Approximation of Developable Surfaces with Cone Spline Surfaces

Stefan Leopoldseder and Helmut Pottmann
Institut für Geometrie
Technische Universität Wien
Wiedner Hauptstrasse 8–10
A–1040 Wien, Austria

December 31, 2003

Abstract

Developable surfaces are modelled with pieces of right circular cones. These cone spline surfaces are well-suited for applications: They possess degree two parametric and implicit representations. Bending sequences and the development can be explicitly computed and the offsets are of the same type. The algorithms are based on elementary analytic and constructive geometry. There appear interesting relations to planar and spherical arc splines.

Keywords: developable surface, rational Bézier representation, NURBS, arc spline, circular spline.

INTRODUCTION

Developable surfaces are special ruled surfaces which can be unfolded or developed into a plane without stretching or tearing. Mathematically speaking, these surfaces can be mapped isometrically into the Euclidean plane. Because of this property, they are of considerable importance to sheet-metal and plate-metal based industries. Applications include windshield design, binder surfaces for sheet metal forming processes, aircraft skins, ship hulls and others (see e.g. [16]).

To include developable surfaces into current CAD/CAM systems, they need to be represented as NURBS surfaces [7, 20]. There are basically two approaches to rational developable surfaces. First, one can express such a surface as a tensor product surface of degree $(1, n)$ and solve the nonlinear side conditions expressing the developability

[1, 2, 8, 15]. Second, we can view the surface as envelope of its one parameter set of tangent planes and thus treat it as a curve in dual projective space [4, 5, 11, 12, 13, 22, 23, 26, 28]. Based on the latter approach, some interpolation and approximation algorithms as well as initial solutions to special applications have been presented recently [12, 13, 23].

The numerical computation of the isometric mapping of a developable surface into the plane has been treated by several authors (see e.g. [6, 10, 14, 27]).

For applications, where the development of the designed surfaces needs to be computed with high accuracy, it is desirable to avoid numerical integration techniques. Therefore, we study here the design of *smooth developable surfaces with pieces of cones of revolution*: these surfaces possess the lowest possible parametric and implicit degree for designing G^1 surfaces, their development and bending into other developable shapes is elementary and their offsets are of the same type. We call these surfaces *cone spline surfaces* henceforth.

During the preparation of the revised version of the present paper, we became aware of a contribution by Redont [25] on cone spline surfaces. However, there appears to be no overlap with our work. Redont studies the surfaces from a purely differential geometric point of view and proposes a global approximation algorithm. Our algorithms are local and based on a careful study of cone spline surfaces using tools from various branches of classical geometry. In this way we can get valuable insight into the degrees of freedom, the variety of feasible solutions and ways for optimizing them.

Our approach to approximation of a given developable surface Γ by a cone spline surface is based on the following strategy. Choose an appropriate sequence of rulings plus tangent planes of Γ. Then each pair of consecutive rulings plus tangent planes (G^1 Hermite data) can be interpolated by a smoothly joined pair of right circular cones. These cones either have the same vertex or they possess parallel axes or a common inscribed sphere. In the first two cases, we get a construction using spherical or planar biarcs. In the third, general case, the cones touch the inscribed sphere, which is computable from the input, along a spherical biarc. Using known results on biarcs [9], we see that there exists a one parameter family of solutions to the present G^1 Hermite problem. We discuss the choice of good solutions within this set. Furthermore, a certain spatial counterpart to the planar osculating arc splines of Meek and Walton [18] is presented. The efficiency and practicality of the proposed algorithms is discussed at hand of several examples.

FUNDAMENTALS AND PROBLEM DESCRIPTION

Developable surfaces can be isometrically mapped (*developed*) into the plane. Assuming sufficient differentiability, they are characterized by the property of possessing vanishing Gaussian curvature. All nonflat developable surfaces are envelopes of one parameter sets of planes. Such a developable surface is either a conical surface, a cylindrical surface, the tangent surface of a twisted curve or a composition of these three surface types. Thus, developable surfaces are ruled surfaces, but with the special property that they possess the same tangent plane at all points of the same generator.

For an analytical treatment, we will work in the projective extension P^3 of real Euclidean 3-space E^3. We use homogeneous Cartesian coordinates (x_0, x_1, x_2, x_3), collected in the 4-vector $\mathbf{X}$. The one dimensional subspace of $\mathbb{R}^4$ spanned by $\mathbf{X}$ is a point in P^3. This point will also be denoted by $\mathbf{X}$ if no ambiguity can result. For points not at infinity, i.e., $x_0 \neq 0$, the corresponding inhomogeneous Cartesian coordinates are $x = x_1/x_0,\ y = x_2/x_0,\ z = x_3/x_0$; they are comprised in $\mathbf{x} = (x, y, z)$.

The inhomogeneous parametric representation of a ruled surface Γ is

$$\mathbf{g}(u, v) = \mathbf{l}(u) + v\mathbf{e}(u), \tag{1}$$

where $\mathbf{l}(u)$ represents a curve on Γ and $\mathbf{e}(u)$ are unit vectors of the generator lines. The condition that (1) represents a developable surface is

$$\det(\dot{\mathbf{l}}, \mathbf{e}, \dot{\mathbf{e}}) = 0. \tag{2}$$

For a cylinder $\mathbf{e}$ is constant, and for a cone we may choose $\mathbf{l} = \mathit{const}$ as the vertex. In a differential geometric treatment, a tangent surface is written in the form (1) with u as arc length of the line of regression $\mathbf{l}$, and $\mathbf{e} = \dot{\mathbf{l}}(u)$. Let $\mathbf{e}, \mathbf{p}, \mathbf{n}$ be the Frenet frame vectors of $\mathbf{l}$, with $\mathbf{p}$ and $\mathbf{n}$ as principal normal and binormal, respectively, and let $\varkappa$ and τ be curvature and torsion of $\mathbf{l}$. Then the *Darboux vector*

$$\mathbf{d}(u) = \tau(u)\mathbf{e}(u) + \varkappa(u)\mathbf{n}(u) \tag{3}$$

defines the axis $\mathbf{a} = \mathbf{l} + \lambda\mathbf{d}$ of a cone of revolution $\Delta(u)$ with vertex $\mathbf{l}$, which touches Γ along the generator $e(u)$. Δ is called *osculating cone*, since it has contact of order 2 with Γ at all regular points of the common ruling. This cone may be considered as the counterpart of the osculating circle of a curve; it determines the curvature behaviour of a developable surface along a ruling. Formula (3) is not applicable for a cone or cylinder Γ. If Γ is a cone, the osculating cone Δ contains the osculating circle of the spherical curve $\mathbf{c}(u) = \mathbf{l} + \mathbf{e}(u)$; for a cylinder, one may compute its intersection curve $\mathbf{c}$ with a plane normal to the generators. Then, the osculating cylinders pass through the osculating circles of $\mathbf{c}$.

The osculating cone or cylinder may degenerate to a plane; the corresponding generator of Γ is then called an *inflection generator*. For a cylinder Γ this happens, if the normal section $\mathbf{c}$ has an inflection point (point with vanishing curvature). A cone Γ has an inflection generator, if the spherical curve $\mathbf{c}$ sends its osculating plane through the vertex. Finally, a tangent surface has an inflection generator if the corresponding point of the line of regression possesses vanishing torsion.

A plane $u_0 + u_1 x + u_2 y + u_3 z = 0$ can be represented by its homogeneous plane coordinates $\mathbf{U} = (u_0, u_1, u_2, u_3)$. We will also use oriented planes and represent them by normalized plane coordinates, where the normal vector (u_1, u_2, u_3) is normalized, $u_1^2 + u_2^2 + u_3^2 = 1$, and determines the orientation.

The "dual approach" to developable surfaces interprets a *developable NURBS surface* as set of its tangent planes $\mathbf{U}(t)$. It can then be written as

$$\mathbf{U}(t) = \sum_{i=0}^{n} \mathbf{U}_i \, N_i^k(t), \tag{4}$$

with the normalized B–spline functions $N_i^k(t)$ of degree k over a given knot vector V. The vectors $\mathbf{U}_i$ are the homogeneous plane coordinate vectors of the *control planes*, also denoted by $\mathbf{U}_i$.

