Convex and Discrete Geometry: Ideas, Problems and Results

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1 Introduction

Convex geometry is an area of mathematics between geometry, analysis and discrete mathematics. Classical discrete geometry is a close relative of convex geometry with strong ties to the geometry of numbers, a branch of number theory. Both areas have numerous relations to other fields of mathematics and its applications. While it is out of reach to describe on one or two dozen pages the main features of convex and discrete geometry, it is well possible to show the flavor of these areas by describing typical ideas, problems and results. This will be done in the following. In particular, we consider

Mixed volumes and the Brunn-Minkowski theorem,
Polar bodies in high dimensions,
Valuations,
Euler's polytope formula,
Lattice polytopes and lattice point enumerators,
Theorems of Minkowski and Minkowski-Hlawka,
Sums of moments,
Koebe's representation theorem.

There is a huge literature on convex and discrete geometry. We mention the books of Fejes Tóth [5], Gruber and Lekkerkerker [14], Erdös, Gruber and Hammer [4], Webster [23], Ziegler [24], Matoušek [19], Barvinok [1], Grünbaum [17], the books in the red Cambridge series by Schneider [21], Gardner [6], Groemer [8], Thompson [22], McMullen and Schulte [20], the green collection of surveys [15], the Handbooks of Convex Geometry [16] and of Computational and Discrete Geometry [7] and the overviews of Klee [18], Gruber [11] and Berger [2]. Two articles of Gruber [10, 12] on convexity are in a similar spirit as the present report.

For precise references for the results considered below, additional material, comments and historical remarks see the author's [13] forthcoming book. To give the reader an idea of the development of convex and discrete geometry, we add for each author the year of publication of the result in question.

2 Mixed Volumes and the Brunn-Minkowski Theorem

A *convex body* is a compact convex set in d-dimensional Euclidean space $\mathbb{E}^d$. The space $\mathcal{C} = \mathcal{C}(\mathbb{E}^d)$ of all convex bodies is endowed with (*Minkowski*) *addition* $+$ which is defined as follows:

$$C + D = \{x + y : x \in C, y \in D\} = \bigcup_{x \in C} (x + D) \text{ for } C, D \in \mathcal{C}.$$

It is easy to show that $C + D$ is again a convex body. With this definition of addition, the space of convex bodies is an Abelian semigroup with cancellation law.

A *major problem* of the Brunn-Minkowski theory is to obtain information on the volume $V(C + D)$ of $C + D$ in terms of information on C and D. The first pertinent result is *Steiner's formula for the volume of parallel bodies* (1840). Let B^d be the solid Euclidean unit ball of $\mathbb{E}^d$. For a convex body C and $\lambda \geq 0$ the convex body $C + \lambda B^d$ is the *parallel body* of C at distance λ. Steiner's formula then says, that there are coefficients $W_0(C), \dots, W_d(C)$, the *quermassintegrals* of C, such that

$$V(C+\lambda B^d)=W_0(C)+\binom{d}{1}W_1(C)\lambda+\cdots+\binom{d}{d}W_d(C)\lambda^d \text{ for } \lambda\geq 0.$$

$W_0(C)=V(C)$, $W_1(C)=S(C)/d$, the surface area of C, and $W_d(C)=V(B^d)$. It is an unsolved *problem of Blaschke* (1916) to characterize the $d+1$ tuples of real numbers which are quermassintegrals of convex bodies, or, in other words, to describe the set

$$\big\{\big(W_0(C),\dots,W_d(C)\big):C\in\mathcal{C}(\mathbb{E}^d)\big\}\subset\mathbb{E}^{d+1}.$$

A substantial refinement of Steiner's formula is *Minkowski's theorem on mixed volumes* (1911): Given convex bodies $C_1,\dots,C_n\in\mathcal{C}$, there are coefficients $V(C_{i_1},\dots,C_{i_d})$, called *mixed volumes*, such that

$$V(\lambda_1C_1+\cdots+\lambda_nC_n)=\sum_{i_1,\dots,i_d=1}^{n}V(C_{i_1},\dots,C_{i_d})\,\lambda_{i_1}\cdots\lambda_{i_d} \text{ for } \lambda_1,\dots,\lambda_n\geq 0.$$

Clearly, the following equalities hold: $V(C,\dots,C)=W_0(C)=V(C), V(C,\dots,C,B^d)=W_1(C)$ $=S(C)/d,\dots,V(C,B^d,\dots,B^d)=W_{d-1}(C),V(B^d,\dots,B^d)=W_d(C)=V(B^d)$ for $C\in\mathcal{C}$.

