SHADOW SYSTEMS OF ASYMMETRIC L_p ZONOTOPES

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Abstract. Shadow systems are used to establish new asymmetric L_p volume product and asymmetric L_p volume ratio inequalities, along with their equality conditions. These inequalities have Reisner's volume product inequality for L_1 zonotopes as a special case. Moreover, uniqueness of the extremals in the symmetric setting is obtained.

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1. Introduction

In the last decades, characterizing the minimizers of the volume product, that is the product of the volumes of a convex body and its polar reciprocal, has become a central quest. The celebrated Blaschke–Santaló inequality characterizes ellipsoids as the maximizers of this functional on convex bodies (compact convex subsets of $\mathbb{R}^n$ with nonempty interior). In contrast, Mahler's longstanding conjecture that its minimizers are the simplices remains open, and also the variant of the conjecture for origin-symmetric convex bodies appears to be extremely difficult to attack. However, due to the strong research interest in the problem, substantial inroads have been made (see e.g. [**1, 3, 6, 7, 10, 13, 18–20, 31, 33, 34, 38, 39, 41**]).

One of the most striking partial results towards Mahler's conjecture for origin-symmetric convex bodies is Reisner's characterization of the minimizers of the volume product among zonotopes and zonoids, that is Minkowski sums of origin-symmetric line segments in $\mathbb{R}^n$, and their limits with respect to the Hausdorff distance [**13, 38**]. Since this important result, Minkowski addition has been absorbed into the more general concept of L_p addition for $p \geq 1$, and it turned out that many key tools of the Brunn–Minkowski theory have L_p analogues; see [**8, 24, 25**]. This progress sparked the rapid development of a new L_p Brunn–Minkowski theory (see e.g. [**9, 12, 21, 23, 26–30, 32, 35–37, 44, 46–48, 51**]).

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However, it took another ten years until an L_p version of Reisner's volume product inequality was established. Using a new ingenious antisymmetrization technique, Campi and Gronchi [**7**] proved an L_p volume product inequality—together with its dual volume ratio inequality—that contains Reisner's result as a special case. Moreover, an even Orlicz extension was discovered very recently (see [**49**]). However, all of these generalizations are restricted to the origin-symmetric setting and the equality conditions remained open. In this article we establish inequalities for the *asymmetric* L_p volume product and the *asymmetric* L_p volume ratio, along with a characterization of the extremals.

The seminal work in the new asymmetric L_p Brunn–Minkowski theory is Ludwig's discovery and characterization [**22**] of both the asymmetric L_p centroid body operator and the asymmetric L_p projection body operator. Soon after that, the asymmetric L_p Brunn–Minkowski theory built up momentum (see e.g. [**14**–**16**]). For instance, the asymmetric L_p centroid body operator turned out to be tailor-made to establish an L_p extension of the Blaschke–Santaló inequality for *all* convex bodies [**15**], whereas earlier work by Lutwak and Zhang had been limited to the origin-symmetric setting [**32**]. However, no reverse isoperimetric inequalities, that is geometric inequalities that have simplices or parallelepipeds as their extremals, have been obtained in the asymmetric L_p framework yet.

In this article we establish such sharp reverse isoperimetric inequalities, along with their equality conditions, for the new notion of *asymmetric L_p zonotope*. These zonotopes are the L_p sums of line segments in $\mathbb{R}^n$ ($n \geq 2$) with one endpoint at the origin. More precisely, if Λ is a finite set of vectors from $\mathbb{R}^n \setminus \{o\}$, then the associated asymmetric L_p zonotope $Z_p^+\Lambda$ is the unique compact convex set with support function (see Section 2)

$$h(Z_p^+\Lambda, u) := \sqrt[p]{\sum_{w\in\Lambda} \langle w, u\rangle_+^p},$$

where $u \in \mathbb{R}^n$ and $\langle\cdot,\cdot\rangle_+ = \max\{0, \langle\cdot,\cdot\rangle\}$ denotes the positive part of the Euclidean scalar product. In particular, the asymmetric L_1 zonotopes $Z_1^+\Lambda$ are the Minkowski sums of line segments with one endpoint at the origin, and we observe in Section 2 that these are, up to translation, the classic origin-symmetric zonotopes. Hence also the asymmetric L_p zonotopes can be used to embed Reisner's inequality into an L_p setting.

Our first main result is an asymmetric L_p volume product inequality that includes, as the special case $p = 1$, Reisner's characterization of parallelepipeds as the minimizers of the volume product among zonotopes. Throughout this article we call a subset of $\mathbb{R}^n$ *spanning* if it spans $\mathbb{R}^n$. We always use $\Lambda_\perp = \{e_1, \dots, e_n\}$ to denote the canonical basis. Moreover, we write $Z_p^{+,*}\Lambda$ for the polar body of $Z_p^+\Lambda$ with respect to the Santaló point (see Section 2 for the definition).

Theorem 1. *Suppose $p \geq 1$ and $\Lambda \subseteq \mathbb{R}^n \setminus \{o\}$ is finite and spanning. Then*

$$V(Z_p^{+,*}\Lambda)V(Z_1^+\Lambda) \geq V(Z_p^{+,*}\Lambda_\perp)V(Z_1^+\Lambda_\perp)$$

with equality for $p > 1$ if and only if Λ is a $\mathrm{GL}(n)$ image of the canonical basis. If $p = 1$, then equality holds if and only if $Z_1^+\Lambda$ is a parallelepiped.

The theorem is an asymmetric counterpart to Campi and Gronchi's volume product inequality for symmetric L_p zonotopes [**7**], which in turn provides a stronger lower bound under the additional symmetry assumption. We show in Section 6 how the key ingredients in the proof of Theorem 1 also yield the Campi–Gronchi inequality together with new equality conditions.

The second main result presented in this article is an asymmetric L_p volume ratio inequality, and a natural dual to Theorem 1. Surprisingly, the extremizers turn out to be very different, and we use the notion of *obtuse set* to describe them (see e.g. [**2**]).

Definition. A set Λ of vectors from $\mathbb{R}^n$ is called *obtuse* if every pair of distinct vectors u, v from Λ satisfies

$$\langle u, v\rangle_+ = 0.$$

For example, the canonical basis $\Lambda_\perp$ and its symmetrization $\Lambda_\perp \cup -\Lambda_\perp$ are obtuse. The next theorem states that precisely the $\mathrm{GL}(n)$ images of obtuse sets are the maximizers of the asymmetric L_p volume ratio.

Theorem 2. *Suppose $p > 1$ and $\Lambda \subseteq \mathbb{R}^n \setminus \{o\}$ is finite and spanning. Then*

$$\frac{V(Z_p^+\Lambda)}{V(Z_1^+\Lambda)} \leq \frac{V(Z_p^+\Lambda_\perp)}{V(Z_1^+\Lambda_\perp)}$$

with equality if and only if Λ is a $\mathrm{GL}(n)$ image of an obtuse set.

The L_p volume ratio inequality for origin-symmetric L_p zonotopes has been established earlier [**7**]. This result due to Campi and Gronchi, together with new equality conditions, is in fact a special case of our Theorem 2 (see Section 6). We remark that a standard limiting argument yields results similar to Theorem 1 and Theorem 2 in the continuous setting, though without equality conditions.

The proofs of our main results make critical use of the notion of *shadow system* (see e.g [**40**,**45**]) and related ideas, techniques and results by Campi–Gronchi [**4**–**7**] and Meyer–Reisner [**33**]; we provide the definition of this notion and more general background material in Section 2. In Section 3 we recall Campi and Gronchi's stepwise reduction of multisets to the canonical basis [**7**], and prove that this process is compatible with the asymmetric L_p volume product and the asymmetric L_p volume ratio. Some preparatory lemmas concerning the equality conditions are presented in Section 4, and we move on to the proof of the main theorems in Section 5. The final section of this article is dedicated to the symmetric versions of Theorem 1 and Theorem 2.

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2. Background material

In the following, we collect the background material that is required for our main theorems. More specifically, we recall classic definitions and results from convex geometry, along with recently discovered properties of shadow systems. In addition to the references indicated below, the reader may wish to consult the monographs [**11**, **42**].

Throughout this article, a convex body is a compact convex subset of $\mathbb{R}^n$ ($n \geq 2$) with nonempty interior. If K is a convex body, then we denote by $V(K)$ its n-dimensional volume and by

$$h(K,u) = \max\{\langle x,u\rangle :\ x \in K\}$$

its support function at $u \in \mathbb{R}^n$. The sublinear support function characterizes a convex body and, conversely, every sublinear function on $\mathbb{R}^n$ is the support function of a nonempty compact convex set. It is an immediate consequence of the definition that convex bodies K, L satisfy $K \subseteq L$ if and only if $h(K,\cdot) \leq h(L,\cdot)$.

Support functions can also be used to introduce a concept of duality: for every interior point s of a convex body K,

$$K^s = \{x \in \mathbb{R}^n :\ h(K-s,x) \leq 1\}$$

defines a convex body that is called the *polar body of K with respect to s*. The unique point s in the interior of K that is uniquely determined by the property that $V(K^s)$ is minimal among all possible choices of s is called the *Santaló point* of K and denoted by $s(K)$. To shorten notation, we adopt the convention to write K^* for $K^{s(K)}$. It is well known that polarization with respect to the Santaló point is translation invariant,

$$(K+y)^* = K^* \text{ for } y \in \mathbb{R}^n,$$

and $\mathrm{GL}(n)$ contravariant,

$$(\phi K)^* = \phi^{-T}K^* \text{ for } \phi \in \mathrm{GL}(n).$$

Here, ϕ^{-T} denotes the inverse of the transpose ϕ^T of the matrix ϕ. In particular, the volume product $V(K^*)V(K)$ is a translation invariant $\mathrm{GL}(n)$ invariant functional on convex bodies, and thus all nondegenerate parallelepipeds have the same volume product.

