteq \mathcal{P}(E).$$

**Definition 2.5.** Let $E$ be the ground set of a matroid $M$.

(i) An element $e$ in $E$ which is in every basis, is called an *isthmus*.

(ii) An element $e$ in $E$ which is in no basis at all, is called a *loop*.

These two definitions are essential to define the Tutte polynomial. An interesting observation is that, based on property (I3), a loop is not in any independent set. This implies that loops are elements of $E$ dependent on themselves. Both of the terms loop and isthmus originate from graph theory and and will be specifically addressed in the context of graphs in Section 2.3.
A matroid which has only one element $e$ is sometimes in literature called a *loop* if $e$ is a loop, and an *isthmus* if $e$ is an isthmus.

The last essential term in matroid theory which will be defined in this thesis before continuing to operations for matroids are *spanning sets*. A subset $S \subseteq E$ is called a spanning set of a matroid $M$ if a basis $B$ of $M$ exists such that $B \subseteq S$.

**Definition 2.6.** Let $\mathcal{S}$ be a family of subsets of $E$ with $E$ being the groundset of the matroid $M$. $\mathcal{S}$ is called *spanning sets* if it satisfies:

(S1) $E \in \mathcal{S}$,

(S2) If $S_1 \in \mathcal{S}$ and $S_1 \subseteq S_2$ then $S_2 \in \mathcal{S}$,

(S3) If $S_1, S_2 \in \mathcal{S}$ and $|S_1| < |S_2|$, then there exists an element $x \in S_2 \setminus S_1$ so that $S_2 \setminus \{x\} \in \mathcal{S}$.

**Lemma 2.7.** *A matroid $M$ has exactly one spanning set, if and only if every element of $M$ is an isthmus. Furthermore, $M$ has exactly one independent set if and only if every element of $M$ is a loop.*

*Proof.* The proof follows from Definition 2.1, 2.5 and 2.6. □

As a brief comment on Lemma 2.7, for the matroid where every element is a loop, the only independent set is the empty set. Similarly, when all elements are isthmuses, the only spanning set is equal to the matroid's only basis and its ground set; furthermore, it holds $\mathcal{I} = \mathcal{P}(E)$.

## 2.2 New Matroids from existing ones

Like for most of the structures in mathematics, operations for matroids can be defined, which create new matroids from already existing ones. Although there are many operations, only three operations will be discussed in this thesis, deletion, contraction and duality. In this section the focus lies on deletion and contraction, both reducing a matroid by one element from the ground set. Duality will be discussed later on in Section 2.4.

**Definition 2.8.** Let $E$ be the ground set of a matroid $M$.

*Deletion* For $e \in E$, where $e$ is not an isthmus, the matroid $M - e$ is defined by the ground set $E \setminus \{e\}$ and the independent sets are all those in $M$ which do not include $e$.

*Contraction* For $e \in E$, where $e$ is not a loop, the matroid $M/e$ is defined by the ground set $E \setminus \{e\}$ and the independent sets are all those in $M$ which contain $e$, however $e$ will be removed from each of the sets.

If $e \in E$ is neither a loop nor an isthmus, the independent sets of $M - e$ and $M/e$ fulfill (I1)-(I3), thus both are matroids. Deletion and contraction split the independent sets of the matroid $M$ into two families, those which contain $e$ and those which do not. These operations also partition the bases of $M$ in the same way:

**Proposition 2.9.** *Let $E$ be the ground set of a matroid $M$.*

*(i) The bases of the matroid $M - e$, where $e \in E$ is not a isthmus, are exactly the bases of $M$ which do not contain $e$.*

*(ii) The bases of the matroid $M/e$, where $e \in E$ is not a loop, are those bases of $M$ which do contain $e$, with $e$ being removed.*

*Proof.* To prove $(i)$ let $\mathcal{I}_{M-e} = \{I \in \mathcal{I}_M \mid e \notin I\}$ be the independent sets of $M - e$, with $\mathcal{I}_M$ denoting the independent sets of $M$. Note that since $e$ is not an isthmus, $M$ has bases that do not contain $e$. Those bases will be included in $\mathcal{I}_{M-e}$. Since $\mathcal{I}_{M-e} \subseteq \mathcal{I}_M$ and the bases of $\mathcal{I}_M$ are the independent sets with the greatest cardinality, the independent sets with the greatest cardinality in $\mathcal{I}_{M-e}$, the bases of $M - e$, are exactly the bases of $M$ which do not contain $e$.

Proposition 2.9 $(ii)$ follows analogously. $\square$

Through Proposition 2.9 it is clear that after the deletion of an element the rank stays the same and the contraction of an element results in a reduction of the rank. A similar observation can be made about the circuits of a matroid.

**Proposition 2.10.** *Let $e \in E$ with $E$ being the ground set of a matroid $M$ and $e$ neither being an isthmus nor a loop. Then the following equations hold:*

(1) **Rank function** *Let $A \subseteq E$ with $e \notin A$. Then*
  (a) Deletion: $r_{M-e}(A) = r_M(A)$
  (b) Contraction: $r_{M/e}(A) = r_M(A \cup e) - 1$

(2) **Circuits**
  (a) Deletion: *$C$ is a circuit of $M - e$ if and only if $e \notin C$ and $C$ is a circuit of $M$.*
  (b) Contraction: *$C$ is a circuit of $M/e$ if and only if*
    *(i) $C \cup \{e\}$ is a circuit in $M$, or*
    *(ii) $C$ is a circuit of $M$ and $C \cup \{e\}$ contains no circuits except $C$.*

The reason for defining deletion of an element $e$ under the condition that $e$ is not an isthmus, is that it results in the matroid $M - e$ having no bases and Proposition 2.9 would not be fulfilled by $M - e$. In addition to that, if $M - e$ has no bases it would follow through the augmentation that $\mathcal{I} = \emptyset$, which would violate (I1).
In similar fashion, a loop $e$ is not included in any independent set, resulting in $M/e$ not having any independent sets. This again violates (I1), therefore $M/e$ is not a matroid.

**Proposition 2.11.** *Let $a, b \in E$ with $E$ being the ground set of a matroid $M$. Assuming everything is well defined, we have*

*(1)* $(M - a) - b = (M - b) - a$,

*(2)* $(M/a)/b = (M/b)/a$,

*(3)* $(M/a) - b = (M - b)/a$.

For the proof of Proposition 2.11 refer to [GM12][Pages 106-107]. Assuming everything is well defined means that for $M - a$ the element $a$ is not an isthmus in $M$ and for $M/a$ $a$ is not a loop in $M$, analogous with the element $b$.

Before finishing this subsection and starting with graph theory, a small example will be discussed to illustrate the subject of deletion and contraction of an element in a matroid.

**Example 2.12.** Consider the *uniform matroid* $U_{2,4}$. In general a uniform matroid $U_{k,n}$ is defined as a matroid with the ground set $E = \{e_1, e_2, \dots, e_n\}$ and $\mathcal{I}$ are all subsets of $E$ having $k$ or fewer elements, with $k \leq n$,

$$U_{k,n} = \{E, \{A \in \mathcal{P}(E) : |A| \leq k\}\}$$

For $U_{2,4}$ we have the ground set $E = \{e_1, e_2, e_3, e_4\}$ and the independent sets are $\emptyset, \{e_i\}$ and $\{e_i, e_j\}$, for $i, j = 1, \dots, 4$, with $i \neq j$. Since this matroid has neither loops nor isthmuses, the operations deletion and contraction are well defined for all elements of $E$.

The matroid $U_{2,4} - e_4$ has the ground set $E' = \{e_1, e_2, e_3\}$ and the independent sets are all those of the original matroid which do not contain $e_4$, which are $\emptyset, \{e_i\}$ and $\{e_i, e_j\}$, for $i, j = 1, \dots, 3$ where $i \neq j$. As one can see, the resulting matroid is $U_{2,3}$.

Now for $U_{2,4}/e_4$, the ground set is again $E' = \{e_1, e_2, e_3\}$. The independent sets of $U_{2,4}$ that do contain $e_4$ are $\{e_4\}, \{e_1, e_4\}, \{e_2, e_4\}$ and $\{e_3, e_4\}$. By removing $e_4$ from each of those sets we get the independent sets $\emptyset, \{e_1\}, \{e_2\}$ and $\{e_3\}$, which gives us exactly the uniform matroid $U_{1,3}$.

Example 2.12 not only illustrates deletion and contraction on a matroid and provides the definition of a uniform matroid, an interesting observation can also be made: reducing the uniform matroid $U_{2,4}$ by deleting or contracting an element of its ground set results in another uniform matroid. To this end the following lemma will be introduced [Oxl92, Pages 106–107, 111].