Each generator of the developable surface follows from (4) as intersection of the plane $\mathbf{U}(t)$ and its derivative $\dot{\mathbf{U}}(t)$. In particular, the boundary rulings of the surface are the intersections of the boundary control planes $\mathbf{U}_0 \cap \mathbf{U}_1$ and $\mathbf{U}_{n-1} \cap \mathbf{U}_n$. The cuspoidal edge or line of regression is obtained as intersection $\mathbf{U}(t) \cap \dot{\mathbf{U}}(t) \cap \ddot{\mathbf{U}}(t)$. In general, it is a Bézier or B–spline curve of degree $3k - 6$. Recently, algorithms for the computation with the dual representation, the conversion to the standard tensor product representation and the solution of interpolation and approximation algorithms have been developed [12, 13, 23].

For $k = 2$, the developable NURBS surface is composed of pieces of quadratic cones or cylinders. Apart from the cone vertices, the surface is G^1, i.e., adjacent cones or cylinders are tangent to each other along the common ruling. Even in this simple case, the development of the surface can, in general, not be given in terms of elementary functions. Therefore, we will now study the case where the *surface is composed of right circular cones or cylinders only.* These surfaces shall be called *cone spline surfaces* henceforth. Figure 1 shows the Bézier planes of a right circular cone segment. Using normalized plane coordinates $\mathbf{U}_0$ and $\mathbf{U}_2$, the Bézier plane $\mathbf{U}_1 = (u_{10}, u_{11}, u_{12}, u_{13})$ contains the boundary generators and has the weight

$$w_1 = \sqrt{u_{11}^2 + u_{12}^2 + u_{13}^2} = \frac{\cos\alpha}{\cos\beta},$$

with α and β denoting the angle between the axis a and $\mathbf{U}_0$ or $\mathbf{U}_1$, respectively. It is

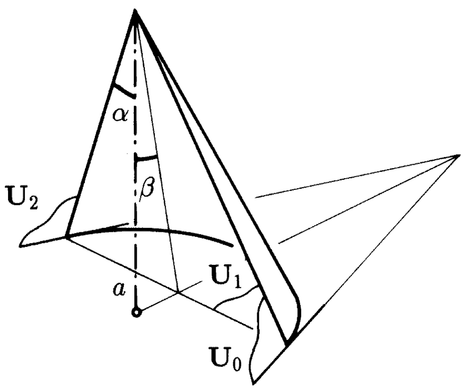


Figure 1: Cone segment

well known that the point set of such a cone segment could also be represented by a rational Bézier tensor product surface of degree (1,2) (see e.g. [20]).

Modeling with cone spline surfaces is thought in the following way. First, the shape of the surface is described in some analytic form, for example (4) with arbitrary degree k. Together with this G^1 developable surface Γ we prescribe a region of interest Ω in which Γ is free of singularities. Ω may be a simply connected region, for example a bounding box or a ball, which makes it computationally easy to decide whether or not a point lies in Ω. We would like to approximate Γ by a cone spline surface Λ which is free of singularities inside Ω.

G^1 HERMITE ELEMENTS

The idea is to select a sequence of rulings e_i of Γ, compute their tangent planes τ_i and interpolate consecutive G^1 elements (e_i, τ_i) with two segments of cones of revolution, which possess the same tangent plane along the common generator. Throughout this paper, we will tacitly allow that the cone may degenerate to a cylinder and mention this case only if necessary. Furthermore, we will simply refer to a segment of a right circular cone, bounded by two generators, as a *cone segment*. The two cone segments which interpolate the given G^1 *Hermite data* form a G^1 *cone pair*.

For computing a practically useful solution, it is necessary to select special generators of Γ and to introduce some orientations.

Since cones of revolution do not have inflection generators, the typical behaviour of an inflection generator can only be represented at a junction of two cones. Therefore, inflection generators belong to the selected generators. We also select those rulings of the given surface Γ along which we have a G^1 junction of two developable surface patches such that the patches lie locally on different sides of the common tangent plane. The remaining rulings should be selected such that the desired accuracy can be achieved. A good strategy is a topic for future research. Here, we will focus on the solution of the Hermite interpolation problem.

Let (e_i, τ_i), $i = 1, 2$, be two consecutive G^1 elements. With each element we associate an orthonormal basis $\mathbf{e}_i, \mathbf{p}_i, \mathbf{n}_i$ in the following way (Fig. 2). τ_i is spanned

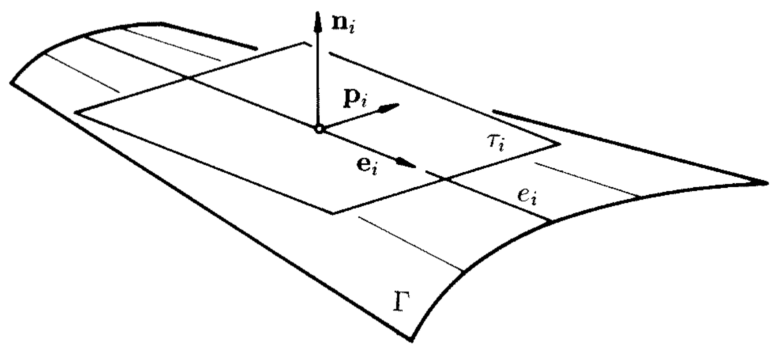


Figure 2: Orthonormal basis

by the generator vector $\mathbf{e}_i$ and the unit vector $\mathbf{p}_i$ normal to $\mathbf{e}_i$. The orientation of $\mathbf{e}_i$ is taken from an orientation of the set of generators of the given surface Γ. The vector $\mathbf{p}_i$ indicates the side on which the interpolant between e_i and e_{i+1} has to connect. Thus, the unit normal $\mathbf{n}_i = \mathbf{e}_i \times \mathbf{p}_i$ always points to the same side of the surface Γ, which is assumed to be regular and orientable in the region of interest.

The general case

If two cones of revolution Λ_1, Λ_2 with different vertices $\mathbf{v}_1$, $\mathbf{v}_2$ possess a common generator and tangent plane, their axes either intersect at a point $\mathbf{m}$ or are parallel. We will now treat the first (general) case. Here $\mathbf{m}$ is the center of a sphere Σ that touches both cones along circles c_1, c_2 (Fig. 3). Σ can already be computed from the two given G^1 elements. Using a point $\mathbf{l}_i$ on e_i, the midpoint $\mathbf{m}$ of Σ is the intersection of the normal planes

$$\nu_i : \quad (\mathbf{x} - \mathbf{l}_i) \cdot \mathbf{p}_i = 0, \qquad i = 1, 2 \tag{5}$$

with the tangent planes' bisector plane

$$\sigma : \quad \mathbf{x} \cdot (\mathbf{n}_1 - \mathbf{n}_2) - \mathbf{l}_1 \cdot \mathbf{n}_1 + \mathbf{l}_2 \cdot \mathbf{n}_2 = 0. \tag{6}$$

Let $\mathbf{a}_i$ be the touching point of Σ and e_i. The desired cone pair will touch Σ along a G^1 pair of circle segments, i.e., a *spherical biarc*. Its end points are $\mathbf{a}_i$, and the end tangent vectors are $\mathbf{p}_i$. The set of biarcs interpolating two points plus tangent vectors has been studied by Fuhs and Stachel [9] and their results can now be applied to the present problem.

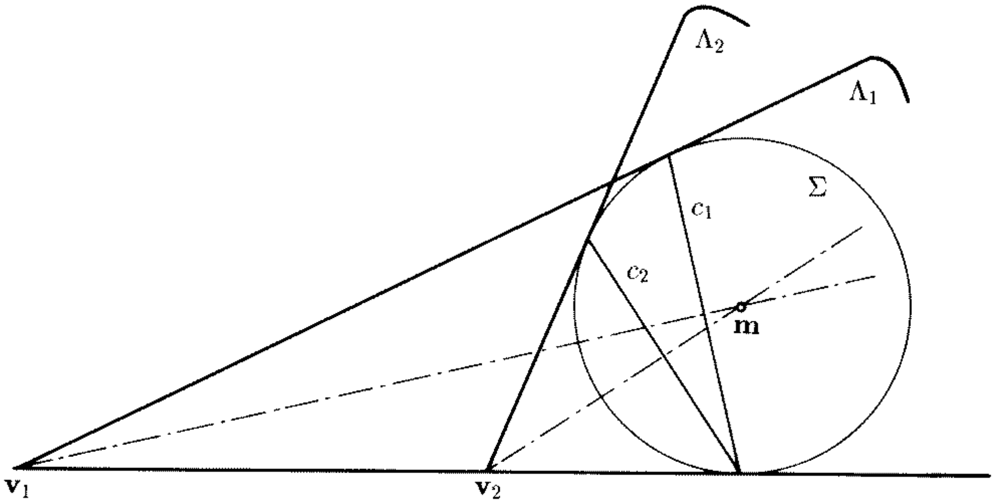


Figure 3: Inscribed sphere of two cones

For the rational Bézier representation of the biarc, we denote the Bézier points of its two circle segments c_1, c_2 by $\mathbf{a}_1, \mathbf{b}_1, \mathbf{c}, \mathbf{b}_2, \mathbf{a}_2$ (Fig. 4) and let $\mathbf{b}_1 = \mathbf{a}_1 + \lambda_1 \mathbf{p}_1$, $\mathbf{b}_2 = \mathbf{a}_2 - \lambda_2 \mathbf{p}_2$. The two legs in the Bézier polygon of a circle possess equal length

