Binary Operations in Spherical Convex Geometry

Florian Besau*, Vienna University of Technology

joint work with

Franz E. Schuster, Vienna University of Technology

Abstract

We establish characterizations of binary operations between convex bodies on the Euclidean unit sphere. Our main result shows that the convex hull is essentially the only non-trivial projection covariant operation between pairs of proper spherical convex bodies contained in open hemispheres. Moreover, we prove that any continuous and projection covariant binary operation between all proper spherical convex bodies must be trivial.

Introduction

In recent years it has been explained why a number of fundamental notions from convex geometric analysis really do have a special place in the theory. For example, Blaschke's classical affine and centro-affine surface areas were given characterizations by Ludwig and Reitzner (Ann. of Math. 172 (2010), 1219–1271) and Haberl and Parapatits (J. Amer. Math. Soc., in press) as unique valuations satisfying certain invariance properties; polar duality and the Legendre transform were characterized by Böröczky and Schneider (Geom. Funct. Anal. 18 (2008) 657–667) and Artstein-Avidan and Milman (Ann. of Math. 169 (2009), 661–674), respectively. These and other results of the same nature not only show that the notions under consideration are characterized by a surprisingly small number of basic properties but also led to the discovery of seminal new notions.

Gardner, Hug, and Weil [**3**] initiated a new line of research whose goal is to enhance our understanding of the fundamental characteristics of known *binary* operations between sets in Euclidean geometry (see also [**4**]). Their main focus is on operations which are projection covariant, that is, the operation can take place before or after projection onto linear subspaces, with the same effect. One impressive example of the results obtained in [**3**] is a characterization of the classical Minkowski addition between convex bodies (compact convex sets) in $\mathbb{R}^n$ as the only projection covariant operation which also satisfies the identity property. In fact, a characterization of all projection covariant operations between origin-symmetric convex bodies was established in [**3**], by proving that such operations are precisely those given by so-called M-addition. This little-known addition was later shown in [**2**] to be intimately related to Orlicz addition, a recent important generalization of Minkowski addition.

Motivated by the results of Gardner, Hug, and Weil we investigate operations between convex sets on the Euclidean unit sphere $\mathbb{S}^n$. Over the last century both Euclidean and affine convex geometry have evolved into a powerful body of results which is well suited to deal with all sorts of problems from geometric analysis. However, the theory of spherical convex bodies is still in its infancy; contributions to spherical convex geometry are scattered and lack a common thread. One reason for this might be the fact that there seems to be no natural analogue of Minkowski addition for spherical convex sets. Our main result underlines this latter statement by showing that projection covariant operations between proper spherical convex sets are either trivial or given (essentially) by the spherical convex hull. The picture changes drastically when operations between convex bodies in a *fixed* open hemisphere are considered. In this case, we establish a one-to-one correspondence between binary operations on spherical convex bodies that are projection covariant with respect to the center of the hemisphere, and projection covariant operations on convex bodies in $\mathbb{R}^n$.

Contact Information

Florian Besau
Email: florian.besau@tuwien.ac.at
Phone: +43 (1) 58801 104762
http://dmg.tuwien.ac.at/fg7/fbesau/

TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

Let $n \geq 2$ and denote by $\mathbb{S}^n$ the n-dimensional Euclidean unit sphere. The usual *spherical distance* between points on $\mathbb{S}^n$ is given by $d(u,v) = \arccos(u \cdot v)$, $u, v \in \mathbb{S}^n$. It induces the *Hausdorff metric* δ_s on closed subsets A, $B \subseteq \mathbb{S}^n$ by

$$\delta_s(A,B) = \min\{0 \leq \lambda \leq \pi : A \subseteq B_\lambda \text{ and } B \subseteq A_\lambda\},$$

where A_λ denotes the set of all points with distance at most λ from A.