**Lemma 2.13.** *Let $U_{k,n}$ be an uniform matroid with the ground set $E$ and $e \in E$ is neither a loop nor an isthmus.*

*$U_{k,n} - e$ results in the uniform matroid $U_{k,n-1}$, as long as $k < n$.*

*$U_{k,n}/e$ results in the uniform matroid $U_{k-1,n-1}$, as long as $k > 0$.*

Uniform matroids of the form $U_{n,n}$ are called *Boolean Algebras* and are denoted $B_n$. Those matroids only have one basis, namely $E$. Since every element is included in every basis, all elements are isthmuses. In contrast to the uniform matroid $U_{0,n}$ where the only independent set is $\emptyset$. Therefore, every element is a loop.
By going over these examples another fascinating property of uniform matroids can be observed:

**Lemma 2.14.** *Consider the uniform matroid $U_{k,n}$ with $0 < k < n$. Then every element of $U_{k,n}$ is neither a loop nor an isthmus.*

*Proof.* Let $U_{k,n}$ be the uniform matroid with $0 < k < n$, the ground set $E = \{e_1, e_2, ..., e_n\}$ and the independent sets $\mathcal{I}$. We want to prove that every $e_i \in E$ for $i = 1, ..., n$ is neither a loop nor an isthmus.

Since $k \geq 1$, every subset of $E$ with one element is included: $\{e_i\} \in \mathcal{I}$ for all $i = 1, ..., n$. Due to the augmentation this yields, that every element is in at least one basis, therefore, no element is a loop.

Now to show that no element of $U_{k,n}$ is an isthmus we consider an arbitrary element $e_i$ with $i \in \{1, 2, ..., n\}$. Since $k < n$ a set $B \subset E$ exists with $k$ elements which does not include $e_i$. Since the matroid is uniform the rank is $k$, hence, $B$ is a basis. Therefore $e_i$ is not an isthmus. □

## 2.3 Matroids resulting from Graphs

Before defining a matroid, an abstract example was given about a matroid associated to a matrix. A matroid $M$ is called *representable* over a field $\mathcal{F}$ if a matrix $A$ exists with entries in $\mathcal{F}$ and the independencies among its columns mirror exactly those in the matroid $M$.
A similar connection can be made with graphs and matroids. Let $G$ be an undirected graph with edge set $E$ and $V$ denoting the set of its vertices. Consider $\mathcal{I}$ as a collection of all subsets of edges in $E$ that are acyclic. Then $M = (E, \mathcal{I})$ is a matroid. $M(G)$ is used to denote the matroid associated with $G$ and will be referred to as the *cycle matroid* of the graph $G$. Matroids which result through a graph are called *graphic*, more precisely, matroids that are isomorphic to the cycle matroid of some graph [Oxl01, Page 5]. Not every matroid is graphic, since not for every matroid $M$ a graph $G$ exists such that $M = M(G)$.

As a reminder, an edge of a graph is defined by its two incident vertices, $e = (v, w) \in E$ with $v, w \in V$. Note that since $G$ is undirected, the edges $(v, w)$ and $(w, v)$ are equivalent. Let $M(G) = (E, \mathcal{I})$ denote the cycle matroid of the graph $G$. For every $A \in \mathcal{I}$, and only for the elements in $\mathcal{I}$, the subgraph $(V, A)$ of $G$ is a forest. Suppose $G$ is connected then it has at least one *spanning tree*. Spanning trees are those subgraphs of $G$ which have the same vertex set as $G$ and are trees, meaning they are acyclic and connected. An important

characteristic of spanning trees is that by adding an edge $e \in E$ which was not yet included, a cycle would be created[1]. Therefore, the bases of $M(G)$ are exactly the edge sets of the spanning trees of $G$. Knowing that every spanning tree of $G$ has the same number of edges, exactly $|V| - 1$ many, it follows clearly that the rank of $M(G)$ is $|V| - 1$. In the case of $G$ not being connected, one considers the spanning trees of each of the connected components of $G$. The rank would then be $|V| - \kappa(G)$, with $\kappa(G)$ being the number of connected components.
In graph theory, it is important to note the coexistence of two terms: *cycles* and *circuits*. Both terms refer to specific structures within graphs, with circuits being closed paths for a graph $G$, whereas, cycles are circuits with only the first and last vertex being the same. Notably, the circuits of $M(G)$ are exactly the edge sets of the cycles in $G$.

A question that might arise is what a cycle matroid looks like if the corresponding graph is not connected. The answer is very simple, because for every graph $G$, which might not be connected, there exists a connected graph $G'$ such that $M(G) = M(G')$. This connected graph can be obtained by merging two vertices from two different connected components of $G$ and repeating this process until we have a connected graph $G'$. Thus in general it can be assumed that the cycle matroid $M(G)$ comes from a connected graph. For an outline of the proof see [GM12, Section 1.4]. One can conclude from the previous statements, that if an isolated vertex, a vertex with no incident edges, were to be added to the graph $G$ resulting in the graph $G'$, the cycle matroid of $G'$ would be the same as for $G$.

**Definition 2.15.** Let $G$ be a graph with the set of edges denoted by $E$.

(i) An edge $e \in E$ is called a *loop*, if the incident vertices are the same, $e = (v, v)$.

(ii) An edge $e \in E$ is called an *isthmus*, if by removing $e$ the number of connected components increases by 1.

Loops and isthmuses were already introduced in the context of matroid theory and as one might have already guessed, a loop (isthmus) in a graph $G$ is a loop (isthmus) of its cycle matroid $M(G)$. Moreover, the operations deletion and contraction can also be defined for graphs:

**Definition 2.16.** Let $G = (V, E)$ be an arbitrary graph.

*Deletion* $G - e$ with $e \in E$, is the graph that has the edge set $E \setminus \{e\}$ and the same vertex set as $G$.

*Contraction* $G/e$ with $e \in E$, where $e$ is not a loop, is the graph with the edge set $E \setminus \{e\}$ and the two vertices which were incident to $e$ are merged.

Since the process of merging two vertices might not be immediately evident, we will demonstrate this. Let $e = (v, w)$ be the edge which is contracted from the graph $G = (V, E)$ with $v \neq w$. The vertex set of the resulting graph is $V \setminus \{v\}$ and all edges which were incident to $v$ in $G$ are now incident to $w$.

[1] That is exactly how *Kruskals Algorithm* operates at finding the Minimal Spanning Tree (MST).

**Proposition 2.17.** *Let $G$ be a graph with $M(G)$ being its cycle matroid. Assuming everything is well defined, both $M(G) - e$ and $M(G)/e$ are graphic matroids and*

$$M(G - e) = M(G) - e$$

$$M(G/e) = M(G)/e$$

For the proof of Proposition 2.17, see [GM12][Pages 157-158]. As for Proposition 2.11, assuming everything is well defined means that for $M(G) - e$ the edge $e$ is not an isthmus for the graph $G$ and for $M(G)/e$ and $M(G/e)$ $e$ is not a loop in $G$.

## 2.4 Duality of Graphs and Matroids

One of the most important characteristic that matroids have is a well-developed theory of duality, which is a helpful tool in solving many problems in matroid theory. Duality is a significant and pervasive concept in mathematics. Next to deletion and contraction it is the most basic matroid operation. First consider duality in graph theory. *Planar* graphs, graphs that can be drawn in the plane without its edges crossing, have duals. For planar graphs it makes sense to define so called *regions* or *faces* which are denoted by $F$, which are those areas surrounded by the edges of the graph [GM12].

For instance, the *complete* graph[2] $K_4$ is a planar graph, however, one can also draw it with its edges crossing. Hence, being planar is a property of the graph itself and not the particular drawing of that graph. Consider a planar graph $G$ drawn without its edges crossing. The dual graph $G^*$ has a vertex for each region of $G$ and two vertices are adjacent by an edge $e^*$ when the corresponding regions share an edge $e$ in $G$.

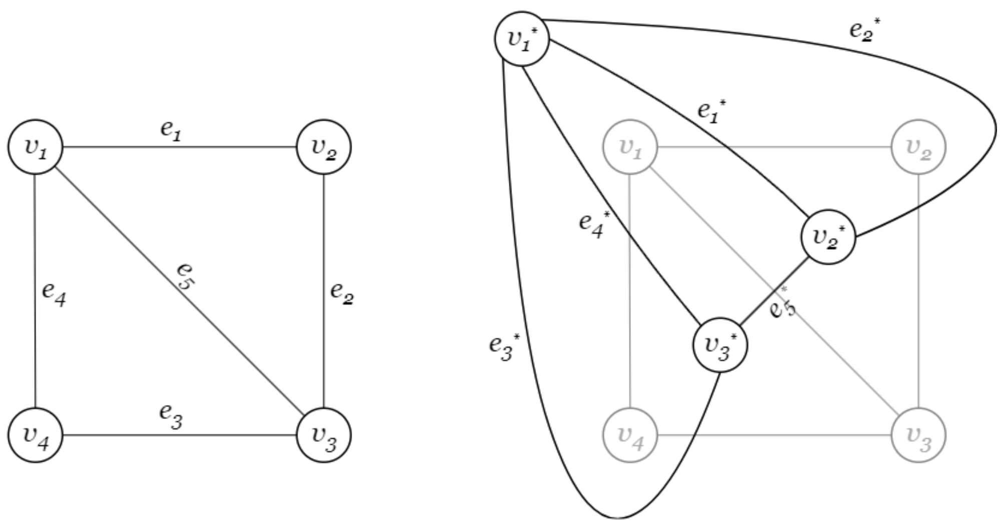



Figure 2.1: Graph $G$ on the left and its dual graph $G^*$ on the right.