In the following, we consider L_p zonotopes associated not only with sets, as defined in the introduction, but more generally with *multisets*, that is sets that may contain more than one copy of an element. More precisely, a multiset Λ is identified with its multiplicity function $\Lambda : \mathbb{R}^n \to \mathbb{N} \cup \{0\}$ that generalizes the characteristic function of sets. We say that a vector is an element of a multiset if the corresponding multiplicity function evaluated at the vector is greater than zero, and call a multiset *finite* if it contains only a finite number of vectors. If these vectors span $\mathbb{R}^n$, then we say that the multiset is *spanning*.

The elementary operations between multisets can be defined using the above identification. For instance, the union $\Lambda_1 \uplus \Lambda_2$ of two multisets Λ_1, Λ_2 has the multiplicity function $\Lambda_1 + \Lambda_2$, and $\Lambda_1 \setminus \Lambda_2$ denotes the multiset with multiplicity function $\max\{\Lambda_1 - \Lambda_2, 0\}$. We write multisets in the usual set notation; that is, $\Lambda = \{v_1, \dots, v_m\}$, where vectors appear according to their multiplicity.

Asymmetric L_p zonotopes associated with finite and spanning multisets $\Lambda = \{v_1, \dots, v_m\}$ are well-defined by their support function ($u \in \mathbb{R}^n$)

$$h(Z_p^+\Lambda, u) = \sqrt[p]{\sum_{i=1}^m \langle v_i, u\rangle_+^p}, \tag{1}$$

because Minkowski's inequality, together with the simple inequality

$$\max\{0, r+s\} \leq \max\{0, r\} + \max\{0, s\} \tag{2}$$

for real numbers r, s, implies that this function is sublinear. Motivated by this definition, we identify multisets that only differ in the number of o-vectors they contain, and work with the representative that contains none. A first simple consequence of definition (1) is that the operator Z_p^+ on finite and spanning multisets is $\mathrm{GL}(n)$ equivariant: $Z_p^+\phi\Lambda = \phi Z_p^+\Lambda$ holds for all $\phi \in \mathrm{GL}(n)$, because

$$h(Z_p^+\phi\Lambda, u)^p = \sum_{i=1}^m \langle \phi v_i, u\rangle_+^p = \sum_{i=1}^m \langle v_i, \phi^T u\rangle_+^p = h(\phi Z_p^+\Lambda, u)^p \tag{3}$$

holds for all $u \in \mathbb{R}^n$. Moreover, for $1 \leq p < q < \infty$, we have that $Z_p^+\Lambda \supseteq Z_q^+\Lambda$; this inclusion follows from the inequality for finite sequences of nonnegative numbers a_i (see [**17**, Theorem 19])

$$\sqrt[p]{\sum_i a_i^p} \geq \sqrt[q]{\sum_i a_i^q}, \tag{4}$$

with equality if and only if only one a_i is positive. In particular, all asymmetric L_p zonotopes associated with finite and spanning multisets have nonempty interior because they contain the L_∞ zonotope, that is the convex hull of the points in Λ and the origin. Before we move on to shadow systems, we remark that the case $p = 1$ of definition (1) is closely related to the origin-symmetric classic zonotope $Z_1\Lambda$, where ($u \in \mathbb{R}^n$)

$$h(Z_1\Lambda, u) = \sum_{i=1}^m |\langle v_i, u\rangle|.$$

More specifically, the computation

$$h(Z_1\Lambda, u) = \sum_{i=1}^m |\langle v_i, u\rangle| = 2\sum_{i=1}^m \langle v_i, u\rangle_+ - \langle \sum_{i=1}^m v_i, u\rangle$$

implies that the two zonotopes $Z_1\Lambda$ and $Z_1^+\Lambda$ are homothetic:

$$Z_1\Lambda = 2Z_1^+\Lambda - \sum_{i=1}^m v_i. \tag{5}$$

In particular, we have that $V(Z_1^{+,*}\Lambda)V(Z_1^+\Lambda) = V(Z_1^*\Lambda)V(Z_1\Lambda)$ holds for all finite and spanning multisets Λ.

The notion of *shadow system* (or linear parameter system) has been introduced by Rogers and Shephard [**40**, **45**] to describe certain types of one-parameter families of points in $\mathbb{R}^n$. For our purposes, all of these families depend on a parameter $t \in [-a^{-1}, 1]$, where a is a positive real number. In this article, we consider shadow systems of *multisets* and shadow systems of *convex bodies.*

Definition.

(i) A shadow system of multisets along the direction $v \in S^{n-1}$ is a family of finite and spanning multisets Λ_t, $t \in [-a^{-1}, 1]$, such that

$$\Lambda_t = \{v_i + t\beta_i v :\ 1 \le i \le m\},$$

where $\Lambda_0 = \{v_1, \ldots, v_m\}$ is a finite and spanning multiset and β_i, $1 \le i \le m$, are real numbers.

(ii) A shadow system of convex bodies along the direction $v \in S^{n-1}$ is a family of convex bodies K_t, $t \in [-a^{-1}, 1]$, such that

$$K_t = \operatorname{conv}\{x + t\beta(x)v :\ x \in M\},$$

where $M \subseteq \mathbb{R}^n$ is a bounded set and β is a real-valued bounded function on M. Here, as usual, $\operatorname{conv}\{x + t\beta(x)v :\ x \in M\}$ denotes the convex hull of the set $\{x + t\beta(x)v :\ x \in M\}$.

By definition, the orthogonal projection of a shadow system of convex bodies along the direction v onto the hyperplane $v^\perp$ is independent of the parameter t. However, it is not hard to see that a one-parameter family of convex bodies with this property is not necessarily a shadow system. Campi and Gronchi [**4**] have shown that shadow systems of convex bodies are distinguished by properties of their graph functions, where, for a convex body K and $v \in S^{n-1}$, the *uppergraph function* $\overline{g}_v(K, \cdot)$ and the *lowergraph function* $\underline{g}_v(K, \cdot)$ of a convex body K are defined by ($x \in v^\perp$)

$$\begin{aligned} \overline{g}_v(K, x) &:= \sup\{\lambda \in \mathbb{R} :\ x + \lambda v \in K\}; \\ \underline{g}_v(K, x) &:= \inf\{\lambda \in \mathbb{R} :\ x + \lambda v \in K\}. \end{aligned} \tag{6}$$

More precisely, they have obtained the following result. Here and throughout the paper, $\cdot|_H$ denotes the orthogonal projection onto a hyperplane H.

Theorem 2.1 ([**4**]). *Let K_t, $t \in [-a^{-1}, 1]$, be a one-parameter family of convex bodies such that $K_t|_{v^\perp}$ is independent of t. Then K_t, $t \in [-a^{-1}, 1]$, is a shadow system of convex bodies along the direction v if and only if for every $x \in K_0|_{v^\perp}$, the functions $t \mapsto \overline{g}_v(K_t, x)$ and $t \mapsto -\underline{g}_v(K_t, x)$ are convex and*

$$\underline{g}_v(K_{\lambda s + \mu t}, x) \le \lambda \overline{g}_v(K_s, x) + \mu \underline{g}_v(K_t, x) \le \overline{g}_v(K_{\lambda s + \mu t}, x) \tag{7}$$

for every $s, t \in [-a^{-1}, 1]$ and $\lambda, \mu \in (0, 1)$ such that $\lambda + \mu = 1$.

Many useful properties of shadow systems of convex bodies stem from the fact that they can be viewed as a family of projections of an $(n+1)$-dimensional convex body onto $\mathbb{R}^n$. For instance, Shephard [**45**] used this fact to prove the convexity of mixed volumes of shadow systems as a function of t. In this article we restrict ourselves to the volume.

Theorem 2.2 ([**40**]). *Suppose K_t, $t \in [-a^{-1}, 1]$, is a shadow system of convex bodies. Then $V(K_t)$ is a convex function of t.*

We remark that this theorem is also an immediate consequence of Theorem 2.1 and Fubini's theorem. A dual analogue of Theorem 2.2 was discovered recently: Campi and Gronchi, in the case of origin-symmetric convex bodies, have established that the volume of polars of shadow systems of convex bodies with respect to the Santaló point is the inverse of a convex function [**6**]. This result has been extended to the general non-symmetric case by Meyer and Reisner [**33**].

Theorem 2.3 ([**33**]). *Suppose K_t, $t \in [-a^{-1}, 1]$, is a shadow system of convex bodies. Then $V(K_t^*)^{-1}$ is a convex function of t.*

In the spirit of this theorem, it is natural to ask for properties of shadow systems that satisfy that the map $t \mapsto V(K_t^*)^{-1}$ is not only convex, but in fact affine on $[-a^{-1}, 1]$. To answer this question, Meyer and Reisner [**33**] have proved a characterization theorem under the additional assumption that the map $t \mapsto V(K_t)$ is affine in t.

We conclude the section by summarizing the implications of this theorem in our setting. In particular, we restrict ourselves to situations where $V(K_t)$ is independent of t. In the formulation of this theorem, as usual, I_{n-1} denotes the $(n-1)$-dimensional identity matrix.