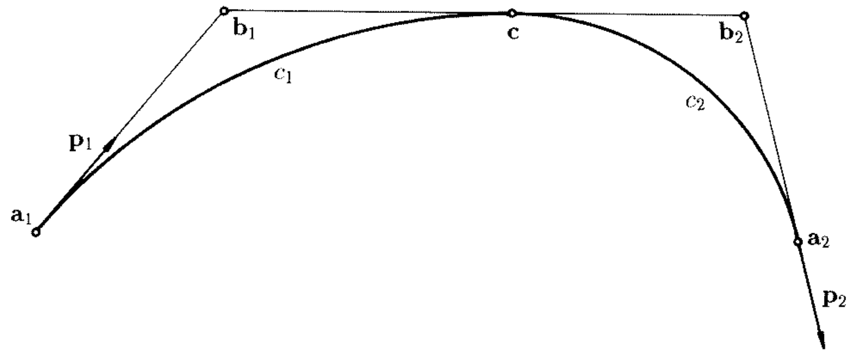


Figure 4: Control polygon of a biarc

and thus an admissible pair of inner Bézier points is characterized by

$$(\mathbf{b}_2 - \mathbf{b}_1)^2 = (\lambda_1 + \lambda_2)^2.$$

This is equivalent to

$$(\mathbf{a}_2 - \mathbf{a}_1)^2 - 2\lambda_1(\mathbf{a}_2 - \mathbf{a}_1) \cdot \mathbf{p}_1 - 2\lambda_2(\mathbf{a}_2 - \mathbf{a}_1) \cdot \mathbf{p}_2 + 2\lambda_1\lambda_2(\mathbf{p}_1 \cdot \mathbf{p}_2 - 1) = 0. \tag{7}$$

We may choose $\mathbf{b}_1(\lambda_1)$ and then compute a unique $\mathbf{b}_2(\lambda_2)$ via (7). To complete the rational Bézier representation, we also need the junction point

$$\mathbf{c} = \frac{\lambda_2 \mathbf{b}_1 + \lambda_1 \mathbf{b}_2}{\lambda_1 + \lambda_2}. \tag{8}$$

Setting the weights at the end points of the two circle segments to 1, the weights w_i at $\mathbf{b}_i$ are computed as

$$|w_i| = \frac{|(\mathbf{b}_i - \mathbf{a}_i) \cdot (\mathbf{c} - \mathbf{a}_i)|}{\|\mathbf{b}_i - \mathbf{a}_i\| \|\mathbf{c} - \mathbf{a}_i\|}.$$

If $\lambda_i > 0$, then one uses the arc contained in the triangle $\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}$ and a positive weight w_i. Otherwise one has to use the complementary arc and a negative weight. The homogeneous coordinates of the Bézier points are

$$\mathbf{A}_i = (1, \mathbf{a}_i),\ \mathbf{B}_i = (w_i, w_i \mathbf{b}_i),\ \mathbf{C} = (1, \mathbf{c}).$$

The Bézier planes of the cone segments are the polar planes of these Bézier points with respect to the sphere $\Sigma : (\mathbf{x} - \mathbf{m})^2 = r^2$. The homogeneous equation of the polar plane of a point with homogeneous coordinates $\mathbf{Y} = (y_0, \mathbf{y})$ is

$$(\mathbf{x} - x_0 \mathbf{m}) \cdot (\mathbf{y} - y_0 \mathbf{m}) = r^2 x_0 y_0.$$

Clearly, a cone pair and the corresponding spherical biarc are connected via this polarity.

As we have a region Ω of interest, an additional problem has to be considered. For

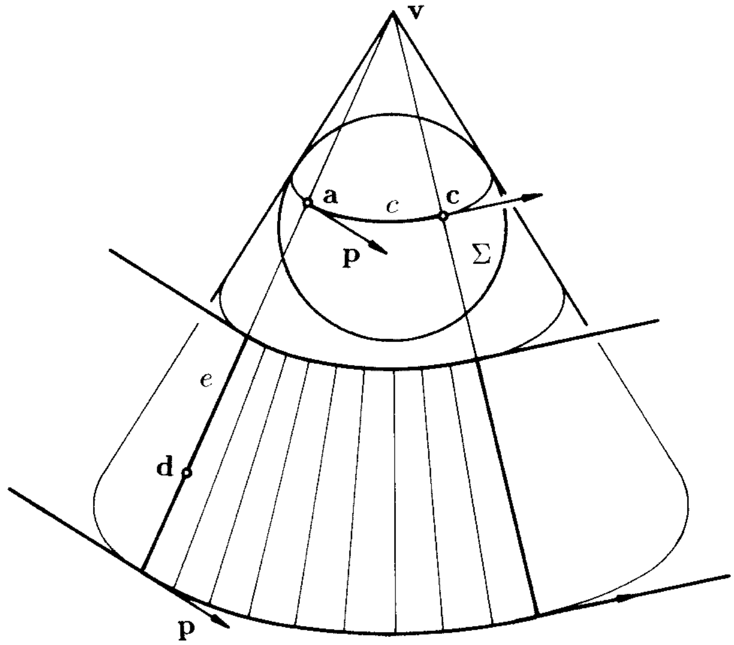


Figure 5: Σ and Ω lying on the same side of $\mathbf{v}$

the moment, let us look at a single cone for which we have computed the corresponding spherical arc c with Bézier points $\mathbf{a}, \mathbf{b}, \mathbf{c}$. As we exclude all solutions with vertex $\mathbf{v}$ in Ω, we have to distinguish between two cases. The first case is shown in Figure 5.

The sphere Σ and the region Ω lie on the same side of the vertex $\mathbf{v}$. Choosing a point $\mathbf{d}$ of the bounding generator e lying in Ω, this is characterized by

$$(\mathbf{a} - \mathbf{v}) \cdot (\mathbf{d} - \mathbf{v}) > 0.$$

Here c directly corresponds to the useful piece of the cone.

If Σ and Ω lie on different sides of $\mathbf{v}$ however, equivalent to

$$(\mathbf{a} - \mathbf{v}) \cdot (\mathbf{d} - \mathbf{v}) < 0,$$

using the computed arc c would lead to sharp edges in the interpolated surface. Figure 6 shows that the arc $\bar{c}$ complementary to c has to be used. This leads to sharp

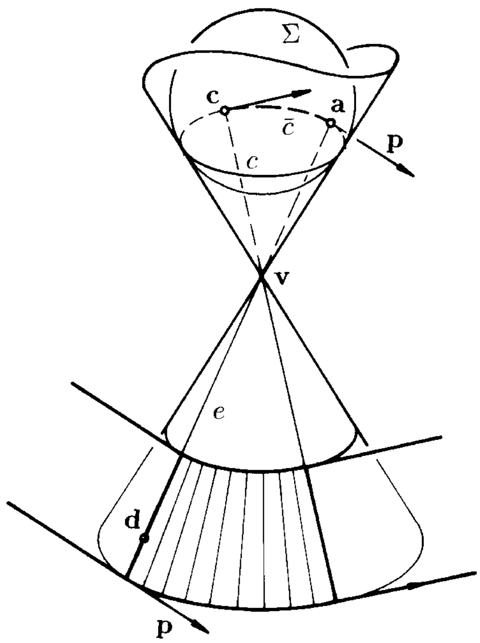


Figure 6: Σ and Ω lying on different sides of $\mathbf{v}$

edges in the spherical arc spline on Σ, but guarantees a smooth surface. The change from c to $\bar{c}$ is easily accomplished by changing the sign of the weight w of the inner Bézier point $\mathbf{B} = (w, w\mathbf{b})$.