[2]A complete graph, denoted $K_n$ with $n$ being the number of vertices, is a graph where every vertex is connected to every other vertex by an edge.

Let $G$ be the graph in Figure[3] 2.1. $G$ has exactly three regions: the two triangles $\{e_1, e_2, e_5\}$ and $\{e_3, e_4, e_5\}$ and the outer region. In each region, denoted as $f_i$, will be a vertex $v_i^*$ of the dual graph $G^*$. For every edge $e$ of $G$ the two vertices in the corresponding regions which are incident to $e$ will be connected.
Evidently, the number of regions of a graph $G$ is the same as the number of vertices of the dual graph $G^*$. The dual graph $G^*$ is always connected and if $G$ itself is connected the dual graph of $G^*$ would be $G$. Furthermore, there is a natural bijection between the edge sets of $G$ and $G^*$, because the dual graph has exactly one edge $e^*$ for every edge $e$ the original graph has. Therefore, it can be assumed that both graphs have the same edge set [EMM08].

**Theorem 2.18.** *Let $G$ be a connected planar graph. Then*

$$|V| - |E| + |F| = 2.$$

Theorem 2.18 is called *Euler's Formula for planar graphs* and has many proofs, some of which can be found in Eppstein's *"Twenty-one Proofs of Euler's Formula: V - E + F = 2"* [Epp13].

In Figure 2.1 one can observe that for a spanning tree of $G$, for example the spanning tree with the edge set $\{e_1, e_2, e_3\}$, the in $G^*$ associated complementary edges, in our case $\{e_4^*, e_5^*\}$, form a spanning tree of $G^*$. This characteristic of spanning trees of dual graphs motivates the definition of *dual matroids*. The dual of a matroid $M$, denoted as $M^*$, is defined on the same ground set as $M$, with the bases of $M^*$ being simply the complements of the bases of $M$:

$$\mathcal{B}(M^*) = \{E \setminus B \ : \ B \in \mathcal{B}(M)\}$$

with $\mathcal{B}(M)$ denoting the set of bases of the matroid $M$.

Forming the dual matroid of $M^*$ would result in the matroid $M$ again, $(M^*)^* = M$. As already mentioned above, for planar graphs on the other hand, $(G^*)^* = G$ is only the case if graph $G$ is connected. If $M$ has rank $r$, meaning the cardinality of its bases is $r$, the dual matroid $M^*$ has bases with cardinality of $|E| - r$. Hence

$$r(M) + r(M^*) = |E|.$$

**Proposition 2.19.** *Let $E$ be the ground set of the matroid $M$. The bases, independent and spanning sets of the dual matroid $M^*$ are determined as follows:*

*(i) $B$ is a basis of $M$, if and only if $E \setminus B$ is a basis of $M^*$.*

*(ii) $I$ is an independent set of $M$, if and only if $E \setminus I$ is a spanning set of $M^*$.*

*(iii) $S$ is a spanning set of $M$, if and only if $E \setminus S$ is an independent set of $M^*$.*

The proof of Proposition 2.19 can be found in [GM12][Pages 116-117].
Taking note of these observations, we can make some intuitive observations for specific

[3]Figure 2.1 was created using `https://app.diagrams.net/`.

instances. Suppose the matroid $M$ has exactly one spanning set. Through Lemma 2.7 we conclude that every element of $M$ is an isthmus and the spanning set (and basis) of $M$ is $E$ itself. With Proposition 2.19 it follows that the dual matroid of $M^*$ has only the empty set as its independent sets (and bases), hence, every element of $M^*$ is a loop. Further, let $e$ be an isthmus in $M$. Since $e$ is in every basis of $M$, it will be in no basis of $M^*$. Therefore, $e$ is a loop of $M^*$ if and only if $e$ is an isthmus of $M$ [Oxl92].

The same applies to planar graphs. Suppose graph $G$ has an isthmus $e$. The regions that are incident to the edge $e$ in $G$ are actually the same, the outer region $f$. Vertex $v^*$ of graph $G^*$ which represents the region $f$ will have a loop for every isthmus $G$ has. On the other hand, if $e$ is a loop, the region which is surrounded only by $e$ itself is adjacent only to one other region. One can gather that for $G^*$ the edge $e$ is an isthmus.

# 3 Tutte Polynomial

William Tutte was born in England in the year 1917. During World War II he was part of the Bletchey Park group as a cryptanalysist, where he was decoding the FISH messages. After the war he studied for a doctorate in mathematics. His original motivation for defining a bivariate polynomial for graphs is based on the chromatic polynomial. That is the reason why he called it originally the dichromatic polynomial, however, it is now known as the *Tutte polynomial*. Henry Crapo extended Tutte's work to matroids, while Thomas Brylawski demonstrated numerous fundamental results regarding the Tutte polynomial for matroids. This chapter introduces the Tutte polynomial defined on a matroid, provides the proof of its well-definedness as well as discusses Tutte-Grothendieck invariants. Furthermore, the chromatic polynomial will be thoroughly covered. Afterwards it elaborates on a specific Tutte-Grothendieck invariant, namley the number of acyclic orientations. To this end the flow polynomial will be introduced as well. This chapter is mostly based on Chapter 9 of [GM12].

## 3.1 Well-definedness and Tutte-Grothendieck invariants

**Definition 3.1.** Let $M$ be a matroid with ground set $E$. The *Tutte polynomial* $t(M;x,y)$ is defined recursively through:

(i) $t(M;x,y) = t(M-e;x,y) + t(M/e;x,y)$, if $e$ is neither a loop nor an isthmus;

(ii) $t(M;x,y) = x \cdot t(M/e;x,y)$, if $e$ is an isthmus;

(iii) $t(M;x,y) = y \cdot t(M-e;x,y)$, if $e$ is a loop;

(iv) $t(M;x,y) = 1$, if $E = \emptyset$.

**Example 3.2.** We want to find the Tutte polynomial of the uniform matroids $U_{2,2}$ and $U_{2,3}$. The matroid $U_{2,2}$ with the ground set $E = \{e_1, e_2\}$ has the independent sets $\emptyset, \{e_1\}, \{e_2\}$ and $\{e_1, e_2\}$. Both elements $e_1$ and $e_2$ are in the only basis the matroid has, therefore, both are isthmuses. The Tutte polynomial can be calculated by

$$\begin{aligned} t(U_{2,2};x,y) &= x \cdot (U_{2,2} - e_2; x, y) \\ &= x \cdot (U_{1,1}; x, y). \end{aligned}$$

Since the uniform matroid $U_{1,1}$ has only the independent set $\{e_1\}$ apart from the empty set, it follows that,

$$\begin{aligned} t(U_{2,2};x,y) &= x \cdot x \cdot (U_{0,0}; x, y) \\ &= x^2. \end{aligned}$$

Let us continue with finding the Tutte polynomial for the matroid $U_{2,3}$. This matroid already encountered us in Example 2.12. For $U_{2,3}$, just like the matroid $U_{2,4}$, all elements are neither isthmuses nor loops, which one might remember from Lemma 2.14.
Using (i) from Definition 3.1 yields,

$$\begin{aligned} t(U_{2,3}; x, y) &= t(U_{2,3} - e_3; x, y) + t(U_{2,3}/e_3; x, y) \\ &= t(U_{2,2}; x, y) + t(U_{1,2}; x, y). \end{aligned}$$

As calculated above, it is known that $t(U_{2,2}; x, y) = x^2$. Since $U_{1,2}$ has only two elements which are again neither an isthmus nor a loop it follows that

$$\begin{aligned} t(U_{2,3}; x, y) &= x^2 + t(U_{1,2} - e_2; x, y) + t(U_{1,2}/e_2; x, y) \\ &= x^2 + t(U_{1,1}; x, y) + t(U_{0,1}; x, y). \end{aligned}$$

The matroid $U_{0,1}$ has only one element which is a loop. $U_{1,1}$, on the other hand, has only an isthmus as an element.
The last element will now be removed, hence we get,

$$\begin{aligned} t(U_{2,3}; x, y) &= x^2 + x \cdot (U_{1,1} - e_1; x, y) + y \cdot (U_{0,1}/e_1; x, y) \\ &= x^2 + x + y. \end{aligned}$$

The order in which an element is deleted and contracted when computing the Tutte polynomial does not matter. Definition 3.1 would not make sense if it did.