Theorem 2.4 ([**33**]). *Suppose K_t, $t \in [-a^{-1}, 1]$, is a shadow system of convex bodies along the direction $v = e_1$ and $V(K_t)$ is independent of t. Then the volume of K_t^* is independent of t if and only if there are a real number α and a vector $z \in \mathbb{R}^{1\times(n-1)}$ such that*

$$K_t = t\alpha e_1 + \begin{pmatrix} 1 & tz \\ o & I_{n-1} \end{pmatrix} K_0.$$

3. Orthogonalization of multisets

Classic symmetrization techniques such as Steiner symmetrization are frequently used to show that affine invariant functionals attain a global extremum at ellipsoids. In contrast, proofs of reverse inequalities, where equality is attained at simplices or parallelepipeds, require some form of antisymmetrization. The main aim of this section is to recall the antisymmetrization technique discovered by Campi and Gronchi [**7**] that allows to transform multisets to the canonical basis, and thus the associated zonotopes to parallelepipeds. Remarkably, it is compatible (see Corollary 3.3) with both the *inverse asymmetric L_p volume product*

$$\frac{1}{V(Z_p^{+,*}\Lambda)V(Z_1^+\Lambda)}$$

and the *asymmetric L_p volume ratio*

$$\frac{V(Z_p^+\Lambda)}{V(Z_1^+\Lambda)}.$$

This transformation is constructed in the following way: let $a > 0$ and suppose $\Lambda = \{v_1, \dots, v_m\}$ is a finite multiset such that $\Lambda \setminus \{v_1\}$ is spanning.

Then set $v = \|v_1\|^{-1}v_1$ and for $t \in [-a^{-1}, 1]$ define

$$\Lambda_t^a = \{w_1(t), \ldots, w_n(t)\}, \text{ where } w_i(t) = \begin{cases} v_1 + ta\|v_1\|v, & i = 1; \\ v_i - t\langle v_i, v\rangle v, & \text{otherwise.} \end{cases} \tag{8}$$

Apart from the fact that $\Lambda_0^a = \Lambda$, note that the vector $w_1(1)$ is orthogonal to the remaining vectors in Λ_1, while $w_1(-a^{-1}) = o$. Moreover, by construction, Λ_t, $t \in [-a^{-1}, 1]$, is a shadow system of multisets along the direction v.

The crucial observation behind this definition is, however, that there is a real number $a > 0$ such that the process preserves the volume of the associated L_1 zonotope. By (5), this important result due to Campi and Gronchi also holds for the asymmetric L_1 zonotope:

Theorem 3.1 ([**7**])**.** *Suppose $\Lambda = \{v_1, \ldots, v_m\}$ is a finite multiset such that $\Lambda \setminus \{v_1\}$ is spanning. Then there exists a positive number $a = a(\Lambda)$ such that $V(Z_1^+\Lambda_t^a)$ is independent of t, where Λ_t^a, $t \in [-a^{-1}, 1]$, is defined by* (8)*.*

Throughout this article, all orthogonalizations of multisets of the form (8) will be defined using this canonical choice of $a = a(\Lambda) > 0$, and we call

$$\Lambda_t := \Lambda_t^a, \qquad t \in [-a^{-1}, 1], \tag{9}$$

the *orthogonalization of* Λ *with respect to* v_1. Hence, by construction, the volume of the asymmetric L_1 zonotope associated with an orthogonalization is a constant function of t.

Conversely, there are simple examples that show that $V(Z_p^+\Lambda_t)$ is not necessarily independent of t if $p > 1$. The volume of these zonotopes and their polars can be controlled using the fact that the family of asymmetric L_p zonotopes associated with a general shadow system of multisets is a shadow system of convex bodies. The proof of this statement exploits the fact that a certain class of operators maps shadow systems to shadow systems (see e.g. [**4**, **5**]). In particular, it is a generalization of a theorem by Campi and Gronchi [**7**] (and their proof) to the asymmetric setting.

Before we formulate and prove this theorem, we establish alternative representations of the graph functions (see e.g. [**4**]). These will make it easier to confirm that the hypotheses of the characterization theorem for shadow systems (Theorem 2.1) are satisfied. Let K be a convex body, $v \in S^{n-1}$, and $x \in v^\perp$. By definition (6) of the uppergraph function and the definition of the support function, it follows that

$$\begin{aligned} \overline{g}_v(K, x) &= \sup\{\lambda \in \mathbb{R} : \langle x + \lambda v, u\rangle \le h(K, u) \text{ for all } u \in \mathbb{R}^n\} \\ &= \sup\{\lambda \in \mathbb{R} : \lambda\langle v, u\rangle \le h(K, u) - \langle x, u\rangle \text{ for all } u \in \mathbb{R}^n\}. \end{aligned}$$

Due to continuity in u, the inequality

$$\lambda\langle v, u\rangle \le h(K, u) - \langle x, u\rangle \tag{10}$$

holds for all $u \in \mathbb{R}^n$ if and only if it holds for all $u \in \mathbb{R}^n \setminus v^\perp$. In fact, by 1-homogeneity in u, it is sufficient to consider $u \in \mathbb{R}^n$ that satisfy $|\langle v, u\rangle| = 1$. Since inequality (10) provides an upper bound on λ precisely if $\langle v, u\rangle > 0$, we may therefore assume that $u = v + w$, where $w \in v^\perp$. Consequently,

$$\overline{g}_v(K, x) = \inf_{w \in v^\perp}\{h(K, v + w) - \langle x, w\rangle\}. \tag{11}$$

Similar arguments also give the dual result for the lowergraph function:

$$\underline{g}_v(K,x) = -\inf_{w\in v^\perp}\{h(K,-v-w)+\langle x,w\rangle\} \tag{12}$$

for all $x \in v^\perp$.

Theorem 3.2. *Suppose $p \geq 1$. If Λ_t, $t \in [-a^{-1},1]$, is a shadow system of multisets along the direction $v \in S^{n-1}$, then $Z_p^+\Lambda_t$, $t \in [-a^{-1},1]$, is a shadow system of convex bodies along the direction v.*

Proof. We adopt the notation used in the formulation of Theorem 2.1: throughout this proof s and t denote real numbers in $[-a^{-1},1]$, x is a point in $Z_p^+\Lambda_0|_{v^\perp}$, and $\lambda,\mu \in (0,1)$ satisfy $\lambda+\mu=1$. Clearly, $Z_p^+\Lambda_t|_{v^\perp}$ is independent of t, so it remains to show that the hypotheses of Theorem 2.1 on properties of the graph functions are satisfied.

By assumption, the shadow system Λ_t is equal to, say, $\{v_1(t),\dots,v_m(t)\}$ where $v_i(t) = v_i + t\beta_i v$. For notational convenience we define

$$\|f\|_p(t) := \sqrt[p]{\sum_{i=1}^m |f(v_i(t))|^p}$$

for real-valued functions f on $\mathbb{R}^n$, and $[\cdot]_+ := \max\{\cdot,0\}$. With these definitions the support function of $Z_p^+\Lambda_t$ can be written in the form ($u\in\mathbb{R}^n$)

$$h(Z_p^+\Lambda_t,u)^p = \sum_{i=1}^m [\langle v_i,u\rangle + t\beta_i\langle v,u\rangle]_+^p = \|\langle\cdot,u\rangle_+\|_p^p(t). \tag{13}$$

To establish the convexity of the uppergraph function as a function of t, we note that, by (11) and (13), $\overline{g}_v(Z_p^+\Lambda_{\lambda s+\mu t},x)$ is equal to

$$\inf_{w_1,w_2\in v^\perp}\{\|\langle\cdot,v+\lambda w_1+\mu w_2\rangle_+\|_p(\lambda s+\mu t) - \langle x,\lambda w_1+\mu w_2\rangle\}.$$

By (13), we have

$$\begin{aligned}&\|\langle\cdot,v+\lambda w_1+\mu w_2\rangle_+\|_p^p(\lambda s+\mu t)\\ &\qquad = \sum_{i=1}^m \left[\lambda\big(\langle v_i,v+w_1\rangle+s\beta_i\big)+\mu\big(\langle v_i,v+w_2\rangle+t\beta_i\big)\right]_+^p,\end{aligned}$$

and therefore Minkowski's inequality implies that

$$\begin{aligned}&\|\langle\cdot,v+\lambda w_1+\mu w_2\rangle_+\|_p(\lambda s+\mu t)\\ &\qquad \leq \lambda\|\langle\cdot,v+w_1\rangle_+\|_p(s)+\mu\|\langle\cdot,v+w_2\rangle_+\|_p(t).\end{aligned} \tag{14}$$

Thus, an application of inequality (14) together with (11) shows that

$$\overline{g}_v(Z_p^+\Lambda_{\lambda s+\mu t},x) \leq \lambda\overline{g}_v(Z_p^+\Lambda_s,x)+\mu\overline{g}_v(Z_p^+\Lambda_t,x). \tag{15}$$

Hence the map $t\mapsto \overline{g}_v(Z_p^+\Lambda_t,x)$ is convex.

Since Λ_t is also a shadow system in direction $-v$, the vector v can be replaced by $-v$ in inequality (15). Therefore, by application of the identity $\overline{g}_{-v}(\cdot,x) = -\underline{g}_v(\cdot,x)$, we obtain that also the map $t\mapsto -\underline{g}_v(Z_p^+\Lambda_t,x)$ is convex.

