Rational blending surfaces between quadrics

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Abstract

Using tools from classical line geometry and the theory of kinematic mappings, it is possible to define an intrinsic control structure for NURBS curves and surfaces on the sphere, the cylinder and on any projectively equivalent quadratic surface. These methods are further used to construct exact C^1 blends between these surfaces, such that interactive design of trim lines and surface tension is possible. The lowest possible degree of a blend that can be achieved with this method is $(4, 3)$.

Keywords: kinematic mapping; line geometry; blending surface; NURBS surface

Blending surfaces are surfaces smoothly joining two other surfaces. There are a variety of different types of blends: *implicit blends*, which are given by a relation of the type $F(x, y, z) = 0$ or *parametric* blends, which are given by an equation of the form $F = (x(u, v), y(u, v), z(u, v))$. Depending on the original surfaces we have to join, we can use *approximate* or *exact* blends. For a survey of different blending methods using parametric surfaces see e.g. (Varady, 1994) and the extensive bibliography given therein, or, for implicit blending, (Hoffmann, 1987) and (Warren, 1989).

It is the aim of this paper to present a method to construct *exact C^1 rational* blending surfaces between quadrics, which are frequently used primitives in solid modeling systems. For this purpose we use quadratic projections which have their origin in the theory of kinematic mappings to define intrinsic control structures for NURBS curves and surfaces on quadrics.

1 The spherical kinematic mapping

In this section we describe well known connections between the group SO_3 of rotations in three-space and the space of quaternions. In euclidean four-space $\mathbb{R}^4$ we have coordinates $(x_0, \ldots, x_3)$. We identify $(a_0, \ldots, a_3) \in \mathbb{R}^4$ with the quaternion $a = a_0 + ia_1 + ja_2 + ka_3$. The conjugate quaternion is denoted by $\bar{a} = a_0 - ia_1 - ja_2 - ka_3$. The canonical scalar product in $\mathbb{R}^4$ gives rise to a norm $N(a) = a_0^2 + a_1^2 + a_2^2 + a_3^2 = a\bar{a}$, which is multiplicative, that is $N(ab) = N(a)N(b)$. We embed $\mathbb{R}^3$ into $\mathbb{R}^4$ by letting $(x, y, z) \mapsto ix + jy + kz$. Then the unit sphere $S^3 = \{a \in \mathbb{R}^4 | N(a) = 1\}$ acts on $\mathbb{R}^3$ by multiplication: If $N(a) = 1$ and $x \in \mathbb{R}^3$, then $ax\bar{a} \in \mathbb{R}^3$. The mapping $x \mapsto ax\bar{a}$ is linear and because of $N(ax\bar{a}) = N(x)$ it is isometric. It can also be shown that it is orientation preserving, which means that it is a rotation. Now it is well known that for all rotations $L \in \mathrm{SO}_3$ there exists an $a \in S^3$ such that precisely a and $-a$ describe L.

This allows us to define a 1-1 mapping σ from the group of rotations onto projective three space P^3, which will be called the *spherical kinematic mapping*. In P^3 we will use homogeneous coordinates, a point $\mathbb{R}a \in P^3$ consists of all scalar multiples of a homogeneous coordinate vector $a = (a_0, \dots, a_3)$. Choose $L \in \mathrm{SO}_3$ and let $a \in S^3$ correspond to L. Then $\sigma(L)$ is $\mathbb{R}a$. Note that σ is well defined.

The scalar product in $\mathbb{R}^4$ induces a distance $d(.,.)$ in P^3 by $\cos d(\mathbb{R}a, \mathbb{R}b) = (ab)/\sqrt{N(a)N(b)}$. Thus P^3 becomes an *elliptic space* according to the Cayley-Klein classification. For details on σ, see for instance (Müller, 1962). The following list of properties can be easily verified:

1. One-parameter subgroups of SO_3 and their cosets are mapped to straight lines.

2. The group-invariant distance in SO_3 which equals the *angle* between two rotations, is mapped to twice the distance in P^3.

3. Fix a unit vector $n \in S^2 \subset \mathbb{R}^3$. For all unit vectors $v \in S^2$ the set of all rotations which map n to v has a straight line as image under σ. These lines form a fibration of P^3, which is known to geometers as an *elliptic net*. The induced fibration in S^3 is known as the *Hopf fibration*.