**Definition 3.3.** Let $M = (E, \mathcal{I})$ be a matroid and $r$ the rank function. The *corank-nullity polynomial* of the matroid $M$ is defined as

$$s(M; u, v) = \sum_{A \subseteq E} u^{r(E) - r(A)} v^{|A| - r(A)}.$$

The *corank* of $A \subseteq E$ is $r(E) - r(A)$. It is the minimal number of elements which have to be added to $A$ such that $A$ is a basis. The *nullity* of $A \subseteq E$ is $|A| - r(A)$, which is the minimal number of elements which have to be removed from $A$ such that $A$ is independent.

**Definition 3.4.** A *matroid invariant* is defined as a function $f : \mathcal{M} \to R$, with $\mathcal{M}$ being the class of all matroids and $R$ a commutative ring. The function $f$ has to satisfy the property

$$M_1 \cong M_2 \implies f(M_1) = f(M_2).$$

Invariants are a familiar and important idea in mathematics. In graph theory the chromatic polynomial is a graph invariant which will be discussed in Section 3.2. Some matroid invariants like the number of bases or the number of independent sets were already introduced throughout this work. What is more, the Tutte polynomial is a well-defined invariant. This will be proven by writing the Tutte polynomial as a closed form using the corank-nullity polynomial which is an invariant in matroid theory as well.

**Theorem 3.5.** *Let $M$ be a matroid. The Tutte polynomial can be obtained through the corank-nullity polynomial, with*

$$t(M;x,y) = s(M;x-1,y-1).$$

*Further, the Tutte polynomial is a well-defined matroid invariant.*

*Proof.* The proof that the corank-nullity polynomial $s(M;x-1,y-1)$ obeys Definition 3.1 for every ground set element $e$, is done by induction on $n = |E|$. To recall, we want to prove

$$t(M;x,y) = \sum_{A\subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}. \tag{3.1}$$

Let $M$ be an arbitrary matroid with ground set $E$. For $E = \emptyset$, it is known through the definition of the Tutte polynomial that it is 1. Since the corank and nullity of the empty set, the only subset of $E$, equals 0, the right hand side of 3.1 is equal to

$$s(M;x-1,y-1) = (x-1)^0(y-1)^0 = 1.$$

Although the base case of our induction was already presented with $E = \emptyset$, we will also look at the case of $E$ only having one element $e$. If $e$ is an isthmus the Tutte polynomial has to be $t(M;x,y) = x$ and $t(M;x,y) = y$ for $e$ being a loop. Let $e$ be an isthmus, then $r(E) = r(\{e\}) = 1$ and $r(\emptyset) = 0$. The nullity of both subsets is 0 and the corank equals 1 for $\{e\}$ and 0 for $\emptyset$, giving us

$$s(M;x-1,y-1) = (x-1)^1(y-1)^0 + (x-1)^0(y-1)^0 = x.$$

For $e$ being a loop, $r(E) = r(\emptyset) = r(\{e\}) = 0$. Therefore the corank of both subsets is equal to 0 and the nullity of $\emptyset$ is 0 and of $\{e\}$ is 1. Computing $s(M;x-1,y-1)$ gives us $y$ and Theorem 3.5 holds for $|E| = 1$.

Let $|E| = n$ with $n \geq 2$ and assume that for $|E| = n-1$ equation 3.1 holds. Let $e \in E$, then three cases have to be considered: $e$ can either be an isthmus, a loop or neither. For the purpose of simplification let $r_d$ and $r_c$ denote the rank functions of $M - e$ and $M/e$ respectively.

*Case 1:* Let $e$ be an isthmus. Removing an isthmus will change the rank of a matroid, $r(E) - 1 = r_c(E \setminus \{e\})$ as well as the rank of those subsets $A$ containing $e$. For every $A \subseteq E \setminus \{e\}$ the rank stays the same. We split the subsets of $E$ into two sets $\mathcal{S}_1$ and $\mathcal{S}_2$, the ones that contain $e$ and the ones that do not. So

$$\begin{aligned} s(M;x-1,y-1) &= \sum_{A\subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)} \\ &= \sum_{A\in\mathcal{S}_1} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)} + \sum_{A\in\mathcal{S}_2} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}. \end{aligned}$$

For any subset $A$ in $\mathcal{S}_1$ the corank and nullity of $A$, computed in $M$, are the same as the ones of $A \setminus \{e\}$ computed in $M/e$. This leads to

$$\sum_{A \in \mathcal{S}_1} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)} = \sum_{B \subseteq E \setminus \{e\}} (x-1)^{r_c(E\setminus\{e\})-r_c(B)}(y-1)^{|B|-r_c(B)}$$
$$= s(M/e; x-1, y-1).$$

Let now be $A \in \mathcal{S}_2$. Since the corank changes to $r(E) - r(A) = r_c(E \setminus \{e\}) + 1 - r_c(A)$ and the nullity stays the same, it follows, that

$$\begin{aligned}\sum_{A \in \mathcal{S}_2} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)} &= \sum_{A \in \mathcal{S}_2} (x-1)^{r_c(E\setminus\{e\})+1-r_c(A)}(y-1)^{|A|-r_c(A)} \\ &= (x-1) \cdot \sum_{A \in \mathcal{S}_2} (x-1)^{r_c(E\setminus\{e\})-r_c(A)}(y-1)^{|A|-r_c(A)} \\ &= (x-1) \cdot s(M/e; x-1, y-1).\end{aligned}$$

Hence,

$$\begin{aligned}s(M; x-1, y-1) &= s(M/e; x-1, y-1) + (x-1) \cdot s(M/e; x-1, y-1) \\ &= x \cdot s(M/e; x-1, y-1) \\ &= x \cdot t(M/e; x, y) \\ &= t(M; x, y)\end{aligned}$$

The equality $s(M; x-1, y-1) = x \cdot s(M/e; x-1, y-1)$ holds due to the induction assumption.

*Case 2:* Suppose $e$ is a loop. Since case 2 is handled similarly to case 1, this case will be kept short. Remember that removing a loop does not change the rank of the matroid $r(E)$ nor of any other subset since it is not contained in any independent set. Let $A \in \mathcal{S}_1$, then since $r(A) = r_d(A \setminus \{e\})$ the corank of $A$ stays the same and the nullity increases by 1, $|A| - r(A) = |A \setminus \{e\}| + 1 - r_d(A \setminus \{e\})$. For $A \in \mathcal{S}_2$ the corank and nullity does not change, hence,

$$\begin{aligned}s(M; x-1, y-1) &= \sum_{A \in \mathcal{S}_1} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)} + \sum_{A \in \mathcal{S}_2} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)} \\ &= (y-1) \cdot s(M-e; x-1, y-1) + s(M-e; x-1, y-1) \\ &= y \cdot s(M-e; x-1, y-1) \\ &= y \cdot t(M-e; x, y) \\ &= t(M; x, y).\end{aligned}$$

*Case 3:* Let $e$ be neither an isthmus nor a loop. It holds that $r_d(A \setminus \{e\}) = r(A)$ for all $A \subseteq E$ and $r_c(A \setminus \{e\}) = r(A) - 1$ for $e \in A$, else the rank after contracting an element stays the same. Again we break up the subsets into $\mathcal{S}_1$ and $\mathcal{S}_2$ the same way as above.

Assume $A \in \mathcal{S}_1$. The corank of $A \setminus \{e\}$ in $M/e$ is $r_c(E \setminus \{e\}) - r_c(A \setminus \{e\}) = r(E) - r(A)$ and for the nullity we get $|A \setminus \{e\}| - r_c(A \setminus \{e\}) = |A| - r(A)$. Both are the same and we get

$$\sum_{A\in\mathcal{S}_1}(x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)} = \sum_{A\in\mathcal{S}_1}(x-1)^{r_c(E\setminus\{e\})-r_c(A\setminus\{e\})}(y-1)^{|A\setminus\{e\}|-r_c(A\setminus\{e\})}$$
$$= s(M/e;x-1,y-1).$$

For $A\in\mathcal{S}_2$, $r(E)=r_d(E\setminus\{e\})$ and $r(A)=r_d(A)$ holds. It follows that

$$\sum_{A\in\mathcal{S}_2}(x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)} = \sum_{A\in\mathcal{S}_2}(x-1)^{r_d(E\setminus\{e\})-r_d(A)}(y-1)^{|A|-r_d(A)}$$
$$= s(M-e;x-1,y-1).$$

With the definition of the Tutte polynomial this yields

$$\begin{aligned} t(M;x,y) &= t(M/e;x,y)+t(M-e;x,y)\\ &= s(M/e;x-1,y-1)+s(M-e;x-1,y-1)\\ &= s(M;x-1,y-1). \end{aligned}$$

The last equality is done by induction.

□

Theorem 3.5 shows the well-definedness of the Tutte polynomial which is very important, since through the recursive Definition 3.1 it is not immediately clear. Since the corank-nullity polynomials of two isomorphic matroids are the same, it is a matroid invariant and consequently the Tutte polynomial is as well. If the Tutte polynomial of a matroid is known, the Tutte polynomial of its dual can be easily calculated by switching $x$ and $y$ [GM12].