We proceed with the proof of the only remaining inequality (7). First, we claim that

$$\underline{g}_v(Z_p^+\Lambda_{\lambda s+\mu t},x) \leq \lambda\overline{g}_v(Z_p^+\Lambda_s,x)+\mu\underline{g}_v(Z_p^+\Lambda_t,x). \tag{16}$$

To see this, put $w = \mu^{-1}(w_1 - \lambda w_2)$ in (12), where $w_1, w_2 \in v^\perp$, to obtain

$$-\mu \underline{g}_v(Z_p^+\Lambda_t, x) = \inf_{w_1,w_2\in v^\perp} \{h(Z_p^+\Lambda_t, -\mu v - w_1 + \lambda w_2) + \langle x, w_1 - \lambda w_2\rangle\}.$$

Then, by expanding the support function in the right-hand side of this equation using (13), we observe that $-\mu \underline{g}_v(Z_p^+\Lambda_t, x)$ is equal to

$$\inf_{w_1,w_2\in v^\perp} \left\{ \sqrt[p]{\sum_{i=1}^m [\langle v_i, -\mu v - w_1 + \lambda w_2\rangle - \beta_i \mu t]_+^p} + \langle x, w_1 - \lambda w_2\rangle \right\}. \tag{17}$$

Since

$$\begin{aligned}\langle v_i, -\mu v - w_1 + \lambda w_2\rangle - \beta_i \mu t \\ = \langle v_i, -v - w_1\rangle - \beta_i(\lambda s + \mu t) + \lambda\langle v_i, v + w_2\rangle + \beta_i \lambda s,\end{aligned}$$

an application of inequality (2), together with Minkowski's inequality and (13), yields that expression (17) is dominated by

$$\begin{aligned}\inf_{w_1,w_2\in v^\perp} \{h(Z_p^+\Lambda_{\lambda s+\mu t}, -v - w_1) + \langle x, w_1\rangle \\ + \lambda h(Z_p^+\Lambda_s, v + w_2) - \lambda\langle x, w_2\rangle\}.\end{aligned}$$

Thus, by (11) and (12),

$$-\mu \underline{g}_v(Z_p^+\Lambda_t, x) \le -\underline{g}_v(Z_p^+\Lambda_{\lambda s+\mu t}, x) + \lambda \overline{g}_v(Z_p^+\Lambda_s, x), \tag{18}$$

which is just the desired inequality (16).

To conclude the proof, we note that the inequality

$$\underline{g}_{-v}(Z_p^+\Lambda_{\lambda s+\mu t}, x) \le \mu \overline{g}_{-v}(Z_p^+\Lambda_t, x) + \lambda \underline{g}_{-v}(Z_p^+\Lambda_s, x).$$

can be derived from inequality (16) by first replacing v by $-v$, then interchanging λ and μ, and finally interchanging s and t. Since

$$\underline{g}_{-v}(\cdot, x) = -\overline{g}_v(\cdot, x) \qquad \text{and} \qquad \overline{g}_{-v}(\cdot, x) = -\underline{g}_v(\cdot, x)$$

the remaining second part of inequality (7) follows immediately. □

We conclude the section with a corollary on the implications of this theorem towards the behavior of the asymmetric L_p volume product and the asymmetric L_p volume ratio.

Corollary 3.3. *Suppose $p \ge 1$ and Λ is a finite and spanning multiset. If Λ_t, $t \in [-a^{-1}, 1]$, is an orthogonalization of Λ defined by (9), then:*

(i) The map $t \mapsto V(Z_p^{+,*}\Lambda_t)^{-1}$ is convex. In particular, the inverse asymmetric L_p volume product associated with Λ_t is a convex function of t.

(ii) The map $t \mapsto V(Z_p^+\Lambda_t)$ is convex. In particular, the asymmetric L_p volume ratio associated with Λ_t is a convex function of t.

Proof. The volume of the asymmetric L_1 zonotope associated with Λ_t is independent of t by construction, and $Z_p^+\Lambda_t$, $t \in [-a^{-1}, 1]$, is a shadow system of convex bodies by Theorem 3.2. Hence the convexity of the volume with respect to t, Theorem 2.2, and its dual statement, Theorem 2.3, yield the assertions. □

4. The equality conditions

The central result in the previous section (Corollary 3.3) implies that the inverse asymmetric L_p volume product and the asymmetric L_p volume ratio are nondecreasing if Λ is replaced by either $\Lambda_{-a^{-1}}$ or Λ_1, because convex functions attain global maxima at the boundary of compact intervals. We establish in Section 5 that iterations of this step yield the inequalities asserted in our main theorems. However, to study the equality conditions, it is necessary to determine under which circumstances applications of Corollary 3.3 strictly increase these functionals.

In this section, we address this issue both directly and indirectly. We motivate our results by considering the special case $\Lambda = \Lambda_\perp \uplus \Lambda_{e_1}$, where Λ_{e_1} is a finite multiset of multiples of the first canonical basis vector e_1. For $p > 1$ we show that multisets Λ of this type have a greater associated asymmetric L_p *volume product* than the canonical basis $\Lambda_\perp$ if Λ_{e_1} is not empty. Moreover, we prove that Λ has the same associated asymmetric L_p *volume ratio* as the canonical basis if and only if Λ_{e_1} contains at most one negative multiple of e_1.

We first look at multisets that contain vectors that point into the same direction (that is vectors that are positive multiples of each other) and show that these are not among the extremizers of both the associated asymmetric L_p volume product and the associated asymmetric L_p volume ratio.

Lemma 4.1. *Suppose $p > 1$ and Λ is a finite and spanning multiset. Replace all vectors in Λ that point in the same direction by their sum, and denote this new multiset by $\overline{\Lambda}$. Then $Z_1^+\Lambda = Z_1^+\overline{\Lambda}$. Moreover, the inequalities*

$$\frac{V(Z_p^+\Lambda)}{V(Z_1^+\Lambda)} \le \frac{V(Z_p^+\overline{\Lambda})}{V(Z_1^+\overline{\Lambda})} \quad \text{and} \quad V(Z_p^{+,*}\Lambda)V(Z_1^+\Lambda) \ge V(Z_p^{+,*}\overline{\Lambda})V(Z_1^+\overline{\Lambda}) \tag{19}$$

hold with equality if and only if $\overline{\Lambda} = \Lambda$.

Proof. The multiset Λ is equal to, say, $\{v_1, \dots, v_m\}$. By construction of $\overline{\Lambda} = \{w_1, \dots, w_k\}$, there is a partition $(I_j)_{j=1}^k$ of $\{1, \dots, m\}$ such that

$$w_j = \sum_{i \in I_j} v_i$$

for $1 \le j \le k$, and the vectors in every $\{v_i : i \in I_j\}$ point in the same direction. Thus, $Z_1^+\Lambda = Z_1^+\overline{\Lambda}$ follows from the computation ($u \in \mathbb{R}^n$)

$$h(Z_1^+\overline{\Lambda}, u) = \sum_{j=1}^k \left[\sum_{i \in I_j} \langle u, v_i\rangle\right]_+ = \sum_{j=1}^k \sum_{i \in I_j} \langle u, v_i\rangle_+ = h(Z_1^+\Lambda, u),$$

where, again, $[\cdot]_+ = \max\{\cdot, 0\}$. Similarly,

$$h(Z_p^+\overline{\Lambda}, u)^p = \sum_{j=1}^k \left[\sum_{i \in I_j} \langle u, v_i\rangle\right]_+^p = \sum_{j=1}^k \left(\sum_{i \in I_j} \langle u, v_i\rangle_+\right)^p.$$

Due to the fact that the L_1 sum dominates the L_p sum (see inequality (4)), we now have

$$h(Z_p^+\overline{\Lambda}, u)^p \geq \sum_{j=1}^{k} \sum_{i\in I_j} \langle u, v_i\rangle_+^p = h(Z_p^+\Lambda, u)^p$$

with equality if and only if all sums over $i \in I_j$ contain at most one positive summand. In particular, if $\Lambda \neq \overline{\Lambda}$ because, say, v_1 and v_2 point in the same direction, then $h(Z_p^+\overline{\Lambda}, v_1) > h(Z_p^+\Lambda, v_1)$. Hence

$$Z_p^+\Lambda \subseteq Z_p^+\overline{\Lambda}$$

with equality if and only if $\overline{\Lambda} = \Lambda$. The first inequality of (19) now follows immediately. Moreover, since polarity (with respect to the origin) reverses set inclusion,

$$Z_p^{+,*}\Lambda = (Z_p^+\Lambda - s(Z_p^+\Lambda))^o \supseteq (Z_p^+\overline{\Lambda} - s(Z_p^+\Lambda))^o$$

with equality if and only if $\Lambda = \overline{\Lambda}$. Thus the estimate

$$V(Z_p^{+,*}\Lambda) \geq V((Z_p^+\overline{\Lambda} - s(Z_p^+\Lambda))^o) \geq V(Z_p^{+,*}\overline{\Lambda})$$

proves the second inequality of (19) together with the asserted equality conditions. $\square$

In particular, the previous lemma implies that a multiset of the form $\Lambda_\perp \uplus \Lambda_{e_1}$ maximizes the associated asymmetric L_p volume ratio only if $\Lambda_{e_1} = \{-\mu e_1\}$, where μ is a nonnegative number. In fact, the asymmetric L_p volume ratio attains the same value for all $\mu \geq 0$, because, more generally, all obtuse sets have the same associated asymmetric L_p volume ratio. Before we formulate and establish this proposition, a few remarks on obtuse sets are in order.

A first observation is that a set that can be written as a disjoint union $\Lambda_\perp \cup \{v_1, \dots, v_\ell\}$ is obtuse if and only if there are disjoint nonempty subsets $I_1, \dots, I_m$ of $\{1, \dots, n\}$ and negative numbers λ_i such that, for every $j \in \{1, \dots, \ell\}$,

$$v_j = \sum_{i\in I_j} \lambda_i e_i. \tag{20}$$

We show in the next lemma that every spanning obtuse set has a linear image of this type, and establish a property of asymmetric L_p zonotopes associated with such sets.

Lemma 4.2. *Suppose $p \geq 1$. If Λ is a spanning obtuse set, then the following three statements hold:*

(i) If $B \subseteq \Lambda$ is a basis, then the vectors in $\Lambda \setminus B$ are pairwise orthogonal and have nonpositive components with respect to the basis B.
(ii) Every $\mathrm{GL}(n)$ image of Λ that contains the canonical basis $\Lambda_\perp$ is obtuse.
(iii) Suppose in addition that Λ contains the canonical basis. For every $y \in Z_p^+\Lambda$ there is a $\phi \in \mathrm{GL}(n)$ such that ϕy has nonnegative coordinates with respect to the canonical basis and $\Lambda_\perp \subseteq \phi\Lambda$.