2 A spherical control structure

We choose a unit vector $n \in S^2$ and define a mapping $\delta : P^3 \to S^2$ by letting

$$\delta(\mathbb{R}a) = av\bar{a}.$$

n can be chosen such that δ has the following representation in homogeneous coordinates: $\delta : \mathbb{R}^4 \to \mathbb{R}^4, x \mapsto y = \delta(x)$,

$$\begin{aligned} y_0 &= x_0^2 + x_1^2 + x_2^2 + x_3^2, \\ y_1 &= 2(x_0x_1 - x_2x_3), \\ y_2 &= 2(x_1x_3 - x_0x_2), \\ y_3 &= x_1^2 + x_2^2 - x_0^2 - x_3^2. \end{aligned} \tag{1}$$

Sometimes we will look at δ as at a mapping $P^3 \to P^3$, sometimes as at a mapping $\mathbb{R}^4 \to \mathbb{R}^4$. When restricted to the unit sphere S^3, δ is called the *Hopf map* $S^3 \to S^2$. It has been proved in (Dietz et al., 1993) that a rational curve or surface of degree $\leq 2m$ contained in the unit sphere has a representation of the form $y = \delta(x)$, where the x_i are univariate or bivariate polynomials of degree less or equal to m (see also Dietz et al., 1995).

δ also is the composition of an elliptic net projection and the inverse of a stereographic projection, and therefore is also called *generalized stereographic projection*. Now the inverse images of points are just the lines mentioned above. If we neglect metric properties of δ, an analogous map exists for all oval quadrics in P^3, which are projectively equivalent to the sphere.

It is now possible to define an intrinsic control structure for NURBS curves on the sphere, as described in (Pottmann, 1995). Choose a sequence $d_0, \dots$ of de Boor points and a sequence $f_0, \dots$ of Farin points, choose p_0 in $\delta^{-1}d_0$ and calculate successively points $q_0, p_1, q_1, p_2, \dots$ such that $q_i = \frac{1}{2}(p_i + p_{i+1})$, $\delta(q_i) = f_i$ and $\delta(p_i) = d_i$. If we use the sequence $p_0, p_1, \dots$ as a sequence of control points for a C^k piecewise polynomial spline curve $p(t)$ of degree n, $\delta(p(t))$ will be a C^k piecewise rational spline curve of degree $2n$, which does not depend on the choice of p_0, because an elliptic net admits a one-parameter group of automorphic collineations which map each line of the net onto itself.

If we want to generate closed NURBS curves on the sphere described by the sequence $d_0, f_0, \dots, d_{n-1}, f_{n-1}, d_n = d_0$, we extend the sequence d_j by defining $d_{i+n} = d_i$ and construct a corresponding control polygon p_i. This does not have to

Figure 1: control structure for rational curves on the sphere

be closed, but if we use a "periodic" knot vector, the δ-image of the resulting polynomial spline curve will be a periodic NURBS curve on the sphere, as is illustrated in figure 1.

3 The kinematic mapping of Blaschke and Grünwald

To perform similar constructions in the case of the unit cylinder Z^2 with equation $x^2 + y^2 = 1$, we can do the following (see Blaschke and Müller, 1956): SO_3 was the group of invertible linear transformations automorphic for the unit sphere. We introduce an *isotropic* scalar product in $\mathbb{R}^3$, that is a symmetric bilinear form of defect 1 and index 0, such as $(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = x_1x_2 + y_1y_2$. Then we define a norm by $\|x\| = \sqrt{x.x}$ and Z^2 is the unit sphere in this metric.

The group G of those invertible linear transformations which are automorphic for Z^2 and leave the z-axis pointwise fixed, is isomorphic to the planar euclidean motion group OA_2. An isomorphism ν is constructed as follows: Let $x = (x_1, x_2, x_3) \in Z^2$ and consider the linear function $f_x(u, v) = -x_3 - x_2u + x_1v$. $f_x \geq 0$ defines a half-plane in $\mathbb{R}^2$, whose boundary is the oriented line $\nu(x)$. The action of OA_2 on oriented lines now equals the ν-image of G.