The vertices $\mathbf{v}_i$ of the two cone segments can be computed by intersecting the tangent plane of Σ at $\mathbf{c}$ with the boundary generators e_i. We can also first compute the vertices in a way analogous to the inner Bézier points. Letting $\mathbf{v}_1 = \mathbf{a}_1 + \mu_1\mathbf{e}_1$, $\mathbf{v}_2 = \mathbf{a}_2 - \mu_2\mathbf{e}_2$, an admissible vertex pair is characterized by

$$(\mathbf{v}_2 - \mathbf{v}_1)^2 = (\mu_1 + \mu_2)^2.$$

Similar to (7), this yields a bilinear relation between μ_1, μ_2, given by

$$(\mathbf{a}_2 - \mathbf{a}_1)^2 - 2\mu_1(\mathbf{a}_2 - \mathbf{a}_1) \cdot \mathbf{e}_1 - 2\mu_2(\mathbf{a}_2 - \mathbf{a}_1) \cdot \mathbf{e}_2 + 2\mu_1\mu_2(\mathbf{e}_1 \cdot \mathbf{e}_2 - 1) = 0. \qquad (9)$$

In order to determine a cone pair within the one parameter set of solutions we can either choose $\mathbf{b}_1(\lambda_1)$ and compute $\mathbf{b}_2(\lambda_2)$ or choose $\mathbf{v}_1(\mu_1)$ and compute $\mathbf{v}_2(\mu_2)$. We see that both mappings $\mathbf{b}_1 \mapsto \mathbf{b}_2$ and $\mathbf{v}_1 \mapsto \mathbf{v}_2$ are projective maps. The connecting rulings $\mathbf{v}_1\mathbf{v}_2$ therefore lie in a ruled quadric Φ. According to Fuhs and Stachel [9], Φ is a hyperboloid of revolution, which touches the sphere Σ along a circle, which the connecting points $\mathbf{c}$ of the spherical biarcs are lying on (Fig. 7). The tangent

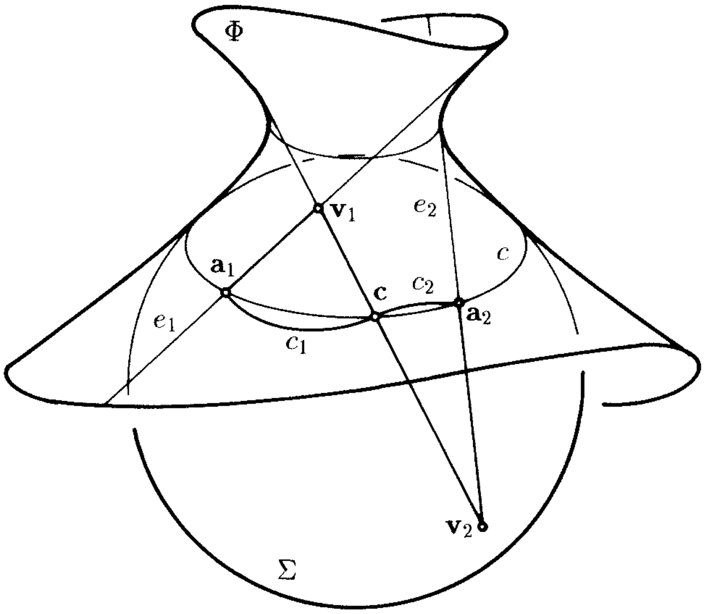


Figure 7: Spherical biarc

planes of Σ (and Φ) along the circle c are the tangent planes of the cone pairs at their junction generators; they envelope a cone of revolution. Discussing degeneracies and special cases later, we summarize:

Theorem 1. *Given two G^1 elements (e_i, τ_i) in general position, there is a one parameter family of cone pairs interpolating these data. The cones possess a common inscribed sphere Σ. The junction generators of the cone pairs form a set of rulings on a hyperboloid of revolution Φ, which touches Σ along a circle. The generators e_1, e_2 lie in the second set of rulings of Φ. The tangent planes of Σ and Φ along this circle are the junction tangent planes of the cone pairs.*

Let us now discuss some special solutions within the one parameter family we have obtained so far.

In case that the surface to be approximated possesses a generator e_i whose singular point is at infinity, we might want to find a solution in which vertex $\mathbf{v}_i$ of the cone pair is at infinity and thus the first of the two cone segments is a *cylindrical* *segment.* In view of (9) this occurs if

$$\mu_2 = \frac{(\mathbf{a}_2 - \mathbf{a}_1) \cdot \mathbf{e}_1}{\mathbf{e}_1 \cdot \mathbf{e}_2 - 1}. \tag{10}$$

The cones which the two segments of a pair are taken from are *congruent*, if $\mu_1 = \mu_2$. Hence, these cone pairs belong to solutions of the quadratic equation, which arises from (9) with $\mu_1 = \mu_2$. It always has two real solutions, which is in accordance with a result by Fuhs and Stachel [9] on spherical biarcs consisting of circles of equal radius.

As practically useful cone pairs we identified those with (locally) *minimal vertex distance* $\|\mathbf{v}_2 - \mathbf{v}_1\| = |\mu_1 + \mu_2|$. Writing equation (9) in the form

$$\mu_2 = \frac{A\mu_1 + B}{C\mu_1 + D}, \tag{11}$$

this cone pair belongs to a solution of

$$C^2\mu_1^2 + 2CD\mu_1 + AD + D^2 - BC = 0. \tag{12}$$

Equation (12) always has two real solutions: According to Theorem 1, we have to find those rulings of the hyperboloid Φ, which intersect the generators e_i in points $\mathbf{v}_i$ with minimal distance from each other. Using the top view of Φ, the images of all rulings of Φ are tangent to the circular silhouette u' of Φ. As all rulings of Φ are constantly sloped, one only has to find those with minimal distance $\mathbf{v}_1'\mathbf{v}_2'$ (Fig. 8). These special solutions of junction generators $\mathbf{v}_1\mathbf{v}_2$ can also be characterized by the

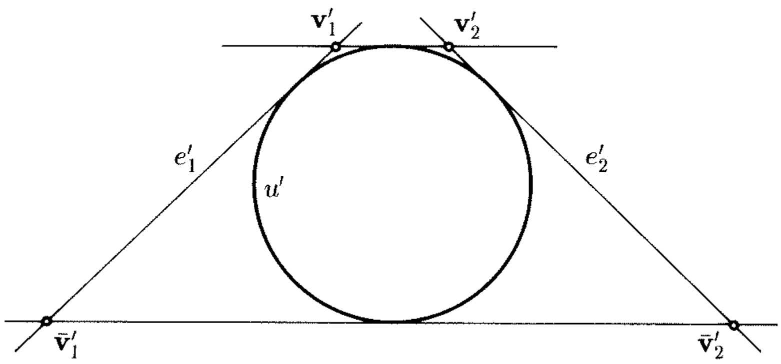


Figure 8: Minimal vertex distance $\mathbf{v}_1\mathbf{v}_2$

fact that the generator $\mathbf{v}_1\mathbf{v}_2$ defines the same angle with both generators e_i. One has to take attention that only one of the two solutions leads to a cone pair useful for applications. This does not always have to be the one with absolute minimal distance, however.

Special cases

Important special cases arise if the developable surface Γ to be approximated (locally) is a general cone, a general cylinder or a surface of constant slope, that is the tangent surface of a curve whose tangent vectors enclose a constant angle with some plane π (Fig. 9). Any osculating cone Δ_i of a developable surface of constant slope has its

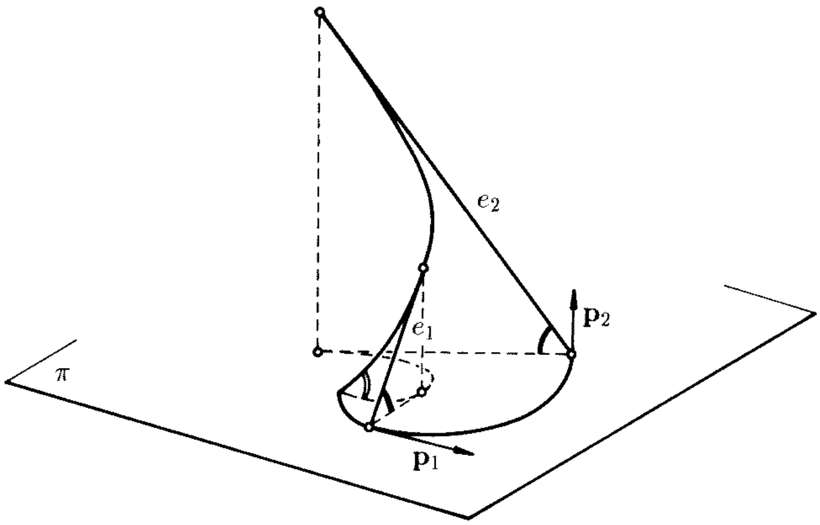


Figure 9: Surface of constant slope

axis normal to π. For any two tangent planes $\tau_i, i = 1, 2$ touching Γ along e_i we have $\mathbf{p}_i$ parallel to π and the normal vectors $\mathbf{n}_i$ enclose a constant angle with π.