Proof. We start with the proof of assertion (i). For this purpose, let ϕ_B denote the $n \times n$ matrix whose columns are the vectors $b_1, \dots, b_n$ in B. If v_1 is a vector in $\Lambda \setminus B$, then there are real numbers λ_i such that

$$v_1 = \sum_{i=1}^{n} \lambda_i b_i.$$

In other words, $v_1 = \phi_B \underline{\lambda}$, where $\underline{\lambda}$ is the vector $(\lambda_1, \dots, \lambda_n)^T$. Since $\langle b_j, v_1 \rangle \leq 0$ for every $j \in \{1, \dots, n\}$ by assumption, we have that

$$(\phi_B^T \phi_B)\underline{\lambda} \text{ has nonpositive components.} \tag{21}$$

The matrix $\phi_B^T \phi_B$ is positive definite with nonpositive off-diagonal entries because Λ is obtuse. Hence (see e.g. [**52**, pp. 42–45]), observation (21) implies that every λ_i is nonpositive. In other words, v_1 has nonpositive components with respect to the basis B. Now let $v_2 = \phi_B \underline{\mu}$ denote a second vector in $\Lambda \setminus B$, where again $\underline{\mu} = (\mu_1, \dots, \mu_n)^T$. Then $(\phi_B^T \phi_B)\underline{\mu}$ has nonpositive components. By assumption,

$$0 \geq \langle v_1, v_2 \rangle = (\phi_B \underline{\lambda})^T (\phi_B \underline{\mu}) = \underline{\lambda}^T (\phi_B^T \phi_B) \underline{\mu} \geq 0;$$

thus v_1 and v_2 are orthogonal.

To prove assertion (ii), let ψ denote a linear map from $\mathrm{GL}(n)$ such that $B := \psi^{-1}\Lambda_\perp \subseteq \Lambda$. We will show that every pair of distinct vectors u, v from Λ satisfies

$$\langle \psi u, \psi v \rangle \leq 0. \tag{22}$$

If we assume that $u, v \in B$, then (22) follows immediately because ψu and ψv are distinct canonical basis vectors. To show that (22) also holds if $u \in \Lambda \setminus B$, we apply the first part of the lemma: a vector u from $\Lambda \setminus B$ has nonpositive coordinates, say λ_i, with respect to the basis B. Consequently, if j is such that $\langle \psi b_j, \psi v \rangle = 1$, then

$$\langle \psi u, \psi v \rangle = \Big\langle \sum_{i=1}^{m} \lambda_i \psi b_i, \psi v \Big\rangle = \lambda_j \leq 0.$$

Only the proof of (22) for $u, v \in \Lambda \setminus B$ remains. Let λ_i and μ_i denote the coordinates with respect to the basis B of u and v, respectively. These coordinates induce a partition $\{B_{u,v}, B_u, B_v, B_0\}$ of B, where

$$\begin{aligned} B_{u,v} &= \{b_i \in B : \ \lambda_i \neq 0 \text{ and } \mu_i \neq 0\}; \\ B_u &= \{b_i \in B : \ \lambda_i \neq 0 \text{ and } \mu_i = 0\}; \\ B_v &= \{b_i \in B : \ \lambda_i = 0 \text{ and } \mu_i \neq 0\}; \\ B_0 &= \{b_i \in B : \ \lambda_i = 0 \text{ and } \mu_i = 0\}. \end{aligned} \tag{23}$$

For every $b_j \in B_{u,v}$ the set $\{v\} \cup B \setminus \{b_j\}$ is a basis. By the first part of the lemma, b_j and u are orthogonal. In particular,

$$u \perp \operatorname{span} B_{u,v}, \tag{24}$$

where $\operatorname{span} B_{u,v}$ denotes the linear hull of $B_{u,v}$. Similarly, the fact that $\{u, v\} \cup B \setminus \{b_j, b_k\}$ is a basis when $b_j \in B_{u,v}$ and $b_k \in B_u$ implies that

$$\operatorname{span} B_{u,v} \perp \operatorname{span} B_u. \tag{25}$$

By (23) and (25), the vector u is the sum of two orthogonal vectors: $u = u_1 + u_2$, where $u_1 \in \operatorname{span} B_{u,v}$ and $u_2 \in \operatorname{span} B_u$. Moreover, by (24), the vector u is orthogonal to u_1. This is only possible if $u_1 = o$; thus $B_{u,v} = \emptyset$. Consequently, the computation

$$\langle \psi u, \psi v \rangle = \sum_{i,j=1}^{n} \lambda_i \mu_j \langle \psi b_i, \psi b_j \rangle = \sum_{i=1}^{n} \lambda_i \mu_i = 0$$

verifies (22) when $u, v \in \Lambda \setminus B$.

For the proof of assertion (iii) write $y = \sum_{i=1}^n y_i e_i$ and assume that, say, $y_k < 0$. By assumption Λ is equal to, say, $\Lambda_\perp \cup \{v_1, \dots, v_\ell\}$, and we make use of (20) to write every v_j as a sum of canonical basis vectors with negative coefficients. Since $h(Z_p^+\Lambda, -e_k) > 0$, there exists a j such that $k \in I_j$. Let $m(j)$ denote an index in I_j where $\max\{\frac{y_i}{\lambda_i} : i \in I_j\}$ is attained. Clearly, $y_{m(j)} < 0$. Now define $\phi \in \mathrm{GL}(n)$ through its action on the basis $\{e_1, \dots, e_{m(j)-1}, v_j, e_{m(j)+1}, \dots, e_n\}$:

$$\phi v_j = e_{m(j)} \qquad \text{and} \qquad \phi e_i = e_i \text{ for } i \neq m(j).$$

Then $\phi\Lambda$ clearly contains the canonical basis. Moreover, substituting the identity

$$e_{m(j)} = \frac{1}{\lambda_{m(j)}} v_j + \sum_{i \in I_j \setminus \{m(j)\}} \frac{-\lambda_i}{\lambda_{m(j)}} e_i$$

into the right-hand side of

$$\phi y = \sum_{i=1}^{n} y_i \phi e_i = y_{m(j)} \phi e_{m(j)} + \sum_{i \notin I_j} y_i e_i + \sum_{i \in I_j \setminus \{m(j)\}} y_i e_i,$$

we obtain

$$\phi y = \frac{y_{m(j)}}{\lambda_{m(j)}} e_{m(j)} + \sum_{i \notin I_j} y_i e_i + \sum_{i \in I_j \setminus \{m(j)\}} \left(y_i - y_{m(j)} \frac{\lambda_i}{\lambda_{m(j)}} \right) e_i.$$

In particular, by definition of $m(j)$, the vector ϕy has nonnegative coordinates with respect to all canonical basis vectors e_i such that $i \in I_j$. The remaining negative coordinates can be dealt with by iterating this argument. □

One of the immediate implications of the above lemma is that a spanning obtuse set contains at least n and not more than $2n$ vectors. Moreover, we are now in a position to prove the "if" part of the equality conditions of Theorem 2.

Proposition 4.3. *Suppose $p \geq 1$ and Λ is a spanning obtuse set. Then*

$$\frac{V(Z_p^+\Lambda)}{V(Z_1^+\Lambda)} = \frac{V(Z_p^+\Lambda_\perp)}{V(Z_1^+\Lambda_\perp)}.$$

Proof. By the $\mathrm{GL}(n)$ invariance of the asymmetric L_p volume ratio and Lemma 4.2, we may assume that $\Lambda = \{v_1, \dots, v_m\}$ contains the canonical basis. In a first step, we establish the dissection formula

$$Z_p^+\Lambda = \bigcup_{1 \leq i_1 < \cdots < i_n \leq m} Z_p^+\{v_{i_1}, \dots, v_{i_n}\}. \tag{26}$$

Note that only the fact that $Z_p^+\Lambda$ is a subset of the right-hand side of (26) requires a proof. To this end, let $y \in Z_p^+\Lambda$. It is sufficient to show that there is a $\phi \in \mathrm{GL}(n)$ such that $y \in Z_p^+\phi^{-1}\Lambda_\perp$ and $\phi^{-1}\Lambda_\perp \subseteq \Lambda$ to establish (26).

It follows from Lemma 4.2 that there is a $\phi \in \mathrm{GL}(n)$ such that ϕy has nonnegative coordinates with respect to the canonical basis and $\Lambda_\perp \subseteq \phi\Lambda$. Moreover, the set $\phi\Lambda$ is obtuse. In particular, say,

$$\phi\Lambda = \Lambda_\perp \cup \{w_1, \dots, w_\ell\}$$

and there are disjoint subsets $I_1, \dots, I_\ell$ of $\{1, \dots, n\}$ and negative numbers λ_i such that, for $1 \le j \le \ell$,

$$w_j = \sum_{i \in I_j} \lambda_i e_i.$$

The support function of the associated asymmetric L_p zonotope is ($u \in \mathbb{R}^n$)

$$h(Z_p^+\phi\Lambda, u)^p = \sum_{i=1}^n \langle e_i, u\rangle_+^p + \sum_{j=1}^\ell \Big\langle \sum_{i\in I_j} \lambda_i e_i, u\Big\rangle_+^p.$$

Applying inequality (2) and Hölder's inequality to the right-hand side of this identity, we obtain

$$\begin{aligned} h(Z_p^+\phi\Lambda, u)^p &\le \sum_{i=1}^n \langle e_i, u\rangle_+^p + \sum_{j=1}^\ell \left(\sum_{i\in I_j} \langle \lambda_i e_i, u\rangle_+\right)^p \\ &\le \sum_{i=1}^n \langle e_i, u\rangle_+^p + \sum_{j=1}^\ell \left(\sum_{i\in I_j} \langle \lambda_i e_i, u\rangle_+^p\right)\left(\sum_{i\in I_j} 1\right)^{p-1}. \end{aligned}$$