The following construction is completely analogous to the spherical case: Let Q^3 equal P^3 without the line $x_0 = x_3 = 0$. The kinematic mapping of Blaschke and Grünwald, which will be denoted by σ for simplicity, is a one-to-one mapping $G = \mathrm{OA}_2 \to Q^3$. In homogeneous coordinates, σ maps a euclidean rotation with center (x_m, y_m) and angle α to the point $\mathbb{R}(-\cot(\alpha/2), x_m, y_m, 1)$ and a euclidean translation with vector (u, v) to the point $\mathbb{R}(1, -v, u, 0)$. This makes σ even smooth.

We introduce a scalar product of defect 2 and index 0 in $\mathbb{R}^4$ by letting $x \cdot y = x_0y_0 + x_3y_3$, which induces a distance $d(.,.)$ in Q^3 by letting $\cos d(\mathbb{R}a, \mathbb{R}b) = a \cdot b / \sqrt{(a \cdot a)(b \cdot b)}$. Then σ has the following properties:

1. One-parameter subgroups of OA_2 and their cosets are mapped to straight lines in Q^3.

2. The group-invariant distance in OA_2 which equals the *angle* between two rotations or translations, is mapped to twice the distance in Q^3.

3. Fix a "unit" vector $n \in Z^2 \subset \mathbb{R}^3$. For all unit vectors $v \in Z^2$ the set of all $L \in \mathrm{OA}_2$ which map n to v has a straight line as image under σ. These lines form a fibration of Q^3, which is known to geometers as a *parabolic net*.

4 A cylindrical control structure

We define an intrinsic control structure for NURBS curves on a cylinder or a cone in projective three-space P^3. First, it is sufficient to consider the unit cylinder $Z^2 : x_1^2 + x_2^2 = x_0^2$, as all singular quadrics with 1 singular point are projectively equivalent. From the Klein model of the Grassmann manifold of lines in three-space, where oval quadrics correspond to elliptic, ruled quadrics to hyperbolic and singular quadrics with one singular point to parabolic nets, we would suppose that there is a quadratic projection $P^3 \to Z^2$ with the lines of a *parabolic* net as inverse images of points.

Figure 2: convex hull property

Figure 3: control points for trimlines

This is accomplished in the following way: Let $x \in Q^3$ and let $L = \sigma^{-1}(x)$ be the preimage of x under the kinematic mapping. From (Strubecker, 1961) it follows that we can choose $n \in Z^2$, such that the mapping $\delta : Q^3 \to Z^2 : \delta(x) = L(n)$, has the following representation in homogeneous coordinates:

$$\begin{aligned} y_0 &= x_3^2 + x_0^2, \\ y_1 &= x_3^2 - x_0^2, \\ y_2 &= -2x_3x_0, \\ y_3 &= 2(x_3x_1 - x_2x_0), \end{aligned} \qquad (2)$$

and thus has the desired properties. For more details on δ, see also (Dietz, 1995).

The cylindrical control structure now is completely anologous to the spherical case. There are, however, some difficulties that do not arise in the case of an elliptic net. The condition $q_i = \frac{1}{2}(p_i + p_{i+1})$ can be fulfilled uniquely if e.g. the lines $\delta^{-1}d_i$, $\delta^{-1}f_i$ and $\delta^{-1}d_{i+1}$ are skew and cannot be fulfilled if two of the lines intersect and the third is skew to them. In an elliptic net, all lines are pairwise skew. A parabolic net consists of line pencils, which correspond to lines on the cylinder. Lines of different pencils are skew. Therefore, if two of the points d_i, f_i, d_{i+1} lie on a common generator line $l \subset Z^2$, the third point must also be in l. This is also clear from the fact that the line segment p_ip_{i+1} is mapped to a quadratic rational curve, which is part of a planar intersection of the cylinder, and if two points of a line lie in a plane, then the whole line is contained in the plane.

It is possible to derive a *variation diminishing property* and a *convex hull property* for δ-images of segments of quadratic Bézier curves c: c is contained in some plane ε which contains some line l belonging to the elliptic or parabolic net. Every line contained in ε does intersect l and therefore $\delta(c)$ contains the point $\delta(l)$. For all conic sections d on the sphere or cylinder through $\delta(l)$ there exists a line $\tilde{d} \subseteq \varepsilon$ with $\delta(\tilde{d}) = d$. Therefore both the variation diminishing property and the convex hull property with respect to the conic sections containing $\delta(l)$ hold (see figure 2). A stereographic projection with center $\delta(l)$ maps all these conics to lines, so in a stereographic image this is just the classical case.