**Theorem 3.6.** *Let $M$ be a matroid with $M^*$ denoting its dual matroid. Then*

$$t(M^*;x,y)=t(M;y,x)$$

The proof of Theorem 3.6 can be found in [GM12, Page 333-334].

Through the Tutte polynomial one can gather a lot of interesting information about the matroid. For example, the number of bases $b(M)$ and the number of independent sets $i(M)$ as well as the number of spanning sets $sp(M)$ are all deducible from the Tutte polynomial. All of them satisfy some properties which we can summarize in the next Theorem:

**Theorem 3.7.** *Let $\mathcal{M}$ be the class of all matroids and $R$ a commutative ring, and let $f:\mathcal{M}\to R$ satisfying*

*(1) $f(M_1)=f(M_2)$, if matroids $M_1$ and $M_2$ are isomorphic;*

*(2) $f(M)=f(M-e)+f(M/e)$, for $e$ neither being an isthmus nor a loop;*

*(3) $f(M)=f(I)\cdot f(M/e)$, for $e$ being an isthmus;*

*(4) $f(M) = f(L) \cdot f(M - e)$, for $e$ being a loop.*

*Then $f(M) = t(M; f(I), f(L))$, with $I$ being an isthmus and $L$ a loop.*

*Proof.* For the proof, an inductive method is once again employed. We begin by demonstrating the theorem's validity for a matroid with only one element. The induction is then completed by utilizing the recursive definition of the Tutte polynomial (see Definition 3.1), combined with the assumed characteristics (2), (3), and (4) of the invariant $f$.

Consider the matroid $M$ with $E$ denoting its ground set.
Let $|E| = 1$ and $e \in E$, then $e$ is either an isthmus or a loop. Suppose $e$ is an isthmus then the Tutte polynomial of $M$ is $t(M; x, y) = x$. Using $f(I)$ as $x$ and $f(L)$ as $y$ we get $t(M; f(I), f(L)) = f(I)$. With (3) it follows that

$$f(M) = f(I) = t(M; f(I), f(L)).$$

The proof for $e$ being a loop proceeds analogously.
For $e \in E$ with $|E| > 1$ we again have to consider three cases: $e$ being a loop, an isthmus or neither. Let $|E| = n$ and assume that for every matroid with a $(n-1)$- element ground set the Theorem 3.7 holds.

*Case 1:* Assume $e$ is an isthmus then

$$\begin{aligned} f(M) &= f(I) \cdot f(M/e) \\ &= f(I) \cdot t(M/e; f(I), f(L)) \end{aligned}$$

and through Definition 3.1 (ii) it follows that

$$f(M) = t(M; f(I), f(L)).$$

*Case 2:* Let $e$ be a loop. Then with Definition 3.1 (iii) and assumption 3.7 (4) $f(M) = t(M; f(I), f(L))$ follows analogously to the first case.

*Case 3:* Lastly suppose $e$ is neither an isthmus nor a loop, then

$$\begin{aligned} f(M) &= f(M - e) + f(M/e) \\ &= t(M - e; f(I), f(L)) + t(M/e; f(I), f(L)) \\ &= t(M; f(I), f(L)). \end{aligned}$$

□

A matroid invariant that satisfies (1)-(4) in Theorem 3.7 is called a *Tutte-Grothendieck (T-G) invariant.*

The number of subsets a loop (isthmus) can have is evidently two: $\emptyset$ and the set containing the loop (isthmus). Let us consider the number of independent sets $i(M)$ for the matroid $M$. The independent set of a loop is only $\emptyset$, hence $i(L) = 1$. For an isthmus $i(I) = 2$. Therefore, the number of independent sets of $M$ can be computed by $i(M) = t(M; 2, 1)$. Similarly, one can compute the number of bases and spanning sets:

**Corollary 3.8.** *Let $M$ be a matroid. Then*

*(1)* $b(M) = t(M; 1, 1) = s(M; 0, 0)$,

*(2)* $i(M) = t(M; 2, 1) = s(M; 1, 0)$,

*(3)* $sp(M) = t(M; 1, 2) = s(M; 0, 1)$,

*(4)* $t(M; 2, 2) = s(M; 1, 1) = 2^{|E|}$.

Although many characteristics of a matroid can be drawn from the Tutte polynomial without having to manually solve for them, it is not possible to reproduce the matroid through it. Matroids which have the same Tutte polynomial are called *Tutte-equivalent*. Isomorphic matroids always have the same Tutte polynomial since it is a matroid invariant. However, cases do present themselves in which two non-isomorphic matroids have the same Tutte polynomial. This interesting observation can be made through [GM12, Example 9.11]. In Gordon and McNulty's example they compare two matroids through their geometric configurations[1], which are not discussed in this thesis. For further information on Tutte-equivalent matroids refer to M. M. Rocha's thesis *"Tutte-Equivalent Matroids"* [Roc18], which explores this topic deeply.

## 3.2 The Chromatic Polynomial

In the year 1852, Francis Guthrie wrote a letter to his brother Frederick, in which he posed the question of whether it was possible to color the regions of a map with a maximum of four colors in such a way that adjacent regions had different colors. This letter gave rise to a touchstone problem in modern combinatorics and graph theory, known as the Four Color Problem. The resolution of this problem by Appel and Haken in 1976 eventually led to the problem being referred to as the Four Color Theorem. Not only did Appel and Haken resolve a 125 year-old conjecture, but also their proof was the first significant mathematical proof that relied heavily on the usage of a computer [GM12].

Think of a geographical map as a planar graph with the boundaries between two regions being the edges, the vertices as points where two or more boundaries meet and the regions of the graph being the regions of the map. By forming the dual of the graph one can consider the coloring of the vertices of the dual graph instead of the regions of the map. Let $G$ be a graph with vertex set $V$. A vertex coloring of $G$ is a map $c : V \to \mathbb{Z}^+$. The vertices are colored *properly* if no vertices which are adjacent are colored the same, meaning if the edge $e = (v, w)$ exists in $G$ then $c(v) \neq c(w)$.

Figure[2] 3.1 is a demonstration on how to represent the map of Australia as a graph $A$ and also its dual graph $A^*$. Both graphs have the same number of edges and the vertices of the dual graph $A^*$ are representing the regions of Australia. An interesting observation is how one handles an island, such as Tasmania, denoted as the vertex $T$ in $A^*$. Since the

[1] Geometric configurations are geometric pictures of matroids consisting of dots, which are the elements of $E$, and lines connecting the dots, which represent dependencies. It is important to note that although they remind of a graph, they are NOT a graph.

[2] Figure 3.1 was made using `https://app.diagrams.net/`.

edges of the original graph $A$ are representing the boarders of the graph, let $e_T$ represent the boarder of Tasmania. As discussed in Section 2.4, the dual of a loop is an isthmus, therefore, $e_T$ is a loop in $A$ and for the dual graph $A^*$ an isthmus. Since Tasmania only shares a boarder with the ocean, or rather the vertex $v_T$ is only adjacent to the vertex $v_O$, it can be colored in any color except the one $v_O$ has.


Figure 3.1: Demonstration on creating the dual graph $A^*$ of Australia.

A very important note is that no graph that has a loop $e = (v, v)$ as an edge can be colored properly. In the case of coloring a graph which represents a geographical map like the one on the right of Figure 3.1, these graphs would never have a loop since that would mean that a country shares a boarder with itself. Graphs which do not have any loops nor multiple edges are called *simple* graphs. For example, both graphs which are shown in Figure 3.2 are simple graphs as well as $A^*$.

The reason why this problem is interesting for this paper is G. D. Birkhoff's approach in 1912. He defined a univariate function $\chi_G(\lambda)$ defined on a graph $G$ which gives the number of proper colorings of $G$ using $\lambda$ or fewer colors. It turns out $\chi_G(\lambda)$ is a polynomial in the variable $\lambda$ and is called the *chromatic polynomial*. The Four Color Theorem can then be rephrased in the following way:

Let $G$ be a planar graph and let $G$ have no loops. Then $\chi_G(4) > 0$.

In Gordon and McNulty's book [GM12] the condition for $G$ to not have any loops was omitted, as it is often automatically assumed that a graph $G$ does not have any loops. However, we believe it should be included for clarity and correctness. While graphs with multiple edges can still be colored properly, for the purpose of graph coloring, we can consider multiple edges as if they were a single edge. With the assumption that we are solely working with simple graphs the condition that $G$ has no loops can be ignored.

Although no proof of the Four Color Theorem was found in this way, the chromatic polynomial has various other applications and can be generalized to a two-variable polynomial defined on matroids which we were already introduced to, the Tutte polynomial.

**Definition 3.9.** Let $G$ be a graph and $\lambda \in \mathbb{Z}^+$. The *chromatic polynomial* $\chi_G(\lambda)$ of $G$ gives the number of proper colorings of $G$ using at most $\lambda$ colors.