There are several relations among mixed volumes, including the fundamental *inequality of A.D.Alexandrov and Fenchel* (1937/36) which says that

$$V(C,D,D_3,\dots,D_d)^2\geq V(C,C,D_3,\dots,D_d)V(D,D,D_3,\dots,D_d)$$

for $C,D,D_3,\dots,D_d\in\mathcal{C}$. This inequality is one of the most important geometric inequalities. It is an *unsolved problem* to determine the equality cases in the Alexandrov-Fenchel inequality.

Numerous other geometric inequalities are simple consequences of the Alexandrov-Fenchel inequality. We state two. First, the *isoperimetric inequality*:

$$\frac{S(C)^d}{V(C)^{d-1}}\geq\frac{S(B^d)^d}{V(B^d)^{d-1}}$$

for $C\in\mathcal{C}$ with non-empty interior. Here equality holds precisely in case where C is a Euclidean ball. There is no corresponding upper bound for the *isoperimetric quotient*

$$\frac{S(C)^d}{V(C)^{d-1}},$$

but Ball (1991) proved the following *reverse isoperimetric inequality*: Let C be an o-symmetric convex body with non-empty interior, where o is the origin. Then there is a non-singular linear transfomation $T:\mathbb{E}^d\to\mathbb{E}^d$ such that

$$\frac{S(TC)^d}{V(TC)^{d-1}}\leq(2d)^d.$$

Secondly, the *Brunn-Minkowski inequality* (1887/96):

$$V(C+D)^{\frac{1}{d}}\geq V(C)^{\frac{1}{d}}+V(D)^{\frac{1}{d}}$$

for $C,D\in\mathcal{C}$. Here equality holds precisely if C and D are in parallel hyperplanes or are positive homothetic. The Brunn-Minkowski inequality, which also yields the isoperimetric inequality, has led to numerous generalizations. We mention the Prékopa-Leindler inequality for integrals (1971/72) and the so-called generalized isoperimetric inequalities together with the related concentration of measure phenomenon on metric probability spaces.

The modern theory of mixed volumes deals with surface and curvature measures.

In the last decades it turned out that there are several bridges between convex and algebraic geometry. Bridgeheads on the convexity side are mixed volumes and the Alexandrov Fenchel inequality, other ones are Newton polytopes of systems of algebraic equations.

3 Polar Bodies in High Dimensions

Given a convex body C in $\mathbb{E}^d$ with the origin in its interior, its *polar body* C^* is defined by

$$C^* = \{y : x \cdot y \leq 1 \text{ for all } x \in C\}.$$

It is easy to see that C^* is again a convex body with the origin in its interior.

In view of applications in the geometry of numbers, it is of interest to give upper and lower bounds for the quantity

$$V(C)V(C^*).$$

Blaschke (1917) for $d = 3$ and Santaló (1949) for general d, showed that for convex bodies C which are symmetric in o,

$$V(C)V(C^*) \leq V(B^d)^2,$$

where equality holds precisely in case where C is a Euclidean ball. Thinking of applications in the geometry of numbers, Mahler (1939) *conjectured* that for an o-symmetric convex body C with non-empty interior the following inequality holds:

$$\frac{4^d}{d!} \leq V(C)V(C^*).$$

He could prove this only with the smaller constant $4^d/d!^2$. While Mahler's conjecture is still open, Bourgain and Milman (1989) showed that there is an absolute constant $c > 0$ such that in all dimensions d for all o-symmetric convex bodies C with non-empty interior one has,

$$\frac{c^d}{d!} \leq V(C)V(C^*).$$

We have chosen this result as a typical one for the local theory of normed spaces. This theory started with a striking theorem of Dvoretzky (1961) which says that any normed space of sufficiently large finite dimension has comparatively large subspaces which are almost Euclidean. Characteristically, local theory results deal with properties of convex bodies, which are independent of the dimension. In many cases this means that the phenomenon in question is determined by an absolute constant.