Therefore,

$$h(Z_p^+\phi\Lambda, u)^p \le \sum_{i=1}^n \langle e_i, u\rangle_+^p + \sum_{j=1}^\ell \sum_{i\in I_j} \langle |I_j|^{\frac{p-1}{p}} \lambda_i e_i, u\rangle_+^p,$$

where $|I_j|$ denotes the cardinality of I_j. Hence there exist nonnegative numbers $\mu_1, \dots, \mu_n$ such that

$$Z_p^+\phi\Lambda \subseteq Z_p^+\{e_1, \dots, e_n, -\mu_1 e_1, \dots, -\mu_n e_n\}.$$

In particular,

$$\phi y \in Z_p^+\{e_1, \dots, e_n, -\mu_1 e_1, \dots, -\mu_n e_n\}. \tag{27}$$

It remains to show that (27) holds when $\mu_1 = \dots = \mu_n = 0$. First, we establish (27) for $\mu_1 = 0$. To this end, let $s \ge 0$ and set

$$\Lambda(s) := \{e_1, \dots, e_n, -se_1, -\mu_2 e_2, \dots, -\mu_n e_n\}$$

For $x \in e_1^\perp \cap e_2^\perp$, by (11), the uppergraph function $\overline{g}_{e_2}(Z_p^+\Lambda(s), x)$ is equal to the infimum of

$$\sqrt[p]{\langle e_1, w\rangle_+^p + \langle -se_1, w\rangle_+^p + \sum_{i=2}^n \langle e_i, e_2 + w\rangle_+^p + \langle -\mu_i e_i, e_2 + w\rangle_+^p} - \langle x, w\rangle$$

over all $w \in e_2^\perp$. The scalar product $\langle x, w\rangle$ does not depend on the first component of w, hence it suffices to compute the infimum over all $w \in e_1^\perp \cap e_2^\perp$. It is now obvious that the uppergraph function $\overline{g}_{e_2}(Z_p^+\Lambda(s), x)$

is independent of s for every $x \in e_1^\perp \cap e_2^\perp$. The same argument applied to the lowergraph function leads to the same conclusion, so we infer that

$$Z_p^+\Lambda(s) \cap e_1^\perp \text{ is independent of } s. \tag{28}$$

Moreover, the support function of $Z_p^+\Lambda(s)$ evaluated at vectors $w \in e_1^\perp$ is a constant function of s. Equivalently,

$$Z_p^+\Lambda(s)|_{e_1^\perp} \text{ is independent of } s. \tag{29}$$

At $s = 1$, the convex body $Z_p^+\Lambda(1)$ is symmetric with respect to reflections in the hyperplane $e_1^\perp$. Together with observations (27), (28), and (29), this implies

$$\phi y|_{e_1^\perp} \in Z_p^+\Lambda(\mu_1)|_{e_1^\perp} = Z_p^+\Lambda(1)|_{e_1^\perp} = Z_p^+\Lambda(1) \cap e_1^\perp = Z_p^+\Lambda(s) \cap e_1^\perp$$

for all s. In particular, $\underline{g}_{e_1}(Z_p^+\Lambda(s), \phi y|_{e_1^\perp})$ is nonpositive for all s. Moreover, by (11), the uppergraph function $\overline{g}_{e_1}(Z_p^+\Lambda(s), \phi y|_{e_1^\perp})$ is independent of s, because, for $w \in e_1^\perp$, $h(Z_p^+\Lambda(s), e_1 + w)$ is. Hence

$$\begin{aligned} \phi y &\in \{\phi y|_{e_1^\perp} + re_1 : \ 0 \le r \le \overline{g}_{e_1}(Z_p^+\Lambda(\mu_1), \phi y|_{e_1^\perp})\} \\ &= \{\phi y|_{e_1^\perp} + re_1 : \ 0 \le r \le \overline{g}_{e_1}(Z_p^+\Lambda(0), \phi y|_{e_1^\perp})\} \subseteq Z_p^+\Lambda(0). \end{aligned}$$

Repeating this argument for $\mu_2, \dots, \mu_n$, we have that ϕy is contained in $Z_p^+\Lambda_\perp$, which completes the proof of (26).

The intersection of any two distinct parts in the dissection (26) of $Z_p^+\Lambda$ has volume zero. To see this, let $\Lambda^1, \Lambda^2 \subseteq \Lambda$ each contain n vectors and assume that $\Lambda^1 \neq \Lambda^2$. If one of these sets is not spanning, then the intersection $Z_p^+\Lambda^1 \cap Z_p^+\Lambda^2$ is a set of volume zero contained in a hyperplane. Otherwise, without loss of generality, $\Lambda^1 = \Lambda_\perp$ and Λ^2 does not contain e_1. In particular, $h(Z_p^+\Lambda^1, -e_1) = 0$. Moreover, due to the assumption that Λ is obtuse, $h(Z_p^+\Lambda^2, e_1) = 0$. Combining these two observations we obtain that the intersection $Z_p^+\Lambda^1 \cap Z_p^+\Lambda^2$ is a set of volume zero contained in the hyperplane $e_1^\perp$.

Consequently, by (26), we have

$$V(Z_p^+\Lambda) = \sum_{1 \le i_1 < \cdots < i_n \le m} V(Z_p^+\{v_{i_1}, \dots, v_{i_n}\}).$$

The $\mathrm{GL}(n)$ equivariance of Z_p^+ together with (26) for $p = 1$ now implies that the right-hand side of this equation is equal to

$$\sum_{1 \le i_1 < \cdots < i_n \le m} \frac{V(Z_p^+\Lambda_\perp)}{V(Z_1^+\Lambda_\perp)} V(Z_1^+\{v_{i_1}, \dots, v_{i_n}\}) = \frac{V(Z_p^+\Lambda_\perp)}{V(Z_1^+\Lambda_\perp)} V(Z_1^+\Lambda).$$

Hence we have proved that

$$V(Z_p^+\Lambda) = \frac{V(Z_p^+\Lambda_\perp)}{V(Z_1^+\Lambda_\perp)} V(Z_1^+\Lambda). \qquad \square$$

The situation is different in the case of the asymmetric L_p volume product. For $p > 1$ the next lemma asserts that sets of the form $\Lambda_\perp \cup \{-\mu e_1\}$, where μ is a positive number, are not among the extremizers of the functional.

Lemma 4.4. *Suppose $p \geq 1$ and $\Lambda = \Lambda_\perp \cup \{-\mu e_1\}$, where $\mu > 0$. Then*

$$V(Z_p^{+,*}\Lambda)V(Z_1^+\Lambda) \geq V(Z_p^{+,*}\Lambda_\perp)V(Z_1^+\Lambda_\perp) \tag{30}$$

with equality if and only if $p = 1$.

Proof. Let Λ_t, $t \in [-a^{-1}, 1]$, denote the orthogonalization of Λ with respect to e_1 defined by (9). Corollary 3.3 asserts that the inverse asymmetric L_p volume product associated with Λ_t is a convex function of t, so we have

$$\frac{1}{V(Z_p^{+,*}\Lambda)V(Z_1^+\Lambda)} \leq \max_{t\in\{-a^{-1},1\}} \frac{1}{V(Z_p^{+,*}\Lambda_t)V(Z_1^+\Lambda_t)}.$$

By the $\mathrm{GL}(n)$ invariance of the asymmetric L_p volume product and the definition of Λ_t, the right-hand side of this inequality is just

$$\frac{1}{V(Z_p^{+,*}\Lambda_\perp)V(Z_1^+\Lambda_\perp)};$$

thus only the equality conditions of inequality (30) remain to be proven.

That equality holds for $p = 1$ is an immediate consequence of the fact that all parallelepipeds have the same volume product, so let $p > 1$ and assume that equality holds. Note that then $V(Z_p^{+,*}\Lambda_t)$ is a constant function of t. By definition (9),

$$\Lambda_t = \{(1 + ta)e_1, +\mu(t - 1)e_1, e_2, \ldots, e_n\}.$$

In particular, Λ_t is a spanning obtuse set for every $t \in [-a^{-1}, 1]$. Hence Proposition 4.3 implies that $V(Z_p^+\Lambda_t)$ is independent of t and Theorem 2.4 guarantees the existence of a real number α and a vector $z \in \mathbb{R}^{1\times(n-1)}$ such that

$$Z_p^+\Lambda_t = t\alpha e_1 + Z_p^+\phi_t\Lambda, \tag{31}$$

where

$$\phi_t = \begin{pmatrix} 1 & tz \\ o & I_{n-1} \end{pmatrix}.$$

Equivalently, for all $u \in \mathbb{R}^n$,

$$h(Z_p^+\Lambda_t, u) = t\alpha\langle e_1, u\rangle + h(Z_p^+\phi_t\Lambda, u). \tag{32}$$

The constant α can be computed: for all $t \in [-a^{-1}, 1]$, the zonotope $Z_p^+\Lambda_t$ is symmetric with respect to permutations of all coordinates except the first. Due to identity (31), this implies that z has $n - 1$ equal components, say ζ. If $\zeta \leq 0$, then by substituting $u = e_1$ and $t = 1$ in equation (32), we obtain

$$1 + a = \alpha + 1;$$

thus $a = \alpha$. If $\zeta > 0$, then the same argument with $t = -a^{-1}$ yields the same conclusion.