5 Construction of Blending surfaces

We can use the control structures defined above to find rational blending surfaces between oval and singular quadrics. The line–geometric model of ruled quadrics, the hyperbolic net together with the hyperbolic motion group, would make it possible to perform analogous constructions, but ruled quadrics do not play such an important role in practice.

The construction is as follows:

1. Define four NURBS curves by their intrinsic control polygons $d_{i1}, c_{i1}, \ldots, d_{i4}, c_{i4}$. (see figure 3) The two outer curves will be the trimlines of a blend.

2. Construct the corresponding polygons $p_{i1}, \ldots$ in $\mathbb{R}^4$, such that the conditions given above are fulfilled, and use $\{p_{i1}, p_{i2}\}$ and $\{p_{i3}, p_{i4}\}$ as control net for $(m, 1)$ piecewise TP-polynomial ruled spline surfaces.

Figure 4: Definition of the transition surface

3. Apply δ (the spherical δ in the case of a sphere, the cylindrical δ in the case of a cylinder). This gives us a $(2m, 2)$-NURBS patch on each of the two quadrics. Calculate the control points of the two surface patches.

4. Select the "outermost two" control point rows and define them to be the control net of a $(2n, 3)$-NURBS transition surface (see figure 4).

The two patches do not depend on the choice of the p_{0j}, but the transition surface does. This could be used as a design parameter together with the choice of the trimlines themselves.

As the boundary curve and the first derivatives of a NURBS surface are completely determined by the first two rows of control points, the so defined transition surface is an exact C^1 blend between the two quadrics. By letting $m = 2$ we achieve the lowest degree which is possible for a C^1-blend constructed with this method: $(4, 3)$.

The trimlines could be given by the designer or be chosen automatically, e.g. the distance to the intersection curve could be chosen as a function of the angle between the surfaces.

So far we have not dealt with the case of closed intersection curves and trimlines. Here some problems arise. As mentioned above, the curves $x(t)$ do not have to be closed for their projections $y(t)$ to be closed and to be of the same differentiability class. For the transition surfaces, however, the situation is different. As C^1- or G^1-boundary conditions for NURBS surfaces are rather complicated, it is desirable to have something closed in $\mathbb{R}^4$ to project.

According to (Pottmann, 1995), a spherical polygon c defines a path in SO_3: First we rotate along c_1, then along c_2, and so on. If we apply the spherical kinematic mapping to this path, we get just the polygon $\mathbb{R}p_0, \dots, \mathbb{R}p_n$ in P^3.

Analogously, a polygon consisting of elliptic segments on the cylinder defines a path in the planar euclidean motion group: The ν-image of an elliptic arc c_i contained in Z^2 equals the set of oriented tangents of an oriented circular arc k_i in the euclidean plane, or, as the degenerate case with radius 0, a sector of a line pencil. We rotate along k_1, then along k_2, and so on. Then we apply the kinematic mapping of Blaschke and Grünwald and get just the polygon $\mathbb{R}p_0, \dots, \mathbb{R}p_n$ in $Q^3 \subset P^3$.

To formulate a closedness condition, it is necessary to define the notion of *angle* between two oriented segments c_1 and c_2 of conic sections on the sphere or cylinder, where the endpoint p of c_1 equals the starting point of c_2. In the case of the sphere the angle is defined to be the euclidean angle between tangent vectors. In the case of the cylinder, the angle is defined to be the *isotropic* angle, which is the difference of the slopes of the tangent vectors to c_i in p. The slope of the vector (x, y, z) equals $k = z/\sqrt{x^2 + y^2}$. It is important for us that this angle is an invariant under the groups SO_3 and OA_2, respectively. Having defined the turning angle at each vertex of a spherical or cylindrical polygon, we define the *total turning number* als the sum of the turning angles.