4 Valuations

A real *valuation* on the space of convex bodies is a function $\phi : \mathcal{C} \to \mathbb{R}$ with the following weak additivity property:

$$\phi(C \cup D) + \phi(C \cap D) = \phi(C) + \phi(D) \text{ for } C, D, C \cup D, C \cap D \in \mathcal{C}$$

$$\text{and } \phi(\emptyset) = 0.$$

Besides real valuations on $\mathcal{C}$, valuations on certain families of sets with values in an Abelian group have been studied. Valuations can be found at many corners of convex and discrete geometry, but it was only Blaschke (1935/37) who first defined valuations and started their systematic study. Examples of valuations are the volume and the surface area, more generally the quermassintegrals, the affine surface area, lattice point enumerators and certain Hamel functions on the space of convex polytopes. Mixed volumes also give rise to valuations. Measure and Jordan measure are valuations on the space of measurable, resp. Jordan measurable sets.

The space $\mathcal{C}$ of convex bodies is endowed with a natural topology. The *main problem* in the theory of valuations is the following: given properties such as continuity or semi-continuity, monotony, translation or rigid motion invariance, describe all valuations with these properties. Here, a function $\phi : \mathcal{C} \to \mathbb{R}$ is *rigid motion invariant* if

$$\phi(\rho C) = \phi(C) \text{ for each } C \in \mathcal{C} \text{ and each rigid motion } \rho : \mathbb{E}^d \to \mathbb{E}^d.$$

The *functional theorem of Hadwiger* (1951), anticipated in vague form by Blaschke (1935/37), can be stated as follows: The continuous, rigid motion invariant valuations on $\mathcal{C}$ are precisely the functions $\phi : \mathcal{C} \to \mathbb{R}$ which have a representation of the form

$$\phi = \lambda_0 W_0 + \cdots + \lambda_d W_d \text{ with } \lambda_0, \ldots, \lambda_d \in \mathbb{R}. \tag{1}$$

A recent *functional theorem of Ludwig and Reitzner* (1999) shows that the upper or lower semi-continuous valuations on $\mathcal{C}$ which are invariant with respect to volume-preserving affinities on $\mathbb{E}^d$ are precisely the functions $\phi : \mathcal{C} \to \mathbb{R}$ of the form

$$\phi = \lambda_0 \chi + \lambda_1 A + \lambda_2 V \text{ with } \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}.$$

Here χ and A denote the Euler characteristic and the affine surface area on $\mathcal{C}$. The latter is a notion of surface area which is invariant with respect to volume-preserving affinities. Originally it was introduced in affine differential geometry. In recent years it turned out to be a valuable tool in convex geometry, for example for the problem of approximation of convex bodies by convex polytopes.

These and similar results are satisfying from an aesthetic point of view. Rota even considered Hadwiger's functional theorem to be one of the ten most beautiful theorems of mathematics in the 20th century.

Typical applications of such functional theorems are as follows: Let f be a real function on $\mathcal{C}$. If it can be shown that f is, say, continuous and rigid motion invariant, the functional theorem implies that it is of the form (1). Possibly, a homogeneity property then yields that f is a multiple of the volume or of some other quermassintegral. The latter may be a relevant geometric result. In this way easy proofs of the *principal kinematic formula* in integral geometry and of a qualitative form of the *Minkowski-Hlawka theorem* in the geometry of numbers can be achieved.

5 Euler's Polytope Formula and Related Topics

Given a convex polytope in $\mathbb{E}^3$ with v vertices, e edges and f facets, *Euler's polytope formula* (1752/53) says that

$$v - e + f = 2$$

It is surprising that this simple result was not known already in antiquity. A hundred years before Euler, Descartes gave a result of which Euler's formula is an easy consequence. Euler's proof has been criticized since it used implicitly an argument of the type of the Jordan curve theorem. This criticism should not be taken too seriously, since for the proof Jordan's theorem is needed only in the easy case of polygonal curves. Legendre gave a beautiful alternative proof, using the well-known area formula for spherical polygons.