A set $A \subseteq \mathbb{S}^n$ is called a *convex* if the radial extendsion

$$\operatorname{rad}(A) := \{\lambda x : \lambda \geq 0, x \in K\} \subseteq \mathbb{R}^{n+1}$$

is a convex set in $\mathbb{R}^{n+1}$. We say $K \subseteq \mathbb{S}^n$ is a *convex body* if K is closed and convex. The space of spherical convex bodies endowed with the Hausdorff metric is denoted by $\mathcal{K}(\mathbb{S}^n)$. We call $K \in \mathcal{K}(\mathbb{S}^n)$ a *proper* spherical convex body if K is contained in an open hemisphere. We write $\mathcal{K}^p(\mathbb{S}^n)$ for the subspace of all proper spherical convex bodies. For a fixed point $u \in \mathbb{S}^n$ we denote by $\mathcal{K}^p_u(\mathbb{S}^n)$ the subspace of spherical convex bodies that are contained in the open hemisphere centered at u. Note that

$$\mathcal{K}^p(\mathbb{S}^n) = \bigcup_{u \in \mathbb{S}^n} \mathcal{K}^p_u(\mathbb{S}^n).$$

The *convex hull* $\operatorname{conv} A$ of a set $A \subseteq \mathbb{S}^n$ is given by the intersection of all convex bodies that contain A, that is,

$$\operatorname{conv} A = \bigcap \{K \in \mathcal{K}(\mathbb{S}^n) : A \subseteq K\}.$$

For $0 \leq k \leq n$, a k-sphere S is a k-dimensional great sub-sphere of $\mathbb{S}^n$, that is, the intersection of a $(k+1)$-dimensional linear subspace $V \subseteq \mathbb{R}^{n+1}$ with $\mathbb{S}^n$. For $K \in \mathcal{K}(\mathbb{S}^n)$ the *spherical projection* $K|S$ is defined by

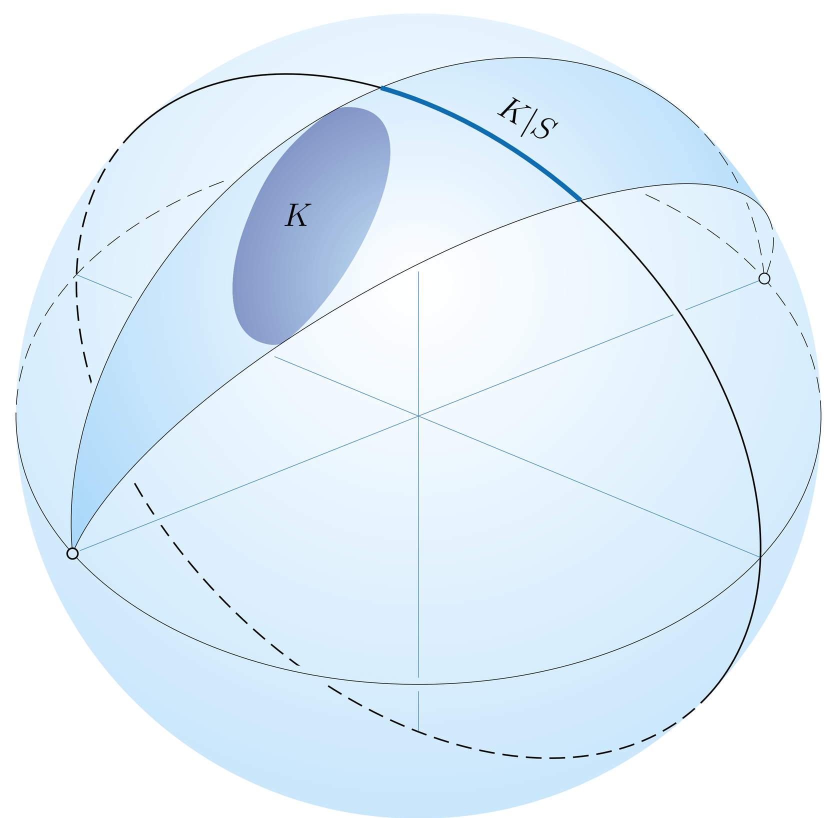
$$K|S := \operatorname{conv}(K \cup S^\circ) \cap S = (\operatorname{pos}(K)|V) \cap \mathbb{S}^n,$$

where $S = V \cap \mathbb{S}^n$ and S° is the $(n-k-1)$-sphere orthogonal to S, that is, $S^\circ = V^\perp \cap \mathbb{S}^n$.