In the following we look at two consecutive G^1 elements $(e_i, \tau_i), i = 1, 2$. Using these data only, we characterize the special cases and derive algorithms to interpolate the given boundary elements with a pair of cone or cylinder segments. These algorithms will prove to be similar to the general case discussed in the previous section.

The cone case

The generators e_1, e_2 intersect in a point $\mathbf{v}$. If additionally $\tau_1 = \tau_2$, we interpolate with a part of τ_1. Otherwise, two cone segments with common vertex $\mathbf{v}$ will be used which intersect a sphere Σ centered in $\mathbf{v}$ in a spherical biarc. The one parameter set of solutions is determined by the end points $\mathbf{a}_i = e_i \cap \Sigma$ and end tangent vectors $\mathbf{p}_i$. The missing Bézier points $\mathbf{b}_1 = \mathbf{a}_1 + \lambda_1 \mathbf{p}_1, \mathbf{c}, \mathbf{b}_2 = \mathbf{a}_2 - \lambda_2 \mathbf{p}_2$ can be computed using formulae (7) and (8).

The cylinder case

The generators e_1, e_2 are parallel and have the same orientation. This case is similar to the first one, with $\mathbf{v}$ being a point at infinity. We interpolate with a planar surface in case of $\tau_1 = \tau_2$, otherwise with a pair of right orthogonal cylinder segments. Intersecting (e_i, τ_i) with a plane normal to e_i we get Hermite elements $(\mathbf{a}_i, \mathbf{p}_i)$ that can be interpolated by a one parameter set of planar arc splines. Obviously (7) and (8) can be used for calculating the inner Bézier points of the planar arc splines.

Surfaces of constant slope

In the following the generators e_1, e_2 have no point in common.

Given two G^1 Hermite elements in general position we found the center m of a common inscribed sphere as intersection of three planes which had normal vectors $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{n}_1 - \mathbf{n}_2$, according to equations (5) and (6). This calculation is not possible if

$$\det(\mathbf{p}_1, \mathbf{p}_2, \mathbf{n}_1 - \mathbf{n}_2) = 0, \tag{13}$$

i.e., the three planes do not intersect in a point. Equation (13) is equivalent to

$$(\mathbf{p}_1 \times \mathbf{p}_2) \cdot \mathbf{n}_1 = (\mathbf{p}_1 \times \mathbf{p}_2) \cdot \mathbf{n}_2.$$

Assuming $\det(\mathbf{p}_1, \mathbf{p}_2) \neq 0$ for the moment, the normal vectors $\mathbf{n}_1, \mathbf{n}_2$ enclose the same angle with a plane π spanned by $\mathbf{p}_1$ and $\mathbf{p}_2$. We will now interpolate the given boundary generators by a cone pair with parallel axes, both being normal to π. The cone pair intersects π in a planar biarc (Fig. 10) with $\mathbf{a}_i = e_i \cap \pi$ as end points and tangent vectors $\mathbf{p}_i$. Admissible Bézier points $\mathbf{b}_1, \mathbf{c}, \mathbf{b}_2$ again follow from (7) and

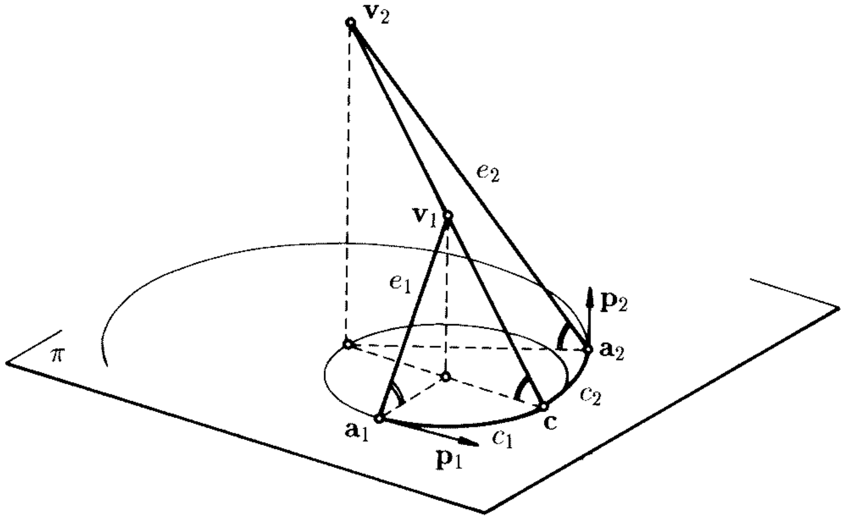


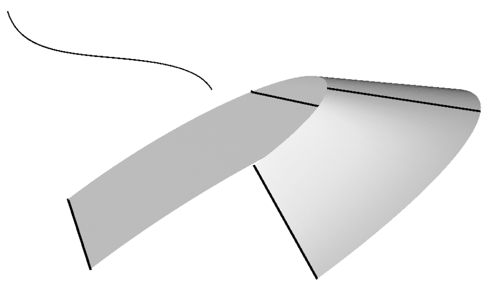
Figure 10: Pair of cone segments

(8). The vertices $\mathbf{v}_1$ and $\mathbf{v}_2$ are the intersection points of the generators e_1 and e_2 with $\nu : \ (\mathbf{x} - \mathbf{c}) \cdot (\mathbf{b}_2 - \mathbf{b}_1) = 0$, which is the plane through the junction point $\mathbf{c}$ perpendicular to the junction tangent in $\mathbf{c}$. One can also compute the vertices directly, using $\mathbf{v}_1 = \mathbf{a}_1 + \mu_1\mathbf{e}_1, \mathbf{v}_2 = \mathbf{a}_2 - \mu_2\mathbf{e}_2$ and equation (9). Minimizing the vertex distance with formula (12) provides congruent cone segments.

In the case of $\det(\mathbf{p}_1, \mathbf{p}_2) = 0$, which was excluded above, the plane π has to be spanned by $\mathbf{p}_1$ and $\mathbf{n}_1 - \mathbf{n}_2$. The remaining algorithms stay the same.

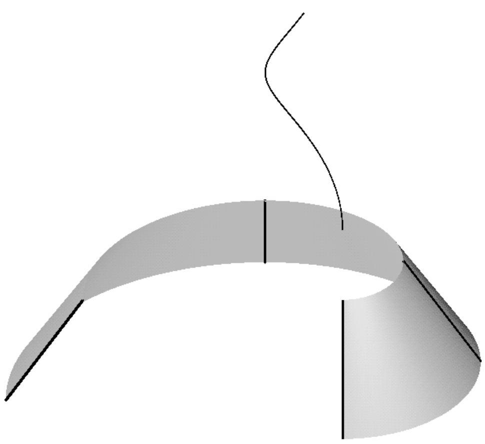
Examples

The first example shall demonstrate the general case. Figure 11 shows a tangent surface Γ. Using four Hermite elements (e_i, τ_i) of Γ as input data we get an approximation by three cone pairs. From the one parameter set of solutions those with minimal vertex distance are chosen. Connecting the six vertices of the cone segments to a polygon we obtain the locus of all the singular points of the cone spline surface. Although the line of regression was not used in the computation of the cone spline, one can see that this curve is approximated perfectly by the vertex polygon.



(a) (b)

Figure 11: (a) Tangent surface, (b) its approximation with 3 cone pairs



(a) (b)

Figure 12: (a) Surface of constant slope, (b) its approximation with 3 cone pairs

The second example illustrates one of our special cases and shows the approximation of the tangent surface of a helical curve (Fig. 12). The approximation of this surface of constant slope with only a small number of cone segments is again very good.