Schläfli (1850/52) extended Euler's formula to all dimensions: Consider a convex polytope P in $\mathbb{E}^d$ and let f_i be the number of its i-dimensional faces for $i = 0, \ldots, d-1$. The vector $f(P) = (f_0, \ldots, f_{d-1})$ is called the *f-vector* of P. Then

$$f_0 - f_1 + - \cdots + (-1)^{d-1} = 1 + (-1)^{d-1}. \tag{2}$$

Unfortunately, Schläfli's proof contained a serious gap, which was filled only in 1970 by means of the *shelling theorem of Bruggesser and Mani*. A topological proof of Poincaré (1893/99) also had a gap, which could be filled by tools from algebraic topology achieved only by the 1930s. The first elementary proof of (2) is due to Hadwiger (1955).

Considering Euler's polytope formula, the *problem* arises to characterize the integer vectors $(f_0, \dots, f_{d-1})$ which are f-vectors of convex polytopes. For $d = 3$ this was solved by Steinitz (1906): An integer vector (v, e, f) is an f-vector of a convex polytope in $\mathbb{E}^d$ precisely in case where

$$v - e + f = 2,\ 4 \le v \le 2f - 4,\ 4 \le f \le 2v - 4.$$

The problem remains unsolved in dimensions greater than 3. For the important special case of convex polytopes where all faces are simplices, McMullen (1971) stated his *g-conjecture* which characterizes the f-vectors. This conjecture was confirmed by Stanley (1980), Billera and Lee (1981) and McMullen (1993), using deep algebraic tools.

Even questions about the f-vectors of convex polytopes which, presumably, are less demanding remain open. An example is the following *question of Ziegler*: Give precise bounds for

$$\frac{f_1 + f_2}{f_0 + f_3 + 20},$$

where (f_0, f_1, f_2, f_3) ranges over all f-vectors of convex polytopes in $\mathbb{E}^4$.

Given a convex polytope, its faces of all dimensions, including the empty face, form a polytopal cell complex in the sense of algebraic topology, the *boundary complex* of the polytope. The *problem* arises, to characterize the polytopal cell complexes which are (isomorphic to) boundary complexes of convex polytopes. For $d = 3$ this problem was solved by Steinitz (1922). His result is usually stated in the following form: Each 3-connected planar graph is isomorphic to the edge graph of a convex polytope in $\mathbb{E}^3$. In higher dimensions this problem is far from a solution.

The Euler polytope theorem and its extensions have numerous applications in many fields of mathematics, in particular in convex and discrete geometry. We state two. First, the average number of edges of a facet of a convex polytope in $\mathbb{E}^3$ is less than 6. This shows that, in particular, there is no convex polytope in $\mathbb{E}^3$ having only hexagonal facets. Second, *Cauchy's rigidity theorem* (1813): Consider a closed, convex polyhedral surface in $\mathbb{E}^3$, such that all facets are rigid and such that the facets are connected along common edges by hinges. Then the surface is still rigid. Minor errors in Cauchy's proof were corrected later on. Connelly (1978) showed that the rigidity does not hold if the convexity assumption is abandoned: There are (non-convex) flexible polytopal spheres. Recently, Sabitov (1998) proved that the volume of a flexible polytopal sphere remains constant while flexing, thus confirming the *bellows conjecture*.

6 Lattice Polytopes and Lattice Point Enumerators

Let $\mathbb{Z}^d$ denote the *integer lattice* in $\mathbb{E}^d$, i.e. the set of all points with integer coordinates. A convex polytope is a *lattice polytope* if all its vertices are points of $\mathbb{Z}^d$. Define the *lattice point enumerators* L and L^o and $L^{\cdot}$ on the space of all convex lattice polytopes P by

$$L(P) = \#(P \cap \mathbb{Z}^d),\ L^o(P) = \#((\text{relative interior of } P) \cap \mathbb{Z}^d),\ L^{\cdot}(P) = L(P) - L^o(P),$$

where # stands for cardinal number.