We will now determine $z = (\zeta, \ldots, \zeta)$. Simple computations based on the representations of the graph functions (11) and (12) show that

$$\overline{g}_{e_1}(Z_p^+\Lambda_1, e_2) = \underline{g}_{e_1}(Z_p^+\Lambda_1, e_2) = 0,$$

or, equivalently,

$$\{e_2\} = Z_p^+\Lambda_1 \cap (\{e_2\} + \mathrm{span}\{e_1\}). \tag{33}$$

Moreover, by (4), the convex body $Z_p^+\phi_1\Lambda$ contains all the points in $\phi_1\Lambda$. In particular, it contains $\zeta e_1 + e_2$. Combining this observation with (31) for $t = 1$ and (33), we obtain that

$$e_2 = (a + \zeta)e_1 + e_2;$$

thus $\zeta = -a$.

Now putting $u = e_1 + e_2$ and $t = -a^{-1}$ in equation (32) leads to the desired contradiction

$$1 = -1 + \sqrt[p]{1^p + 2^p + \sum_{i=3}^{n} 1^p}. \qquad \square$$

We conclude this section with a lemma due to Reisner [**38**] on the equality conditions of Theorem 1 when $p = 1$. Here, we give a new proof using shadow systems.

Lemma 4.5. *Suppose Λ is a finite and spanning multiset. If $Z_1^+\Lambda$ is not a parallelepiped, then Λ is not a minimizer of $V(Z_1^{+,*}\Lambda)V(Z_1^+\Lambda)$.*

Proof. Since the convex bodies $Z_1^+\Lambda$ and $Z_1\Lambda$ are homothetic by (5), we may prove the lemma for Z_1 instead of Z_1^+.

Up to a linear transformation Λ is equal to, say, $\Lambda_\perp \uplus \{v_1, \ldots, v_\ell\}$ where $\ell \geq 1$. By the invariance of $Z_1\Lambda$ under reflections of vectors in the origin, and Lemma 4.1, we may assume that Λ contains no pair of parallel vectors. Without loss of generality, $\Lambda \setminus \{e_1\}$ is still spanning. Let Λ_t, $t \in [-a^{-1}, 1]$, denote the orthogonalization of Λ with respect to e_1. Then, by (5) and Corollary 3.3,

$$\frac{1}{V(Z_1^*\Lambda)V(Z_1\Lambda)} \leq \max_{t\in\{-a^{-1},1\}} \frac{1}{V(Z_1^*\Lambda_t)V(Z_1\Lambda_t)} \tag{34}$$

with equality if and only if $V(Z_1^*\Lambda_t)$ is independent of t.

The proof of the lemma is complete if we can show that (34) is a strict inequality. Assume that equality holds. By Theorem 2.4 for $t = 1$, there is a constant α and a matrix $\phi \in \mathrm{GL}(n)$,

$$\phi = \begin{pmatrix} 1 & z \\ o & I_{n-1} \end{pmatrix} \text{ for some } z \in \mathbb{R}^{1\times(n-1)},$$

such that

$$\begin{aligned} Z_1(\{(1+a)e_1, e_2, \ldots e_n,\} \uplus \{v_1|_{e_1^\perp}, \ldots, v_\ell|_{e_1^\perp}\}) \\ = \alpha e_1 + Z_1(\{e_1, \phi e_2, \ldots, \phi e_n\} \uplus \{\phi v_1, \ldots, \phi v_\ell\}). \end{aligned} \tag{35}$$

Moreover, $\alpha = 0$ because L_1 zonotopes are origin-symmetric. Note that the two zonotopes in equation (35) are generated by multisets that contain only one vector parallel to e_1 each. Because these two vectors, that is $(1 + a)e_1$ and e_1, are not equal, and the generating measures of zonoids are unique (see [**42**, Theorem 3.5.3]), we have arrived at a contradiction. $\square$

5. Proofs of the main theorems

We are now in a position to establish the main theorems. In fact, we prove that these do not only hold for sets, but also for multisets. The core arguments in both proofs are similar, so we collect them in a lemma.

Lemma 5.1. *Suppose Φ is a real-valued $\mathrm{GL}(n)$ invariant function on finite and spanning multisets. Moreover, assume that $\Phi(\Lambda_t)$ is a convex function of t whenever Λ_t, $t \in [-a^{-1}, 1]$, is an orthogonalization of a multiset Λ defined by (9). Then, for every finite and spanning multiset Λ, there exists a multiset Λ_{e_1} of multiples of e_1 such that*

$$\Phi(\Lambda) \leq \Phi(\Lambda_\perp \uplus \Lambda_{e_1}). \tag{36}$$

Moreover,

- *(i) if Λ is not a $\mathrm{GL}(n)$ image of $\Lambda_\perp$ and equality holds in (36), then Λ_{e_1} is not the empty set.*
- *(ii) if Λ is not a $\mathrm{GL}(n)$ image of an obtuse set and equality holds in (36), then Λ_{e_1} contains a positive multiple of e_1.*

Proof. By the $\mathrm{GL}(n)$ invariance of Φ, we may assume that the multiset Λ contains the canonical basis. We will transform this Λ into a multiset of the form $\Lambda_\perp \uplus \Lambda_{e_1}$ in a finite number of steps and start with the construction of one step of this reduction process.

If $1 \leq i \leq n$ is such that $\Lambda \setminus \{e_i\}$ is spanning, then let Λ_t, $t \in [-a^{-1}, 1]$, denote the orthogonalization of Λ with respect to e_i; see (9). By assumption, the map $t \mapsto \Phi(\Lambda_t)$ is convex and hence attains its global maximum at one of the endpoints of the interval. If the maximum is attained at $t = 1$, then we define

$$\Lambda[i] := \psi_1 \Lambda_1, \tag{37}$$

where $\psi_1 \in \mathrm{GL}(n)$ is the linear map that rescales the ith canonical basis vector such that $\Lambda[i]$ again contains the canonical basis $\Lambda_\perp$. If this function attains its unique global maximum at $t = -a^{-1}$, then we choose any $\psi_{-a^{-1}} \in \mathrm{GL}(n)$ that fixes all canonical basis vectors except e_i such that $\Lambda_\perp \subseteq \psi_{-a^{-1}} \Lambda_{-a^{-1}}$ and set

$$\Lambda[i] := \psi_{-a^{-1}} \Lambda_{-a^{-1}}.$$

Finally, we define $\Lambda[i] := \Lambda$ in situations where $\Lambda \setminus \{e_i\}$ is not spanning. Note that, in all cases, $\Lambda[i]$ contains the canonical basis and the value of Φ is not decreased by this transformation:

$$\Phi(\Lambda) \leq \Phi(\Lambda[i]). \tag{38}$$

Moreover, equality holds in inequality (38) only if $\Lambda[i] \setminus \{e_i\}$ is just the orthogonal projection of $\Lambda \setminus \{e_i\}$ onto $e_i^\perp$.

We will now apply iterations of this process to a given multiset Λ to prove the lemma. For this purpose, set $\Lambda^0 := \Lambda$ and define the multisets Λ^{i+1}, $i \geq 0$, inductively:

$$\Lambda^{i+1} = \Lambda^i[n] \cdots [2].$$

By construction, there is a finite index $i_\perp$ such that $\Lambda^{i_\perp}$ only contains the canonical basis and multiples of e_1. Also, repeated applications of inequality (38) yield

$$\Phi(\Lambda) = \Phi(\Lambda^0) \leq \cdots \leq \Phi(\Lambda^{i_\perp}), \tag{39}$$

which is just inequality (36).

If Λ is not a $\mathrm{GL}(n)$ image of $\Lambda_\perp$, then, by the $\mathrm{GL}(n)$ invariance of Φ, we may assume that $\Lambda_\perp \subseteq \Lambda$ and that $\Lambda \setminus \Lambda_\perp$ contains a vector with a nonzero first component. Under the assumption that equality holds in inequality (39), the multiset $\Lambda^{i_\perp}$ contains the projection of this vector onto e_1. Hence there is a nonzero multiple of e_1 in $\Lambda^{i_\perp} \setminus \Lambda_\perp$.

If Λ is not a $\mathrm{GL}(n)$ image of an obtuse set, then we may assume that $\Lambda_\perp \subseteq \Lambda$ and $\Lambda \setminus \Lambda_\perp$ contains a vector with a positive first component. As before, equality in inequality (39) implies that this positive component is preserved. □

We proceed with a proof of the following slight refinement of Theorem 1.

Theorem 5.2. *Suppose $p \geq 1$ and Λ is a finite and spanning multiset. Then*

$$V(Z_p^{+,*}\Lambda)V(Z_1^+\Lambda) \geq V(Z_p^{+,*}\Lambda_\perp)V(Z_1^+\Lambda_\perp)$$

with equality for $p > 1$ if and only if Λ is a $\mathrm{GL}(n)$ image of the canonical basis $\Lambda_\perp$. If $p = 1$, then equality holds if and only if $Z_1^+\Lambda$ is a parallelepiped.

Proof. First let $p > 1$. The asymmetric L_p volume product is $\mathrm{GL}(n)$ invariant, so there is nothing to show if Λ is a $\mathrm{GL}(n)$ image of the canonical basis. Otherwise, by Corollary 3.3, the inverse asymmetric L_p volume product

$$\mathcal{P}\Lambda := \frac{1}{V(Z_p^{+,*}\Lambda)V(Z_1^+\Lambda)}$$

satisfies the hypotheses of Lemma 5.1. This lemma asserts that there is a multiset Λ_{e_1} of multiples of e_1 such that

$$\mathcal{P}\Lambda \leq \mathcal{P}(\Lambda_\perp \uplus \Lambda_{e_1}) \tag{40}$$

with equality only if Λ_{e_1} is not empty. By Lemma 4.1,

$$\mathcal{P}(\Lambda_\perp \uplus \Lambda_{e_1}) \leq \mathcal{P}(\overline{\Lambda_\perp \uplus \Lambda_{e_1}}) \tag{41}$$

with equality if and only if $\Lambda_{e_1} = \{-\mu e_1\}$, where $\mu \geq 0$. If Λ_{e_1} is empty or contains only positive multiples of e_1, then $\overline{\Lambda_\perp \uplus \Lambda_{e_1}}$ is a $\mathrm{GL}(n)$ image of $\Lambda_\perp$, and inequalities (40) and (41) yield the desired inequality together with its equality conditions.