PROPOSITION 1: *Let $p_0, \dots, p_n$ be the vertices of a polygon in $\mathbb{R}^4$ and let the spherical or cylindrical δ-image c, consisting of segments $c_1, \dots, c_n$ be closed. Then $\mathbb{R}p_0 = \mathbb{R}p_n$ if and only if the total turning angle of c equals $2k\pi$ for the spherical and 0 for the cylindrical case*

PROOF: From the corresponding kinematic mapping it is clear that c is closed if and only if the motion corresponding to the endpoint of the last segment is the identity mapping. A motion with one point fixed is uniquely determined by the image of one tangent vector. On the sphere this condition if fulfilled if and ony if

Figure 5: Example of blending surface between oval quadrics

Figure 6: Example of blending surface between oval and singular quadric

a tangent vector rotates about an angle of $2k\pi$ during the motion. On the cylinder this is equivalent to the condition that after moving a tangent vector around its slope has not changed, which in our terms is expressed by a vanishing total turning angle. □

For design purposes it is best to leave the c_i unchanged and change the f_i such that the total turning angle will equal the desired value and the polygon $\mathbb{R}p_i$ will be closed. To achieve closedness of the polygon p_i in $\mathbb{R}^4$, we first change the f_i to be the group-midpoint of p_i and p_{i+1} in the one-parameter subgroup determined by c_i. Without loss of generality $\|p_0\| = 1$. Then $\|p_i\| = 1$ for all i and $q_i = \lambda(p_i + p_{i+1})$. The rotation along c_i is mapped to a segment of a "great circle" starting at p_i, having its midpoint at q_i and ending at p_{i+1}.

As both the unit spheres S^3 for the spherical and Z^3 for the cylindrical case are twofold coverings of P^3 and Q^3, resp., with fibers consisting of pairs of antipodal points, for the case $c_n = c_0$ the endpoint p_n will be either p_0 or $-p_0$.

Let p_i and q_i be two such polygons. If the corresponding paths in the appropriate motion group are homotopic, $p_0 = \pm p_n$ if and only if $q_0 = \pm q_n$. This gives us

PROPOSITION 2: *Let $c = c_1, \ldots, c_n$ and $d = d_1, \ldots, d_n$ be two closed spherical polygons satisfying the angle condition. After changing the total turning angle of at most one of the two polygons there exist δ-preimages $p_0, \ldots$ and $q_0, \ldots$ such that the δ-image of a piecewise polynomial $(m, 1)$ spline surface defined by the p_i and the q_i is a closed C^1 piecewise polynomial $(2m, 2)$ spline surface in $\mathbb{R}^4$.*

PROOF: The homotopy relation will be denoted by $\sim$. For details on homotopy and coverings, see e.g. (Bredon, 1993). If for the corresponding paths $\tilde{c}$ and $\tilde{d}$ in the spherical motion group $\tilde{c} \sim \tilde{d}$ holds, either both $p_n = p_0$ and $q_n = q_0$ or both $p_n = -p_0$ and $q_n = -q_0$. As $\delta(x) = \delta(-x)$, the proposition then follows. If $\tilde{c} \not\sim \tilde{d}$, w.l.o.g. we can assume $p_n = p_0$ and $q_n = -q_0$. Now we construct a spherical polygon $d'_0, \ldots, d'_n$ with the same starting points and endpoints as $d_0, \ldots, d_n$ by adding a total turning angle of 2π. Let μ denote the rotation with angle 2π and axis $\mathbb{R}\delta(p_0)$. Then $\mu \not\sim 0$ and $\tilde{d}' \sim \tilde{d} \star \mu$ imply $\tilde{d}' \sim \tilde{c}$. □

PROPOSITION 3: *Let c and d be two closed* homotopic *cylindrical polygons not containing a line segment and satisfying the angle condition. Then there exist δ-preimages $p_0, \ldots$ and $q_0, \ldots$ with the same properties as in Prop. 2.*

PROOF: If the cylindrical polygons are homotopic, the corresponding paths in the euclidean motion group are homotopic. Therefore either both $p_n = p_0$ and $q_n = q_0$ or both $p_n = -p_0$ and $q_n = -q_0$. As $\delta(x) = \delta(-x)$, the proposition follows. □

Now let two oval or singular quadratic surfaces and on each of them two closed polygons satisfying the angle condition be given. It follows from propositions 2 and 3 that after changing the total turning number of at most one of them, the construction described in figure 4 can be applied. Figures 5 and 6 show some examples.

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