A simple *theorem of Pick* (1899) is the following: Let P be a Jordan lattice polygon, that is a planar, not necessarily convex polygon whose boundary is a Jordan polygon and such that all vertices of P are in $\mathbb{Z}^2$. Then

$$V(P) = L(P) - \frac{1}{2} L^{\cdot}(P) - 1.$$

Here, by V we mean the area in $\mathbb{E}^2$. Examples show that no direct extension of this result to higher dimensions is possible. Considering, besides the lattice polytope P, lattice polytopes of the form nP where $n = 1, \dots, d$, Reeve (1957/59) and Macdonald (1963) proved the following result: Let P be a convex lattice polytope with non-empty interior. Then

$$\text{(i)}\ \ d!V(P) = L(dP) - \binom{d}{1} L\big((d-1)P\big) + - \cdots + (-1)^{d-1} \binom{d}{d-1} L(P) + (-1)^d,$$

$$\begin{aligned}\text{(ii)}\ \ \frac{(d-1)d!}{2} V(P) &= M\big((d-1)P\big) - \binom{d-1}{1} M\big((d-2)P\big) - + \cdots \\ &\cdots + (-1)^{d-2} \binom{d-1}{d-2} M(P) + \frac{1}{2} + \frac{1}{2}(-1)^d, \\ &\text{where } M(P) = L(P) - \frac{1}{2} L^{\cdot}(P) = \frac{1}{2}\big(L(P) + L^o(P)\big).\end{aligned}$$

These results lead to the study of the quantities $L(nP), L^o(nP), L^{\cdot}(nP)$ for $n \in \mathbb{N}$, where P is a lattice polytope. Ehrhart (1967) proved the following *polynomiality theorem*:

$$L(nP) \text{ is a polynomial of degree } d \text{ in } n \in \mathbb{N}.$$

The constant term in this polynomial is 1 and the leading coefficient equals $V(P)$. More general is the *lattice point theorem of McMullen and Bernstein* (1975/76). It resembles Minkowski's theorem on mixed volumes: Let $P_1, \dots, P_m$ be convex lattice polytopes. Then

$$L(n_1 P_1 + \cdots + n_m P_m) \text{ is a polynomial in } n_1, \dots, n_m \in \mathbb{N}.$$

The above lattice point enumerators are valuations on the space of all lattice polytopes in $\mathbb{E}^d$ which, in addition, are *integer unimodular invariant*. This means: If U is an integer $d \times d$ matrix with determinant ± 1 and $u \in \mathbb{Z}^d$, then $L(UP + u) = L(P)$ for each convex lattice polytope P, and similarly for L^o and $L^{\cdot}$. The *valuation theorem of Betke and Kneser (1985)* describes these valuations: The integer unimodular valuations on the space of convex lattice polytopes in $\mathbb{E}^d$ with ordinary addition and multiplication with real numbers form a real vector space of dimension $d + 1$. This space has a basis $\{L_0, \dots, L_d\}$ such that

$$L_i(nP) = n^i L_i(P) \text{ and } L(nP) = L_0(P) + L_1(P)n + \cdots + L_d(P)n^d$$

for convex lattice polytopes P and $n \in \mathbb{N}$.

These results indicate the rich structure of the space of lattice polytopes. Lattice polytopes play a prominent role in several areas of mathematics and its applications, for example in optimization and crystallography. Newton polytopes, that is convex lattice polytopes determined by polynomials and systems of polynomials in one or several variables, convey important information on the polynomials, resp. on the systems of polynomials.

7 Theorems of Minkowski and Minkowski-Hlawka

Let $f : \mathbb{E}^d \to [0, \infty)$, say, a positive definite quadratic form, and let $c > 0$ be a constant. The problems arise, first, to find out whether the inequality

$$f(u) \leq c$$

has integer solutions different from o and, secondly, to determine in the positive case such solutions. These are questions of Diophantine approximation which led to the development of the geometry of numbers and which, in recent years, have been studied in the context of algorithmic geometry.