The fact that the chromatic polynomial is indeed a polynomial in $\lambda$ will become evident later through its connection to the Tutte polynomial. Two isomorphic graphs, meaning that there is a one-to-one correspondence between the vertices of those two graphs that preserves adjacency, have the same chromatic polynomial. Therefore, the chromatic polynomial is a graph invariant.
The *chromatic number* is the smallest amount of colors which are needed for a proper coloring of the graph $G$. The chromatic polynomial $\chi_G(\lambda)$ is bigger than 0 if and only if $G$ has a chromatic number of at most $\lambda \in \mathbb{Z}^+$. Since a graph without edges has no adjacent vertices, the chromatic number is 1. In no other case it can be 1.

**Lemma 3.10.** *Let $G$ be a graph. $G$ has the chromatic number at most $k$ if and only if $G$ is $k$-partite.*

The proof of Lemma 3.10 is trivial, however, it is important to emphasize that trees have the chromatic number of at most[3] 2. This result is derived from the fact that trees are bipartite graphs.

**Example 3.11.** Before discussing the connection to the Tutte polynomial, two small but interesting examples will be presented to get a feeling of the chromatic polynomial on graphs. Consider the complete graph $K_4$ in Figure 3.2a. Since every vertex is adjacent to every other vertex it is clear that the chromatic number has to be 4. In general the chromatic number for complete graphs $K_n$ is clearly $n$. Determining the chromatic polynomial $\chi_{K_4}(\lambda)$ can be done greedily by first analyzing how many colors a vertex could be colored in.

Let us start with coloring vertex $v_1$. Since no vertex is colored yet, $\lambda$ possible colors can be used to color it. Vertex $v_2$ is adjacent to $v_1$ thus it can be colored in any color except the color $v_1$ was assigned to, therefore only $\lambda - 1$ ways are left. The same logic applies to $v_3$ which is adjacent to $v_1$ and $v_2$, thus having $\lambda - 2$ options and vertex $v_4$ has $\lambda - 3$. The chromatic polynomial is the product of these possibilities,

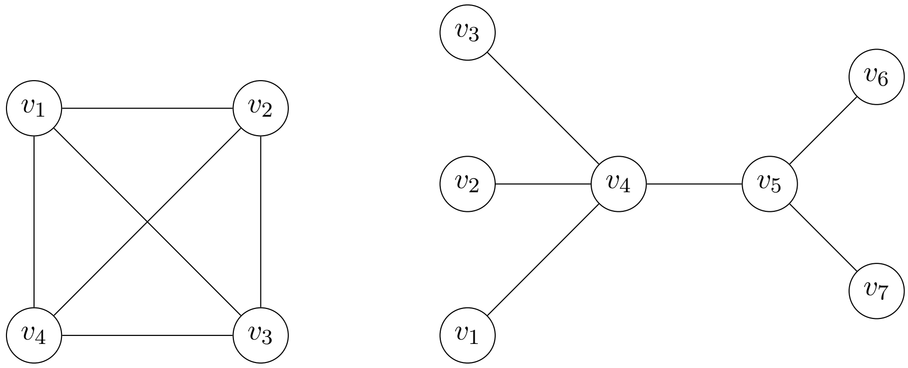
$$\chi_{K_4}(\lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3).$$

It is clear that for every number of colors $\lambda$ below 4 the chromatic polynomial would return 0, showing that it is not possible to color the complete graph $K_4$ properly with less than 4 colors. Setting $\lambda = 4$ reveals that there are 24 possibilities to color the graph $K_4$ with 4 colors. The chromatic polynomial of any complete graph $K_n$ can be computed by

$$\chi_{K_n}(\lambda) = \prod_{i=1}^{n}(\lambda - i + 1).$$

[3] The chromatic number of trees is always 2 except when it has no edges, which means in this case consisting of just one vertex.

This can be shown by induction since the complete graph $K_n$ is equal to $K_{n+1}$ but one vertex removed and all its incident edges.




(a) Complete Graph $K_4$ (b) Tree $T$

Figure 3.2: Two graphs for Example 3.11.

Now consider the tree $T$ shown in Figure 3.2b. The leaf $v_1$ can be colored in $\lambda$ ways. The only adjacent vertex to $v_1$ is $v_4$ which has $\lambda - 1$. Vertices $v_2$ and $v_3$ can have any color, also the one $v_1$ is colored in, except the one $v_4$ got, therefore again $\lambda - 1$. This process can be continued with the vertices $v_5, v_6$ and $v_7$ which all have $\lambda - 1$ choices. This results in the chromatic polynomial being

$$\chi_T(\lambda) = \lambda(\lambda - 1)^6.$$

The vertices of any tree can be arranged in such a way that the first vertex to be colored is a leaf, and each successive vertex to be colored is adjacent to exactly one vertex that has already been colored. Therefore the chromatic polynomial of any tree with $n$ vertices $T_n$ is

$$\chi_{T_n}(\lambda) = \lambda(\lambda - 1)^{n-1}.$$

This approves our comment from earlier that trees with more than one node have a chromatic number of 2, since $\chi_{T_n}(1) = 0$.

**Proposition 3.12.** *Let $G = (V, E)$ be a simple graph and $e$ an edge in $E$. Then*

$$\chi_G(\lambda) = \chi_{G-e}(\lambda) - \chi_{G/e}(\lambda).$$

*Proof.* Let $e = (v, w)$ be an arbitrary edge in $G$ with $v$ and $w$ being its endpoints. As for the graph $G - e$ the two vertices $v$ and $w$ are not adjacent anymore and for every proper coloring of $G - e$ the vertices $v$ and $w$ can be colored either in the same color or a different one.

If they are colored differently, the coloring is also a proper coloring of $G$. In fact every proper coloring of $G$ is a proper coloring of $G - e$ with $v$ and $w$ being differently colored.

On the contrary, if the vertices $v$ and $w$ are being assigned the same color when coloring $G - e$, the coloring corresponds to a proper coloring of the graph $G/e$.

Thus $\chi_{G-e}(\lambda) = \chi_G(\lambda) + \chi_{G/e}(\lambda)$. $\square$

Since the chromatic polynomial of a graph $G$ is defined on its vertices, a connection between the Tutte polynomial of the cycle matroid $M(G)$ needs some adjustment involving the vertices, since the Tutte polynomial is defined on the edges of $G$.

**Theorem 3.13.** *Let $G = (V, E)$ be a connected graph. Then*

$$\chi_G(\lambda) = (-1)^{|V|-1}\lambda \cdot t(M(G); 1-\lambda, 0).$$

*Proof.* For the proof we want to find an appropriate T-G invariant based on the chromatic polynomial of the graph $G$. Through that we can apply Theorem 3.7. However, first we need to have an understanding of the chromatic polynomial of a connected graph which has only one edge, either being a loop or an isthmus.

The chromatic polynomial of the connected[4] graph $I$ with only an isthmus as an edge is $\chi_I(\lambda) = \lambda(\lambda-1)$ and of a graph $L$ with only a loop $\chi_L(\lambda) = 0$. As already mentioned, if a graph $G$ has a loop as an edge, it cannot be colored properly, meaning its chromatic polynomial has to be 0 since $\chi_G(\lambda) = \chi_L(\lambda) \cdot \chi_{G-e}(\lambda)$.

Now let us consider a connected graph $G$ where $e$ represents an isthmus, and $v$ and $w$ denote the vertices incident to the edge $e$. The graph $G-e$ consists of exactly two connected components. These subgraphs of $G$ will be denoted as $G_v$ for the one containing the vertex $v$ and $G_w$ for the other one. It follows that the chromatic polynomial of the graph $G-e$ can be calculated by $\chi_{G-e}(\lambda) = \chi_{G_v}(\lambda) \cdot \chi_{G_w}(\lambda)$ and with Proposition 3.12 we get[5]

$$\chi_{G_v}(\lambda) \cdot \chi_{G_w}(\lambda) = \chi_G(\lambda) + \chi_{G/e}(\lambda).$$

The subgraph $G_v$ can be properly colored in $\chi_{G_v}(\lambda)$ different ways. For $G_w$ let us start with vertex $w$ which can be colored in $\lambda$ ways. $\chi_{G_w}(\lambda)$ can then be classified in $\lambda$ classes each having the size $\frac{\chi_{G_w}(\lambda)}{\lambda}$. Only one of these classes gives a proper coloring of the graph $G/e$, where the vertices $v$ and $w$ have the same color. Hence,

$$\chi_{G/e}(\lambda) = \frac{\chi_{G_v}(\lambda) \cdot \chi_{G_w}(\lambda)}{\lambda} = \frac{\chi_G(\lambda)}{\lambda} + \frac{\chi_{G/e}(\lambda)}{\lambda}.$$

Therefore, if $e$ is an isthmus the chromatic polynomial of $G$ is

$$\chi_G(\lambda) = (\lambda - 1)\chi_{G/e}(\lambda).$$

Through this observation we now define a function $f_G$ based on $\chi_G(\lambda)$, as

$$f_G(\lambda) := \frac{(-1)^{|V|-1}\chi_G(\lambda)}{\lambda}.$$

[4] If we would consider $I$ not being connected, meaning it has isolated vertices and those two which are connected by the isthmus as its connected components, the chromatic polynomial would be $\lambda^{\kappa(I)}(\lambda-1)$, with $\kappa(I)$ denoting the number of vertices.