Otherwise, by Lemma 4.4,

$$\mathcal{P}(\overline{\Lambda_\perp \uplus \Lambda_{e_1}}) < \mathcal{P}\Lambda_\perp. \tag{42}$$

Hence, combining inequalities (40), (41), and (42), we obtain the asserted inequality,

$$\mathcal{P}\Lambda < \mathcal{P}\Lambda_\perp. \tag{43}$$

Now let $p = 1$. Since all parallelepipeds have the same volume product, and inequality (40) is also valid for $p = 1$, we obtain that parallelepipeds are minimizers of the asymmetric L_1 volume product. That they are the only possible minimizers has already been shown in Lemma 4.5. □

The proof of Theorem 2 for multisets is very similar. Since the application of Lemma 4.4 is now replaced by an application of Proposition 4.3, we obtain different equality conditions.

Theorem 5.3. *Suppose $p > 1$ and Λ is a finite and spanning multiset. Then*
$$\frac{V(Z_p^+\Lambda)}{V(Z_1^+\Lambda)} \leq \frac{V(Z_p^+\Lambda_\perp)}{V(Z_1^+\Lambda_\perp)}$$
with equality if and only if Λ is a $\mathrm{GL}(n)$ image of an obtuse set.

Proof. By Proposition 4.3, we may assume that Λ is not a $\mathrm{GL}(n)$ image of an obtuse set. For brevity of notation, let $\mathcal{R}$ denote the asymmetric L_p volume ratio;
$$\mathcal{R}\Lambda = \frac{V(Z_p^+\Lambda)}{V(Z_1^+\Lambda)}.$$
By Corollary 3.3, $\mathcal{R}$ satisfies the hypotheses of Lemma 5.1. Thus
$$\mathcal{R}\Lambda \leq \mathcal{R}(\Lambda_\perp \uplus \Lambda_{e_1}), \tag{44}$$
where Λ_{e_1} only contains multiples of e_1. Moreover, there is at least one vector with a positive first component in Λ_{e_1} unless inequality (44) is a strict inequality. Now, by Lemma 4.1,
$$\mathcal{R}(\Lambda_\perp \uplus \Lambda_{e_1}) \leq \mathcal{R}(\overline{\Lambda_\perp \uplus \Lambda_{e_1}}). \tag{45}$$
Note that the set $\overline{\Lambda_\perp \uplus \Lambda_{e_1}}$ is obtuse. So, by Proposition 4.3, combining inequalities (44) and (45) yields
$$\mathcal{R}\Lambda \leq \mathcal{R}(\Lambda_\perp \uplus \Lambda_{e_1}) \leq \mathcal{R}(\overline{\Lambda_\perp \uplus \Lambda_{e_1}}) = \mathcal{R}\Lambda_\perp.$$
We have in fact proved that $\mathcal{R}\Lambda < \mathcal{R}\Lambda_\perp$, as desired, because Lemma 4.1 also implies that equality cannot hold in inequalities (44) and (45) simultaneously. ☐

6. Symmetric L_p zonotopes

In this final section we show how our asymmetric extension of the Campi–Gronchi approach also yields the reverse affine isoperimetric inequalities for the *symmetric L_p zonotopes* obtained in [**7**]—along with new equality conditions. These zonotopes have been introduced by Schneider and Weil in [**43**].

The symmetric L_p zonotope $Z_p\Lambda$ associated with a finite and spanning multiset $\Lambda = \{v_1, \ldots, v_m\}$ is the convex body with support function $(u \in \mathbb{R}^n)$
$$h(Z_p\Lambda, u) = \sqrt[p]{\sum_{i=1}^m |\langle v_i, u\rangle|^p}.$$

Again, we denote by $Z_p^*\Lambda$ the polar body of $Z_p\Lambda$ with respect to the Santaló point. Since every $Z_p\Lambda$ is origin-symmetric, the Santaló point of these zonotopes in fact always lies at the origin.

Moreover, the symmetric L_p zonotopes are closely related to the asymmetric L_p zonotopes: the observation
$$\sum_{i=1}^m |\langle v_i, u\rangle|^p = \sum_{i=1}^m \left(\langle v_i, u\rangle_+^p + \langle -v_i, u\rangle_+^p\right)$$

implies that $Z_p\Lambda = Z_p^+(\Lambda \uplus -\Lambda)$. In particular, the operator Z_p is $\mathrm{GL}(n)$ equivariant and the symmetric L_p volume product $V(Z_p^*\Lambda)V(Z_1\Lambda)$ is $\mathrm{GL}(n)$ invariant. Reisner (for $p = 1$; see [**38**]) and Campi and Gronchi (for $p \geq 1$; see [**7**]) have proved that this functional attains its minimum at the canonical basis. Moreover, Lutwak, Yang, and Zhang have established inequalities closely related to the case $p = 2$ [**28**]. We extend Campi and Gronchi's theorem to multisets and settle the missing equality conditions for $p > 1$.

Theorem 6.1. *Suppose $p \geq 1$ and Λ is a finite and spanning multiset. Then*

$$V(Z_p^*\Lambda)V(Z_1\Lambda) \geq V(Z_p^*\Lambda_\perp)V(Z_1\Lambda_\perp)$$

with equality for $p > 1$ if and only if Λ is a $\mathrm{GL}(n)$ *image of the canonical basis $\Lambda_\perp$. If $p = 1$, then equality holds if and only if $Z_1\Lambda$ is a parallelepiped.*

Proof. By (5), the case $p = 1$ is just the assertion of Theorem 5.2, so let $p > 1$. We may assume that Λ is not a $\mathrm{GL}(n)$ image of the canonical basis. If Λ_t denotes an orthogonalization of Λ defined by (9), then $\Lambda_t \uplus -\Lambda_t$ is a shadow system of multisets. Hence Theorem 3.2 implies that $Z_p\Lambda_t = Z_p^+(\Lambda_t \uplus -\Lambda_t)$ is a shadow system of convex bodies. By Theorem 2.3 it follows that the map

$$\Lambda \mapsto \frac{1}{V(Z_p^*\Lambda)V(Z_1\Lambda)}$$

satisfies the assumptions of Lemma 5.1, which, in turn, guarantees the existence of a multiset Λ_{e_1} of multiples of e_1 such that

$$V(Z_p^*\Lambda)V(Z_1\Lambda) \geq V(Z_p^*(\Lambda_\perp \uplus \Lambda_{e_1}))V(Z_1(\Lambda_\perp \uplus \Lambda_{e_1})),$$

with equality only if Λ_{e_1} is not the empty set. The right-hand side of this inequality is equal to

$$V(Z_p^{+,*}(\Lambda_\perp \uplus -\Lambda_\perp \uplus \Lambda_{e_1} \uplus -\Lambda_{e_1}))V(Z_1^+(\Lambda_\perp \uplus -\Lambda_\perp \uplus \Lambda_{e_1} \uplus -\Lambda_{e_1})),$$

and, by Lemma 4.1, dominates $V(Z_p^*\Lambda_\perp)V(Z_1\Lambda_\perp)$ with equality if and only if Λ_{e_1} is the empty set. Thus we have proved

$$V(Z_p^*\Lambda)V(Z_1\Lambda) \geq V(Z_p^*\Lambda_\perp)V(Z_1\Lambda_\perp)$$

together with the desired equality conditions. □

Our last result is Campi and Gronchi's [**7**] symmetric L_p volume ratio inequality. Due to the fact that the upper bound of the asymmetric L_p volume ratio is also attained at the obtuse set $\Lambda_\perp \cup -\Lambda_\perp$, it is a direct consequence of Theorem 2. We provide the slightly more general version for multisets, which follows from Theorem 5.3. Note that the equality conditions of Theorem 6.1, and its dual, Theorem 6.2, are the same in the symmetric setting. Again, the case $p = 2$ is closely related to work of Lutwak, Yang, and Zhang [**28**].

Theorem 6.2. *Suppose $p > 1$ and Λ is a finite and spanning multiset. Then*

$$\frac{V(Z_p\Lambda)}{V(Z_1\Lambda)} \leq \frac{V(Z_p\Lambda_\perp)}{V(Z_1\Lambda_\perp)}$$

with equality for $p > 1$ if and only if Λ is a $\mathrm{GL}(n)$ *image of the canonical basis $\Lambda_\perp$.*

Proof. We may assume that Λ contains the canonical basis $\Lambda_\perp$. Since $Z_q\Lambda = Z_q^+(\Lambda \uplus -\Lambda)$ for all $q \geq 1$, we observe that Theorem 5.3 implies

$$\frac{V(Z_p\Lambda)}{V(Z_1\Lambda)} = \frac{V(Z_p^+(\Lambda \uplus -\Lambda))}{V(Z_1^+(\Lambda \uplus -\Lambda))} \leq \frac{V(Z_p^+\Lambda_\perp)}{V(Z_1^+\Lambda_\perp)}. \tag{46}$$

Equality in inequality (46) holds if and only if $\Lambda \uplus -\Lambda$ is a $\mathrm{GL}(n)$ image of an obtuse set, that is, if and only if $\Lambda = \Lambda_\perp$. All obtuse sets have the same associated asymmetric L_p volume ratio, hence the right-hand side of inequality (46) is equal to

$$\frac{V(Z_p^+(\Lambda_\perp \uplus -\Lambda_\perp))}{V(Z_1^+(\Lambda_\perp \uplus -\Lambda_\perp))} = \frac{V(Z_p\Lambda_\perp)}{V(Z_1\Lambda_\perp)}.$$

This observation, together with inequality (46), completes the proof. □

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