OSCULATING CONE SPLINES

Recently, Meek and Walton [18] have studied the approximation of plane curves with *osculating arc splines*. These are circular splines which contain a sequence of segments of osculating circles of the curve to be approximated. Between two consecutive osculating circles one circle segment is built in. It has been shown that the method results in curves with a smaller error than those produced from biarcs (see [17, 18]).

We will now investigate the analogue for cone spline surfaces. Given a developable surface Γ that is neither a conical or cylindrical surface nor a surface of constant slope, we compute a sequence of osculating cones Δ_i and smoothly join consecutive cones by a cone Δ.

Let us assume that Δ_1, Δ_2 are not cylinders. If a cone Δ smoothly joins these two cones, we may perform translations which take their vertices to the origin. The resulting three cones $\Lambda_1, \Lambda, \Lambda_2$ are again smoothly joined. The relevant segments intersect the unit sphere Σ in a spherical triarc, formed by segments of circles $\mathbf{c}_1, \mathbf{c}, \mathbf{c}_2$ (Fig. 13). From the given cones Δ_i we can compute the planes $\mathbf{U}_i$ of the circles $\mathbf{c}_i$.

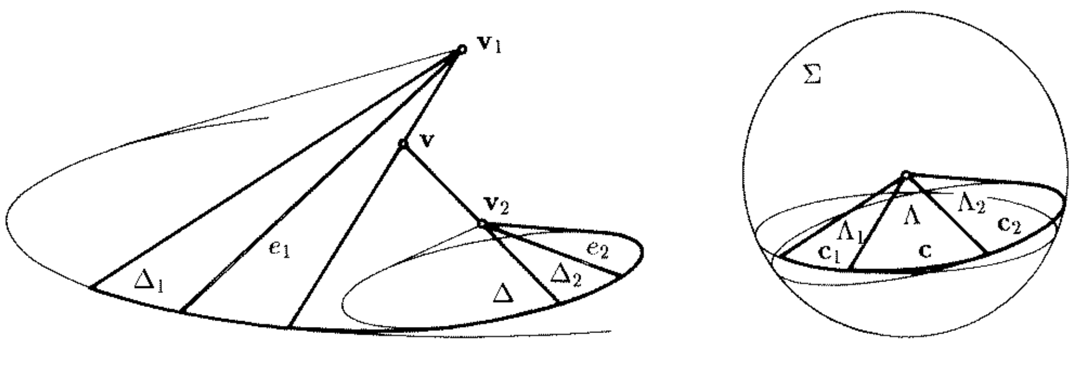


Figure 13: Joining cones Δ_i and Λ_i

The poles $\mathbf{Q}_i$ of $\mathbf{U}_i$ with respect to Σ are the vertices of cones Λ_i^* which touch Σ along the circles $\mathbf{c}_i$. If $\mathbf{c}_1$ is touched by a circle $\mathbf{c}$, the pole $\mathbf{Q}$ of the plane $\mathbf{U}$ of $\mathbf{c}$ must lie on Λ_1^*. Therefore the vertex $\mathbf{Q}$ of a cone which touches Σ along a filling circle $\mathbf{c}$, must lie on the intersection curve $\mathbf{l}$ of the cones Λ_1^*, Λ_2^*. Because of the common inscribed sphere of these two cones, their intersection curve is, in general, composed of two conics $\mathbf{l}_1$ and $\mathbf{l}_2$. Projecting the conics $\mathbf{l}_i$ from the origin yields two quadratic cones whose generators are the axes of possible intermediate cones Λ.

In case of Λ_1^* and Λ_2^* touching each other at a common generator the intersection curve consists of a conic $\mathbf{l}_1$ and the common generator $\mathbf{l}_2$. Points $\mathbf{Q}$ of $\mathbf{l}_2$ do not lead to useful solutions of intermediate cones Δ.

To compute the planes $\mathbf{V}_i$ of the conics $\mathbf{l}_i$, we first note that they must lie in the pencil spanned by $\mathbf{U}_1 = (u_{10} : u_{11} : u_{12} : u_{13})$ and $\mathbf{U}_2 = (u_{10} : u_{11} : u_{12} : u_{13})$, hence

$$\mathbf{V}_i = \mathbf{U}_1 + \lambda_i \mathbf{U}_2.$$

With Figure 14 one verifies that the cross ratio $\mathrm{cr}(\mathbf{U}_1, \mathbf{U}_2, \mathbf{V}_1, \mathbf{V}_2)$ equals -1 and that $\mathbf{V}_1, \mathbf{V}_2$ are conjugate with respect to Σ, i.e., the pole of $\mathbf{V}_1$ with respect to Σ lies in $\mathbf{V}_2$. This leads to

$$\lambda_{1,2} = \pm\sqrt{\frac{u_{11}^2 + u_{12}^2 + u_{13}^2 - u_{10}^2}{u_{21}^2 + u_{22}^2 + u_{23}^2 - u_{20}^2}}.$$

The orientation of the set of generators of the osculating cones Δ_i in the sense of increasing parameters leads to orientations of $\mathbf{c}_i$. Thus only one of the two conics $\mathbf{l}_i$, say $\mathbf{l}_1$, can lead to suitable solutions.

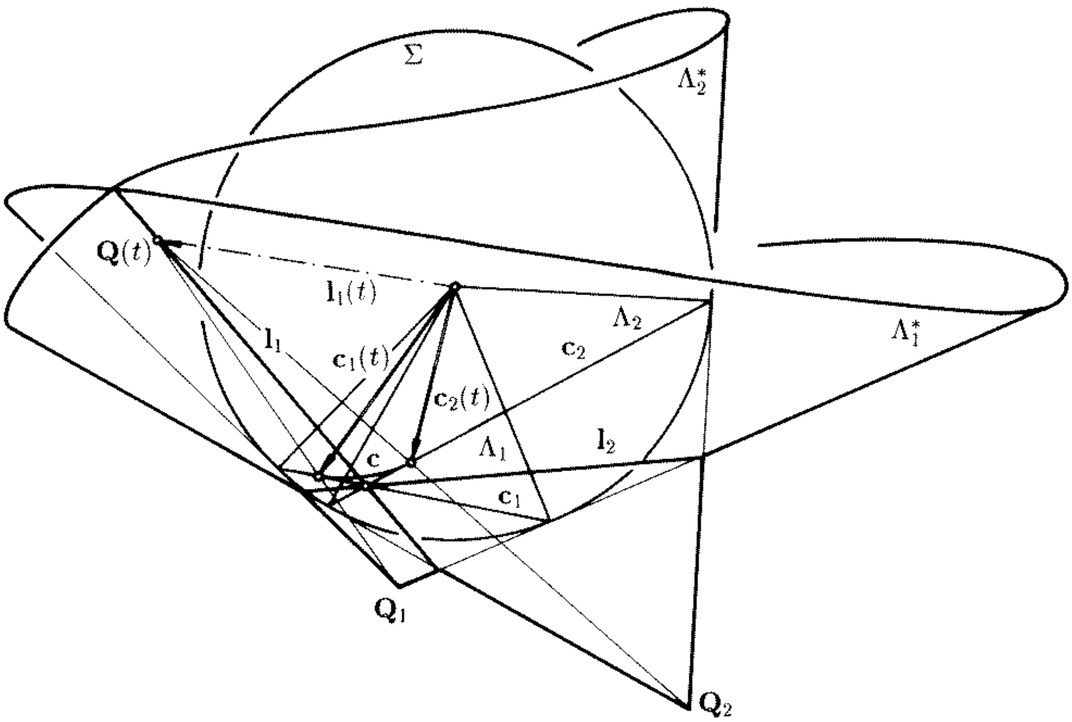


Figure 14: Construction of corresponding generators $\mathbf{c}_1(t)$, $\mathbf{c}_2(t)$

The computation of $\mathbf{l}_1$ and the resulting cones Λ and Δ may proceed as follows. We represent any segment of the circle $\mathbf{c}_1$ as a rational quadratic Bézier curve (see Piegl and Tiller [20]). Its Bézier points shall have the correctly normalized homogeneous coordinates $\mathbf{B}_i^1$, $i = 0, 1, 2$. Then a homogeneous representation of the circle is given by

$$\mathbf{C}_1(t) = (1-t)^2\mathbf{B}_0^1 + 2t(1-t)\mathbf{B}_1^1 + t^2\mathbf{B}_2^1.$$