[5] Although Proposition 3.12 assumes that the graph has to be simple, it still works in this case. Since we assumed $e = (v, w)$ is an isthmus, meaning not a loop, $v \neq w$, nor do more edges exist which are incident to $v$ and $w$, the proof is still correct.

If $f_G(\lambda)$ is a T-G invariant, we can apply Theorem 3.7 and get $f_G(\lambda) = t(M(G); f_I(\lambda), f_L(\lambda))$. Let $f_G(\lambda) = f(M(G); \lambda)$ and $f : \mathcal{M}_{graph} \to \mathbb{Z}[\lambda]$, where $\mathcal{M}_{graph}$ denotes the class of all graphic matroids.

*(1)* Let us consider a graphic matroid $M' = M(G')$ which is isomorphic to $M(G)$. Their structural properties ensure that the associated graphs $G$ and $G'$, though possibly not identical, are graph isomorphic, since we only consider connected graphs. The chromatic polynomials of isomorphic graphs are the same. It follows that $f(M(G); \lambda) = f(M'; \lambda)$ and $f_G$ is a matroid invariant for graphic matroids.

*(2)* As mentioned above if $G$ has a loop $e$ as an edge or only consist of a loop the chromatic polynomial is equal to the null-polynomial, therefore $f_L(\lambda) = 0$ and
$f_G(\lambda) = f_L(\lambda) \cdot f_{G-e}(\lambda) = 0$.

*(3)* Let $I$ be the graph with only an isthmus as an edge. Since $\chi_I(\lambda) = \lambda(\lambda - 1)$ it follows that $f_I(\lambda) = \frac{(-1)\lambda(\lambda-1)}{\lambda} = 1 - \lambda$.
Now consider the graph $G$ from before with the isthmus $e$ as an edge and the chromatic polynomial written as $\chi_G(\lambda) = (\lambda - 1)\chi_{G/e}(\lambda)$. This gives us

$$\begin{aligned} f_G(\lambda) &= \frac{(-1)^{|V|-1}\chi_G(\lambda)}{\lambda} \\ &= \frac{(-1)^{|V|-1}(\lambda - 1)\chi_{G/e}(\lambda)}{\lambda} \\ &= (1 - \lambda) \cdot \frac{(-1)^{|V|-2}\chi_{G/e}(\lambda)}{\lambda} \\ &= f_I(\lambda) \cdot f_{G/e}(\lambda). \end{aligned}$$

*(4)* Let $e$ be neither an isthmus nor a loop. Then with Proposition 3.12 we get

$$\begin{aligned} f_G(\lambda) &= \frac{(-1)^{|V|-1}\chi_G(\lambda)}{\lambda} \\ &= \frac{(-1)^{|V|-1}(\chi_{G-e}(\lambda) - \chi_{G/e}(\lambda))}{\lambda} \\ &= \frac{(-1)^{|V|-1}\chi_{G-e}(\lambda)}{\lambda} + \frac{(-1)^{|V|-2}\chi_{G/e}(\lambda)}{\lambda} \\ &= f_{G-e}(\lambda) + f_{G/e}(\lambda). \end{aligned}$$

Since $f_G(\lambda)$ satisfies all of the necessary requirements, it is a matroid T-G invariant and through Theorem 3.7 it follows that

$$f_G(\lambda) = t(M(G); 1 - \lambda, 0),$$

hence,

$$t(M(G); 1-\lambda, 0) = \frac{(-1)^{|V|-1}\chi_G(\lambda)}{\lambda}$$
$$\chi_G(\lambda) = (-1)^{|V|-1}\lambda \cdot t(M(G); 1-\lambda, 0).$$

□

Using the Tutte polynomial representation, it becomes evident that the chromatic polynomial is, indeed, a polynomial:

**Proposition 3.14.** *The chromatic polynomial $\chi_G(\lambda)$ is a polynomial in $\lambda$.*

**Example 3.15.** Consider the uniform matroid $U_{2,3}$. Since this matroid is graphic, a graph $G$ can be derived such that $M(G) = U_{2,3}$. The matroid has 3 elements in its ground set, therefore, its corresponding graph has 3 edges and none are isthmuses or loops as known from Lemma 2.14. The rank of the matroid is 2. This yields that the spanning trees of $G$ need to have exactly 2 edges and $G$ must have at least 3 vertices. If we consider only connected graphs, $G$ is the complete graph[6] $K_3$ in Figure 3.3.

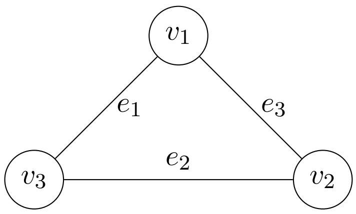



Figure 3.3: Graph $K_3$ with cycle matroid $U_{2,3}$.

The Tutte polynomial of $U_{2,3}$ is $t(U_{2,3}; x, y) = x^2 + x + y$, as calculated in Example 3.2. According to Theorem 3.13 the chromatic polynomial of $K_3$ is

$$\begin{aligned}\chi_{K_3}(\lambda) &= (-1)^{3-1}\lambda \cdot ((1-\lambda)^2 + (1-\lambda) + 0) \\ &= \lambda(\lambda-1)(\lambda-2).\end{aligned}$$

As mentioned in Example 3.11 the chromatic polynomial of the complete graph with three vertices is exactly

$$\chi_{K_3}(\lambda) = \prod_{i=1}^{3}(\lambda - i + 1),$$

which agrees with the computation through Theorem 3.13.

**Remark.** If $G$ is not connected, one can modify the evaluation from Theorem 3.13 like this

$$\chi_G(\lambda) = \lambda^{\kappa(G)}(-1)^{|V|-\kappa(G)}t(M(G); 1-\lambda, 0),$$

with $\kappa(G)$ being the number of connected components of $G$.

[6] $K_3$ is also called the triangle $C_3$. $C_n$ denotes the graph with $n$ vertices which forms a cycle.

## 3.3 Acyclic Orientations and Nowhere-Zero Flow

A graph $G$ can be *oriented* by giving every edge $e$ in the graph a direction. Suppose $e = (v, w)$ the edge directed from $v$ to $w$ then the vertex $v$ is called tail and $w$ head of the edge. Let $\mathcal{O}$ denote an orientation of $G$ and $G_{\mathcal{O}}$ the directed graph. An orientation which does not include any directed cycles is called an *acyclic orientation* of graph $G$. There exist exactly $2^{|E|}$ possibilities to orient a graph, with $E$ being the ground set of the graph. The exact number of acyclic orientations of a graph can be calculated through the Tutte polynomial, since not only isomorphic graphs have the same number of acyclic orientations, but graphs with isomorphic cycle matroids as well. Therefore, the number of acyclic orientations is a T-G invariate. Most of the material that is discussed in this section can be found in [GM12] and also some input was taken from [EMM08].

**Theorem 3.16.** *Let $G$ be a graph with $a(G)$ denoting the number of acyclic orientations. Then $a(G) = t(M(G); 2, 0)$.*

The proof of Theorem 3.16 can be found in [GM12, Pages 339-340]. Further, calculating the Tutte polynomial of the cycle matroid $M(G)$ with $x = 0$ and $y = 2$ gives us the exact number of cycle orientations of the graph $G$, namely $t(M(G); 0, 2)$.

**Example 3.17.** Suppose we have the graph $K_3$ from Figure 3.3, which has the uniform matroid $U_{2,3}$ as its cycle matroid. $K_3$ can be oriented in $2^3 = 8$ different ways. The only cyclic orientations are those demonstrated in Figure 3.4. This leaves the graph with having 6 acyclic orientations.


Figure 3.4: Cyclic orientations of Graph $K_3$.

We now consider $U_{2,3}$, the cycle matroid of the graph $K_3$. As computed in Example 3.2 the Tutte polynomial is $t(U_{2,3}; x, y) = x^2 + x + y$. By inserting 2 as $x$ and 0 as $y$ we get $t(U_{2,3}; 2, 0) = 6$ as the number of acyclic orientations and naturally $t(U_{2,3}; 0, 2) = 2$ is the number of cyclic orientations.

Consider the graph $G$ with the edge set $E$ and the finite abelian group $H$. Define $H$ as an additive group with 0 as its identity element and associate an element from $H$ with each directed edge of $G_{\mathcal{O}}$. Let $w : E \to H$ with $e \mapsto w(e)$ be the function which assigns a so called *weight* to every edge in $E$. An $H$-flow is defined as an assignment of elements to the edges however for every vertex $v$ the sum of the weights of the edges which have $v$ as head is equal to the sum of weights of the edges which have $v$ as tail. This characteristic is called Kirchhoff's current law. An $H$-flow is *nowhere-zero* if for every $e$ in $E$ it holds that $w(e) \neq 0$.