A *lattice* L in $\mathbb{E}^d$ is the system of all integer linear combinations of d linearly independent vectors. These vectors are said to form a basis of L and the absolute value of their determinant is the *determinant* $d(L)$ of L. An example of a lattice is the integer lattice $\mathbb{Z}^d$. It has determinant 1. The first of the above problems amounts to the question whether the set $C = \{x : f(x) \leq c\}$

contains a point $u \neq o$ of the integer lattice. An answer, which in many cases is satisfactory, is given by the *fundamental theorem of Minkowski* (1893): Let C be a convex body with center o and let L be a lattice in $\mathbb{E}^d$. If $V(C) \geq 2^d d(L)$, then C contains at least one pair of points $\pm l \neq o$ of L. On the other hand, we have the following *theorem of Minkowski-Hlawka* (1944): Let J be a Jordan measurable set in $\mathbb{E}^d$ where $V(J) \geq 1$. Then there is a lattice L with $d(L) = 1$ such that J contains no point $l \neq o$ of L.

In order to state more geometric versions of these results, we need the following definitions: Let C be a convex body with center o and L a lattice in $\mathbb{E}^d$. The family $\{C + l : l \in L\}$ of translates of C by lattice vectors is said to be a *lattice packing* of C with packing lattice L if no two of the translates overlap. Its *density* then is the proportion of space covered by the bodies of the packing. In this terminology the fundamental theorem of Minkowski is almost trivial. It simply says that the density of a lattice packing of C is at most one. The Minkowski-Hlawka theorem readily implies that for each convex body C with center o there is a lattice packing with density at least 2^{-d}.

For many years it was thought that the tiny lower bound 2^{-d} for the maximum density of a lattice packing of a convex body with center o was far from the truth. Now it is believed by many mathematicians that for general C no essential refinement of this bound is possible, not even in the case where C is a solid Euclidean ball. This is even more surprising, since all known proofs of the Minkowski-Hlawka theorem are based on mean value arguments, so one is led to think - presumably erroneously - that essential refinements are feasible.

Considering the results of Minkowski and Minkowski-Hlawka, in recent years the questions were considered to specify algorithms to find points of a given lattice in a given convex body and to find lattices which provide dense lattice packings.

A surprisingly efficient algorithm of Betke and Henk (2000) finds densest lattice packings of convex polytopes in dimension 3. As a consequence of the LLL-reduction algorithm for positive definite quadratic forms, Lenstra, Lenstra and Lovász (1982) specified a polynomial algorithm to find lattice points in a convex body of large volume in $\mathbb{E}^d$. Good binary error-correcting codes may be considered as subsets of the set of vertices of the unit cube in $\mathbb{E}^d$ such that any two vertices of such a subset have large distance. Thus suitable balls with centers at these vertices do not overlap, hence give rise to a finite packing of balls. This packing may be continued periodically to give a packing of balls in $\mathbb{E}^d$ which, in some cases, is even a lattice packing. This relation between error-correcting codes and periodic or even lattice packing of balls was discovered by Leech and Sloane (1964/71) and culminated in the work of Rush (1989), who finally constructed in this way lattice packings of balls with density $2^{-d+o(d)}$ as $d \to \infty$, thus reaching the Minkowski-Hlawka bound. Unfortunately, the codes used by Rush are not given in an constructive manner.

We believe that the following *heuristic principle* holds in many contexts: If a situation is sufficiently complicated, then - cum grano salis - the average object is extremal. Here 'sufficiently complicated' may mean 'of sufficiently high dimension' or 'with sufficiently many parameters'. It seems that the Minkowski-Hlawka density bound for lattice packings of balls provides one such example, other examples can be found in the local theory of normed spaces.

8 Sums of Moments

Let J be a Jordan measurable subset of $\mathbb{E}^d$. We consider the *problems* to determine or, at least, to estimate for $n = 1, 2, \dots$ the minimum

$$\min_{\{c_1,\dots,c_n\}\subset\mathbb{E}^d} \int_J \min_{\{c_1,\dots,c_n\}} \{||x - c_i||^2\}\, dx$$

and to describe the n-tuples $\{c_1, \dots, c_n\}$ for which it is attained, the *minimizing configurations*. The integral may be interpreted as the volume above sea level of a mountain landscape with n valleys which have the form of pieces of paraboloids of revolution and with deepest points at

$c_1, \ldots, c_n$.