For fixed $u \in \mathbb{S}^n$ we call a binary operation $*\colon \mathcal{K}^p(\mathbb{S}^n) \times \mathcal{K}^p(\mathbb{S}^n) \to \mathcal{K}^p(\mathbb{S}^n)$ *u-projection covariant* if for all k-spheres S, $0 \leq k \leq n-1$, with $u \in S$ and for all K, $L \in \mathcal{K}^p_u(\mathbb{S}^n)$, we have

$$(K|S) * (L|S) = (K * L)|S.$$

We call $*$ *projection covariant* if $*$ is u-projection covariant for all $u \in \mathbb{S}^n$.



Projection of a spherical convex body K onto a great sub-sphere S.

Main Results

The main objective of our work was to characterize projection covariant operations between spherical convex bodies. Our first result shows that such operations between *all* proper spherical convex bodies are of a very simple form.

Theorem 1. *An operation $*\colon \mathcal{K}^p(\mathbb{S}^n) \times \mathcal{K}^p(\mathbb{S}^n) \to \mathcal{K}^p(\mathbb{S}^n)$ between proper convex bodies is projection covariant and continuous with respect to the Hausdorff metric if and only if either $K*L = K$, or $K*L = -K$, or $K*L = L$, or $K*L = -L$ for all $K, L \in \mathcal{K}^p(\mathbb{S}^n)$.*

The binary operations between spherical convex bodies from Theorem 1 will be called *trivial*. The following example shows that the continuity assumption cannot be omitted.

Example. Consider the set $\mathcal{C} \subset \mathcal{K}^p(\mathbb{S}^n) \times \mathcal{K}^p(\mathbb{S}^n)$ of all pairs (K, L) such that K and L are contained in some open hemisphere, that is,

$$\mathcal{C} = \bigcup_{u \in \mathbb{S}^n} \left(\mathcal{K}^p_u(\mathbb{S}^n) \times \mathcal{K}^p_u(\mathbb{S}^n)\right).$$

Define a binary operation $*\colon \mathcal{K}^p(\mathbb{S}^n) \times \mathcal{K}^p(\mathbb{S}^n) \to \mathcal{K}^p(\mathbb{S}^n)$ by

$$K * L := \begin{cases} K & \text{if } (K,L) \in \mathcal{C}, \\ L & \text{if } (K,L) \notin \mathcal{C}. \end{cases}$$

Clearly, $*$ is not continuous but by our definition it is projection covariant.

The proof of Theorem 1 relies on ideas by Gardner, Hug, and Weil. The critical tool to transfer their techniques to the sphere is the *gnomonic projection* which establishes the following correspondence between projection covariant operations in $\mathcal{K}(\mathbb{R}^n)$, the space of compact, convex sets in $\mathbb{R}^n$, and u-projection covariant operations on $\mathcal{K}^p_u(\mathbb{S}^n)$.

Theorem 2. *For every $u \in \mathbb{S}^n$, there is a one-to-one correspondence between u-projection covariant operations $*\colon \mathcal{K}^p_u(\mathbb{S}^n) \times \mathcal{K}^p_u(\mathbb{S}^n) \to \mathcal{K}^p_u(\mathbb{S}^n)$ and projection covariant binary operations $\bar{*}\colon \mathcal{K}(\mathbb{R}^n) \times \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n)$. Moreover, every such u-projection covariant binary operation $*$ is continuous in the Hausdorff metric.*

Note that by Theorem 2 every projection covariant operation $*$ on $\mathcal{C}$ is also automatically continuous.

Finally, as our main result, we prove that the only *non-trivial* projection covariant operation on the set $\mathcal{C}$ is essentially the spherical convex hull.

Theorem 3. *An operation $*\colon \mathcal{C} \to \mathcal{K}^p(\mathbb{S}^n)$ is non-trivial and projection covariant if and only if either $K * L = \operatorname{conv}(K \cup L)$ or $K * L = -\operatorname{conv}(K \cup L)$ for all $(K, L) \in \mathcal{C}$.*

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