We use a projective parameter line P^1 described by homogeneous parameters t_0, t_1 with $t = t_1/t_0$ and parameterize the entire circle by

$$\mathbf{C}_1(t_0, t_1) = (t_0 - t_1)^2\mathbf{B}_0^1 + 2t_1(t_0 - t_1)\mathbf{B}_1^1 + t_1^2\mathbf{B}_2^1.$$

Projecting $\mathbf{C}_1$ from the center $\mathbf{Q}_1$ onto the plane $\mathbf{V}_1$ yields the conic $\mathbf{l}_1$. Using the projective invariance of the rational Bézier representation, we can represent $\mathbf{l}_1$ by applying the projection with matrix A_1 to the Bézier points $\mathbf{B}_i^1$. We get the Bézier points $\mathbf{B}_i^l = A_1 \cdot \mathbf{B}_i^1$ of $\mathbf{l}_1$. The projection from the plane $\mathbf{V}_1$ to $\mathbf{U}_2$ with center $\mathbf{Q}_2$ yields a Bézier representation of the circle $\mathbf{c}_2$ with Bézier points $\mathbf{B}_i^2$. To each $t = t_1 : t_0$, the points $\mathbf{C}_1(t), \mathbf{C}_2(t)$ determine the generators with vectors $\mathbf{c}_1(t), \mathbf{c}_2(t)$, along which a joining cone $\Lambda(t)$ with axis vector $\mathbf{l}_1(t)$ is touching the cones Λ_1, Λ_2.

If we translate the cones Λ_i back to their position Δ_i, not all $\Lambda(t)$ lead to solutions Δ of our problem. For a solution cone Δ, the generators $\mathbf{v}_i + u\mathbf{c}_i(t),\ u \in \mathbb{R}$ of the cones Δ_i must intersect at the vertex $\mathbf{v}$ of Δ. Thus, only those t which satisfy the intersection condition

$$\det(\mathbf{c}_1(t), \mathbf{c}_2(t), \mathbf{v}_1 - \mathbf{v}_2) = 0 \tag{14}$$

belong to possible joining cones Δ. Equation (14) is a homogeneous quartic polynomial in t_0, t_1, which has up to four real solutions $t = t_1 : t_0$. Note that it is necessary to work with homogeneous parameters, since $t = 1 : 0$ may lead to a useful solution.

We will now look at the four possible solutions in detail. If $\mathbf{c}_1(t) = \mathbf{c}_2(t)$ is the vector of a common generator of Λ_1 and Λ_2 we get a solution for equation (14) which cannot be used to construct a joining cone Δ of Δ_i, though. As there are two common points of the two circles $\mathbf{c}_1, \mathbf{c}_2$, counted algebraically, there only remain two solutions for joining cones Δ which need not be real for arbitrary spatial position of cones Δ_i. In our problem the cones Δ_i are osculating cones of a given developable surface Γ, however, touching Γ along generators e_i. It can be shown that for any generator $e_1 = e(u_1)$ to parameter u_1 there is an interval of generators $e_2 = e(u_1 + \Delta u)$ such that the corresponding osculating cones Δ_i can be joined by a cone Δ. Additionally this solution is consistent with the present approximation problem, meaning that the joining generators $\mathbf{v}_i + u\mathbf{c}_i(t)$ of Δ and Δ_i lie appropriate on the cones Δ_i in the sense of increasing parameters (see Figure 13).

To complete the discussion of the general case, we consider the case of Δ_i, say Δ_1, being an osculating cylinder. Let $\mathbf{c}_1$ determine the direction of its generators. The line $\lambda\mathbf{c}_1,\ \lambda \in \mathbb{R}$ can be interpreted as a degenerated cone Λ_1 which intersects Σ in the point $\mathbf{Q}_1$. Λ_1^* degenerates into the tangent plane of Σ at $\mathbf{Q}_1$. The intersection curve $\mathbf{l}_1 = \Lambda_1^* \cap \Lambda_2^*$ contains the points $\mathbf{Q}$ corresponding to filling circles $\mathbf{c}$. Again a projective mapping of corresponding generators of Δ_1 and Δ_2 is determined which leads to the solutions for Δ.

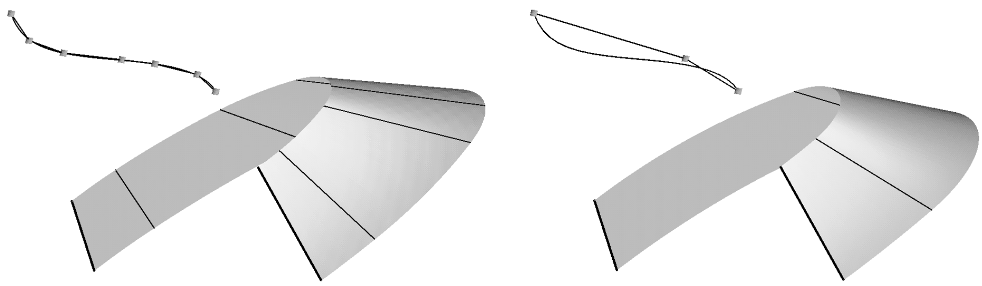
Special cases

The present investigation also contains a treatment of *spherical osculating arc splines*. As in the planar case, there is a one parameter set of intermediate arcs between consecutive osculating circles. In analogy to Meek and Walton [18], one could show that appropriate choices within this solution set will yield a high accuracy approximation scheme.

Of course, this spherical scheme is appropriate for approximating general cones with cone spline surfaces and the planar scheme can be used for approximating cylindrical surfaces as well as surfaces of constant slope (Fig. 9). In the last case all osculating cones Δ_i have parallel axes and the same opening angle. Two consecutive cones Δ_1, Δ_2 in general intersect in a conic $\mathbf{l}_1$, which contains the possible vertices of connecting cone segments.

Examples

We apply the algorithm of osculating cone splines to the example in Figure 11. Using four generators e_i plus osculating cones Δ_i of our tangent surface as input data, we compute the three filling cone segments and get an approximation of Γ with seven cone segments (Fig. 15), one more than we got using four Hermite elements (e_i, τ_i).



(a) (b)

Figure 15: Approximation of the developable surface of Fig. 11(a) with (a) 4, (b) 2 osculating cones as input data

Note that the vertices $\mathbf{v}_i$ of the given cones Δ_i must lie on the line of regression. Using only two osculating cones Δ_1, Δ_2 of the same surface Γ results in a satisfying approximation quality even with few input data.

BENDING SEQUENCES AND DEVELOPMENT

One major advantage of approximating a given developable surface by a cone spline surface is the simple development that does not need numerical integration. A cone segment Λ_0 is determined by the angle α_0 between generators and axis and the segment angle φ_0 (Fig. 16). The development of such a cone segment can be realized

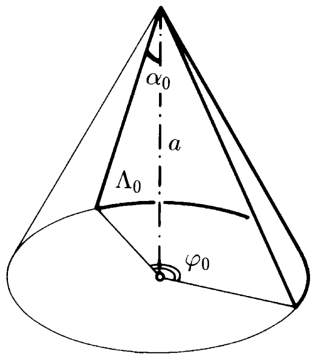


Figure 16: Cone segment

by a continuous set of cone segments Λ_i with segment angle φ_i and angle $\alpha_i \to 0$, satisfying

$$\varphi_i \sin \alpha_i = \varphi_0 \sin \alpha_0.$$

Let Λ_n be the development of Λ_0, i.e., $\sin \alpha_n = 1$, then we get

$$\varphi_n = \varphi_0 \sin \alpha_0.$$

Figure 17 shows a bending sequence of a cone spline surface in which all segments are flattened out simultaneously. The distance between successive vertices stays the same and a spatial arc spline as boundary curve is transferred into a planar arc spline of the development.

FUTURE RESEARCH

Our Hermite–type approximation algorithms are based on data coming from a developable surface that shall be approximated by a cone spline. We need to study good choices for this input. This is one of our current research topics and is connected to error estimates, for which we have some preliminary results.

Another topic are other Hermite-like schemes. One might for example interpolate two tangent planes plus generators and points of regression with three cone segments. There is a two parameter set of solutions for this triple and appropriate selection algorithms may lead to good approximation results.