Over the last 60 years different versions of these problems appeared in various fields. We state three pertinent results. The first is the *inequality for sums of moments of L.Fejes Tóth* (1952) which is as follows: Let $f : [0, \infty) \to [0, \infty)$ be monotone increasing and let H be a convex 3,4,5 or 6-gon. Then

$$\min_{\{c_1,\ldots,c_n\} \in \mathbb{E}^2} \int\limits_H \min_{\{c_1,\ldots,c_n\}} \{f(||x - c_i||)\}\, dx \geq \int\limits_{H_n} f(||x||)\, dx,$$

where H_n is a regular hexagon with center at the origin and area equal to $V(H)/n$. The second result is due to the author (2004) and refines an earlier result of Zador (1982): Let $f : [0, \infty) \to [0, \infty)$ satisfy a certain growth condition. Then there are positive constants div and α depending on f and d, resp. on f, such that

$$\min_{\{c_1,\ldots,c_n\} \in \mathbb{E}^d} \int\limits_J \min_{\{c_1,\ldots,c_n\}} \{f(||x - c_i||)\}\, dx \sim \operatorname{div} V(J)^{\frac{d+\alpha}{d}} f\Big(\frac{1}{n^{\frac{1}{d}}}\Big) \text{ as } n \to \infty.$$

Third, let $C_n = \{c_1, \ldots, c_n\}$, $n = 1, 2, ...$, be a sequence of minimizing configurations for $d = 2$. Then a result of Gruber (2001), reproved by G.Fejes Tóth (2001), shows that asymptotically as $n \to \infty$, C_n is a 'regular hexagonal pattern'. This confirms a *conjecture of Gersho* (1979) in case $d = 2$. For $d > 2$ Gersho's conjecture remains open. We doubt that it is true for sufficiently large dimensions.

These results have numerous applications. We mention two, one in discrete and one in convex geometry: The maximum density of a packing in $\mathbb{E}^2$ with circular discs, all of the same radius, equals

$$\frac{\pi}{\sqrt{12}}.$$

The convex polytopes P_n in $\mathbb{E}^3$ with n facets and minimum isoperimetric quotient have isoperimetric quotient

$$\frac{S(P_n)^d}{V(P_n)^{d-1}} \sim 36\pi + 20\sqrt{3}\pi^2 \frac{1}{n} = 113.09... + \frac{341.98...}{n} \text{ as } n \to \infty$$

and in an asymptotic sense their facets are regular hexagons, all of the same size. (The simple consequence of Euler's polytope formula on the average number of edges of a facet of a convex polytope in $\mathbb{E}^3$ mentioned in Section 5 shows that not all facets can be hexagons.) Other applications deal with data transmission, numerical integration, probability and approximation theory.

9 Koebe's Representation Theorem for Planar Graphs

Given a (finite) planar graph $\mathcal{G}$, Koebe (1936) showed in his *representation theorem for planar graphs* that one may assign to each vertex of $\mathcal{G}$ a circular disc such that these discs form a packing in $\mathbb{E}^2$. Two discs touch precisely in the case where the corresponding vertices are connected by an edge. This result was re-discovered by Andreev (1970) and by Thurston (1978). The latter specified an algorithm how to find such packings. An essential refinement of Koebe's theorem is due to Brightwell and Scheinerman (1993): Let $\mathcal{G}$ be a 3-connected planar graph. Then to each vertex there corresponds a circular disc. These discs form a packing such that any two discs touch precisely in the case where the corresponding vertices are connected by an edge. Dually, to each country of the graph corresponds a circular disc such that these discs also form a packing and any two discs touch precisely in the case where the corresponding countries have a common edge. Finally, for any edge of $\mathcal{G}$ the discs corresponding to its vertices and the discs corresponding to the adjacent countries have a common point where the vertex circles intersect the country circles orthogonally.

Koebe's theorem, its generalizations and the algorithm of Thurston have numerous applications, in particular in graph theory. We consider two deep applications. First, by stereographic projection, the Brightwell-Scheinerman theorem easily yields the Steinitz representation theorem for convex polytopes mentioned in Section 5. Secondly, let D be a simply connected domain in the complex plane. Then the algorithm of Thurston permits to construct arbitrarily precise piecewise linear approximations to the analytic function which maps D onto the unit disc.

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