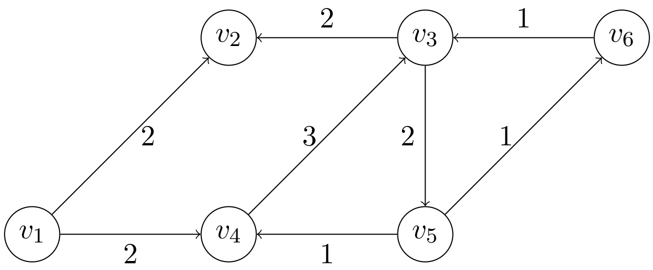



Figure 3.5: Oriented graph $G$ with a nowhere-zero flow with $H = \mathbb{Z}_4$.

For instance, consider the graph $G$ with an orientation shown in Figure 3.5. The edges were assigned weights from $\mathbb{Z}_4$ and no edge has the weight 0. As an example of Kirchhoff's current law, the only two edges incident to vertex $v_1$ are directed from $v_1$ and have each a weight of 2. Adding up all the weights would return 4. In $\mathbb{Z}_4$ this translates to 0 which is equal to the total sum of weights of edges directed in.

**Proposition 3.18.** *For a given graph $G$ and finite abelian group $H$, the number of nowhere-zero flows is independent of the orientation $\mathcal{O}$.*

*Proof.* It is possible to transform a given orientation $\mathcal{O}$ into any other orientation $\mathcal{O}'$ by reversing the direction of some edges. Let us assume $e$ is an edge in the graph. If we already have a nowhere-zero flow using a specific orientation $\mathcal{O}$, then by replacing the weight $w(e)$ with its inverse $-w(e)$ in the group H, we can generate a nowhere-zero flow in orientation $\mathcal{O}'$ with the reverse direction of $e$. In this manner, we establish a bijection between nowhere-zero flows that use $\mathcal{O}$ and those that use $\mathcal{O}'$. □

Define $\chi_G^*(k)$ as the number of nowhere-zero flows in $G$ with weights as elements in $H$ and $|H| = k$. The polynomial $\chi_G^*(k)$ will be called the *flow polynomial*.

**Theorem 3.19.** *Let $G$ be a connected graph and $H$ a finite abelian group. Then,*

$$\chi_G^*(k) = (-1)^{|V|+|E|+1} t(M(G); 0, 1-k).$$

*Proof.* Proving Theorem 3.19 is similar to the proof of Theorem 3.13, therefore, we only provide a sketch of the proof.

The flow polynomial of a graph $L$, which has only a loop as an edge is $\chi_L^*(k) = k - 1$. As for a graph $I$ with only an isthmus as an edge the flow polynomial is 0 as a polynomial, $\chi_I^*(k) = 0$. Graphs which have an isthmus (or more) cannot have a nowhere-zero flow. This is also the reason why the Tutte evaluation $\chi_G^*(k) = (-1)^{|V|+|E|+1} t(M(G); 0, 1-k)$ has $x = 0$. On the other hand, for the chromatic polynomial the Tutte evaluation $\chi_G(\lambda) = (-1)^{|V|-1}\lambda \cdot t(M(G); 1-\lambda, 0)$ has $y = 0$, since graphs with loops cannot be colored properly.

We continue by showing

$$\chi_G^*(k) = \chi_{G/e}^*(k) - \chi_{G-e}^*(k),$$

and that

$$f_G(k) = (-1)^{|V|+|E|+1} \chi_G^*(k)$$

is a T-G invariant. This gives us

$$f_I(k) = (-1)^4 \chi_I^*(k) = 0$$

and

$$f_L(k) = (-1)^3 \chi_L^*(k) = 1 - k.$$

With Theorem 3.7 it follows

$$\begin{aligned} f_G(k) &= t(M(G); f_I(k), f_L(k)) \\ t(M(G); 0, 1-k) &= (-1)^{|V|+|E|+1} \chi_G^*(k) \\ \chi_G^*(k) &= (-1)^{|V|+|E|+1} t(M(G); 0, 1-k). \end{aligned}$$

□

If the graph $G$ is not connected, one can calculate its flow polynomial by inserting $|V| + |E| + \kappa(G)$ instead of $|V| + |E| + 1$, with $\kappa(G)$ denoting the number of connected components of $G$. An important statement about the connection of the chromatic and flow polynomial is incorrectly stated in Gordon and McNulty's book [GM12, Corollary 9.34] and does not include a proof. The correct equation is expressed in Corollary 3.20 followed by its proof.

**Corollary 3.20.** *Let $G$ be a connected planar graph with $G^*$ as its connected planar dual graph. Then,*

$$\chi_{G^*}(k) = k \cdot \chi_G^*(k).$$

*Proof.* To prove Corollary 3.20 we will be using Theorem 3.19,

$$\chi_G^*(k) = (-1)^{|V|+|E|+1} t(M(G); 0, 1-k).$$

With Theorem 3.13 and Theorem 3.6 it follows,

$$\begin{aligned} \chi_{G^*}(k) &= (-1)^{|V^*|-1} k \cdot t(M(G^*); 1-k, 0) \\ &= (-1)^{|F|-1} k \cdot t(M(G); 0, 1-k). \end{aligned}$$

As discussed in Section 2.4, the number of vertices of the graph $G^*$ is equal to the number of regions $F$ of the original graph $G$. By using Euler's Formula 2.18 $|V| - |E| + |F| = 2$ the number of regions of $G$ can be calculated by $|F| = |E| - |V| + 2$ and with the fact that the sum of two numbers is even if and only if the difference is even we get

$$\begin{aligned} \chi_{G^*}(k) &= k \cdot (-1)^{|V|+|E|+1} t(M(G); 0, 1-k) \\ &= k \cdot \chi_G^*(k) \end{aligned}$$

□

To come back to our initial point, when dealing with planar graphs $G$, one can analyze the Four Color Theorem in the context of nowhere-zero flows.

**Corollary 3.21.** *If $G$ is a planar graph and no edge of $G$ is an isthmus, then $G$ has a nowhere-zero $\mathbb{Z}_4$ flow.*

*Proof.* This proof will just be a sketch.
Since $G$ is planar, its dual graph $G^*$ exists and is also planar. $G^*$ does not have any loops since $G$ does not have any isthmuses, therefore, we can apply the Four Color Theorem on the dual graph $G^*$, to show $\chi_{G^*}(4) > 0$. With Corollary 3.20 it holds that $\chi_G^*(4) > 4$. $\square$

One note to add is that Corollary 3.21 was based on [GM12, Corollary 9.35], however, the condition that no edge of $G$ can be an isthmus was added, since without it, the Corollary would not be correct, due to the fact that a graph with an isthmus cannot have a nowhere-zero flow. Although no proofs for the Four Color Theorem arose from flows, the flow polynomial and the chromatic polynomial, they still offer numerous other information and insights. For recent work in the field of flows see Jackson's [Jac07] and Dong and Koh's [DK07].

# 4 Conclusion

In conclusion, the world of matroid theory offers a broad spectrum of applications. One of many is the Tutte polynomial which has applications in computer science, engineering, optimization, physics, biology and knot theory. It is not only significantly enriching the common understanding of matroids but also of graphs. The integration of Tutte-Grothendieck invariants which can be calculated through the Tutte polynomial, provides an efficient means to explore and solve for matroid properties, simplifying complex tasks. Moreover, insights on graphs can be made through the Tutte polynomial without lots of calculation by considering the cycle matroid of a graph, for example the number of acyclic orientations, spanning trees and proper colorings.

A particularly remarkable relationship is the one between the Tutte polynomial and the chromatic polynomial. By employing the recursive definition of the Tutte polynomial and using the cycle matroid of the graph, it becomes feasible to compute the chromatic polynomial, a graph invariant that states the number of proper colorings of a graph. The flow polynomial is another polynomial that can be computed by utilizing the Tutte polynomial. It states the number of nowhere-zero flows in a graph, presenting a bridge between combinatorics and graph theory. As one was able to see, through the Tutte polynomial, it was possible to find a connection between the chromatic and the flow polynomial. Profound interconnections like these can reshape our perception of longstanding problems like the Four Color Theorem once was. By reframing it through the lens of the flow polynomial, we gain a fresh insight into this famous theorem.

For those eager to explore more mathematical developments which utilize the Tutte polynomial further, refer to *Universal Tutte Polynomial* by O. Bernardi, T. Kálmán and A. Postnikov [BKP22] and another collaborative effort of X. Guan, W. Yang and X. Jin for *On the polymatroid Tutte polynomial* [GYJ24] as well as *On chromatic and flow polynomial unique graphs* [DWY08] by the authors Y. Duan, H. Wu and Q. Yu which offer valuable insights not covered in this thesis.

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