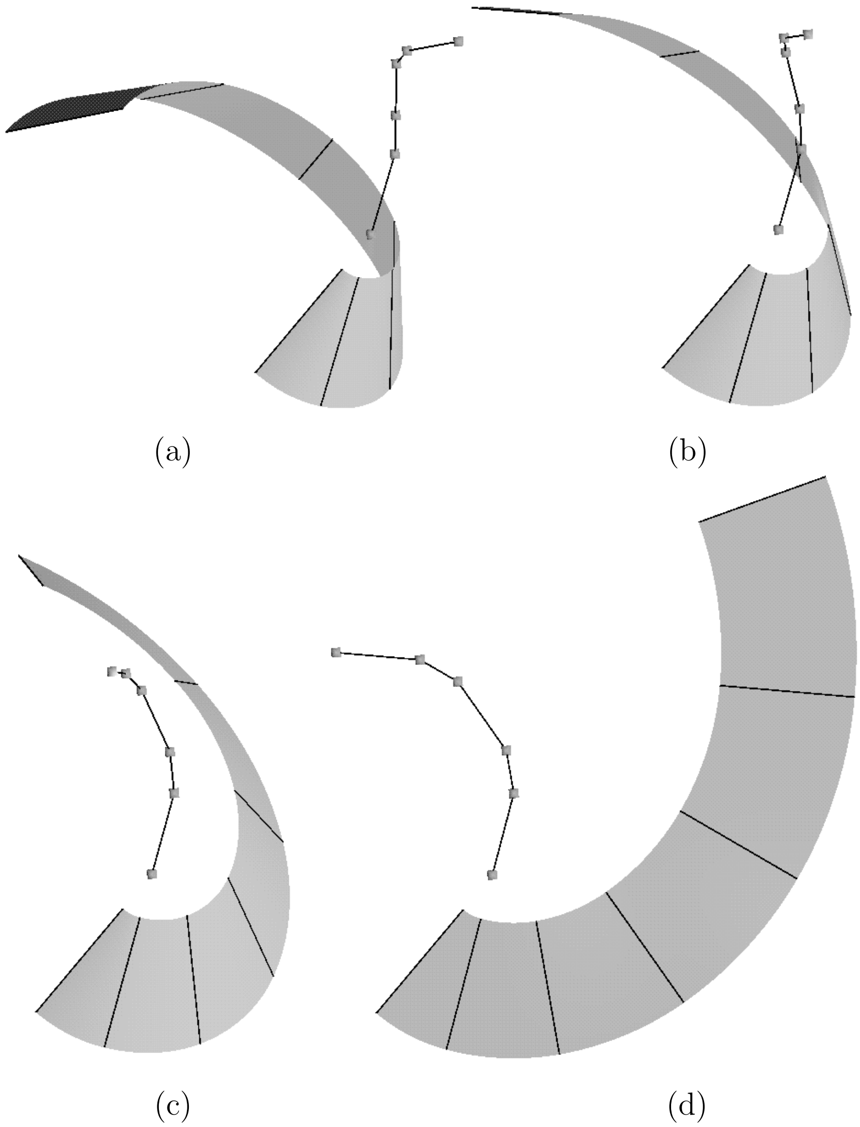


Figure 17: Bending sequence

A natural extension of the above ideas would be to find a class of *low degree rational developable surfaces with elementary development* which allows to model G^2 surfaces or at least G^1 surfaces with a G^1 line of regression. In the classical geometric literature, there appear — apart from the trivial case of certain cones and cylinders — just a few examples of *rational surfaces with elementary development*, namely certain developable surfaces of constant slope [3, 19, 21]. We are not aware of an investigation of surfaces with elementary development from the geometric design point of view. Recent work on surface design with rational plane rolling motions [24] contains ideas to achieve this goal. The methods come from theoretical kinematics and line geometry.

Acknowledgements

This work has been supported in part by grant No. P12252-MAT of the Austrian Science Foundation.

References

[1] Aumann, G., Interpolation with developable Bézier patches, Computer Aided Geometric Design **8** (1991), 409-420.

[2] Aumann, G., A closer look at developable Bézier surfaces, J. of Theoretical Graphics and Computing **7** (1994), 12–26.

[3] Blaschke, W., Bemerkungen über allgemeine Schraubenlinien, Monatsh. Math. Phys. **19** (1908), 188–204.

[4] Bodduluri, R.M.C. and Ravani, B., Geometric design and fabrication of developable surfaces, ASME Adv. Design Autom. **2** (1992), 243–250.

[5] Bodduluri, R.M.C. and Ravani, B., Design of developable surfaces using duality between plane and point geometries, Computer Aided Design **25** (1993), 621–632.

[6] Clements, J.C. and Leon, L.J., A fast accurate algorithm for the isometric mapping of a developable surface, SIAM J. Math. Anal. **18** (1987), 966–971.

[7] Farin, G., *Curves and Surfaces for Computer Aided Geometric design*, Academic Press, Boston, MA, 3rd ed., 1992

[8] Frey, W.H. and Bindschadler, D., Computer aided design of a class of developable Bézier surfaces, General Motors R&D Publication 8057 (1993).

[9] Fuhs, W. and Stachel, H., Circular pipe connections, Computers & Graphics **12** (1988), 53–57.

[10] Gurunathan, B. and Dhande, S.G., Algorithms for development of certain classes of ruled surfaces, Computers & Graphics **11** (1987), 105–112.

[11] Hoschek, J., Dual Bézier curves and surfaces, in *Surfaces in Computer Aided Geometric Design*, R.E. Barnhill and W. Boehm, eds., North Holland, 1993, 147–156.

[12] Hoschek, J., Pottmann, H., Interpolation and approximation with developable B–spline surfaces, in *Mathematical Methods for Curves and Surfaces*, M. Daehlen, T. Lyche and L.L. Schumaker, eds., Vanderbilt University Press, Nashville, 1995, 255–264.

[13] Hoschek, J., Schneider, M., Interpolation and approximation with developable surfaces, in *Curves and Surfaces with Applications in CAGD*, A. Le Méhauté, C. Rabut and L.L. Schumaker, eds., Vanderbilt University Press, Nashville, 1997, 185–202.

[14] Kreyszig, E., A new standard isometry of developable surfaces in CAD/CAM, SIAM J. Math. Anal. **25** (1994), 174–178.

[15] Lang, J. and Röschel, O., Developable $(1, n)$ Bézier surfaces, Computer Aided Geometric Design **9** (1992), 291–298.

[16] Mancewicz, M.J. and Frey, W.H., Developable surfaces: properties, representations and methods of design, General Motors R&D Publication 7637 (1992).

[17] Meek, D.S. and Walton, D.J., Approximating smooth planar curves by arc splines, J. Computational and Applied Mathematics **59** (1995), 221–232.

[18] Meek, D.S. and Walton, D.J., Planar osculating arc splines, Computer Aided Geometric Design **13** (1996), 653–671.

[19] Meirer, K., Zur Verebnung kubischer Böschungstorsen, Anzeiger d. Österr. Akad. d. Wiss. **105** (1968), 37–48.

[20] Piegl, L. and Tiller, W., *The NURBS book*, Springer, 1995.

[21] Pottmann, H., Zur Geometrie höherer Planetenumschwungbewegungen, Monatsh. Math. **97** (1984), 141–156.

[22] Pottmann, H., Studying NURBS curves and surfaces with classical geometry, in *Mathematical Methods for Curves and Surfaces*, M. Daehlen, T. Lyche and L.L. Schumaker, eds., Vanderbilt University Press, Nashville, 1995, 413–438.

[23] Pottmann, H., Farin, G., Developable rational Bezier and B–spline surfaces, Computer Aided Geometric Design **12** (1995), 513–531.

[24] Pottmann, H. and Wagner, M., Principal surfaces, in *The Mathematics of Surfaces VII*, T.N.T. Goodman, ed., Information Geometers Ltd., 1997.

[25] Redont, P., Representation and deformation of developable surfaces, Computer Aided Design **21** (1989), 13–20.

[26] Schneider, M., Approximation von Blechhalterflächen mit Torsen in B–Spline Darstellung, Diplomarbeit, TH Darmstadt, 1996.

[27] Weiss, G. and Furtner, P., Computer–aided treatment of developable surfaces, Computers & Graphics **12** (1988), 39–51.

[28] Vatter, R., Approximation von Datenpunkten mit Torsen in B–Spline Darstellung, Diplomarbeit, TH Darmstadt, 1996.