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Stability and slicing inequalities for intersection bodies

Alexander Koldobsky · Dan Ma

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Abstract We prove a generalization of the hyperplane inequality for intersection bodies, where volume is replaced by an arbitrary measure μ with even continuous density and sections are of arbitrary dimension $n-k,\ 1 \le k < n$. If K is a generalized k-intersection body, then

$$\mu(K) \ \le \ \frac{n}{n-k} c_{n,k} \max_H \mu(K \cap H) \mathrm{Vol}_n(K)^{k/n}.$$

Here $c_{n,k} = |B_2^n|^{(n-k)/n}/|B_2^{n-k}| < 1,\ |B_2^n|$ is the volume of the unit Euclidean ball, and maximum is taken over all $(n-k)$-dimensional subspaces of $\mathbb{R}^n$. The constant is optimal, and for each intersection body the inequality holds for every k. We also prove a stronger "difference" inequality. The proof is based on stability in the lower dimensional Busemann–Petty problem for arbitrary measures in the following sense. Let $\varepsilon > 0,\ 1 \le k < n$. Suppose that K and L are origin-symmetric star bodies in $\mathbb{R}^n$, and K is a generalized k-intersection body. If for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$

$$\mu(K \cap H) \le \mu(L \cap H) + \varepsilon,$$

then

$$\mu(K) \le \mu(L) + \frac{n}{n-k} c_{n,k} \mathrm{Vol}_n(K)^{k/n} \varepsilon.$$

Keywords Convex bodies · Volume · Sections

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A. Koldobsky (✉) · D. Ma
Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
e-mail: koldobskiya@missouri.edu

D. Ma
e-mail: madan516@gmail.com

1 Introduction

The Busemann–Petty problem, posed in 1956 in [7], asks the following question. Suppose that K and L are origin-symmetric convex bodies in $\mathbb{R}^n$ so that

$$\mathrm{Vol}_{n-1}(K \cap \xi^{\perp}) \le \mathrm{Vol}_{n-1}(L \cap \xi^{\perp}), \quad \forall \xi \in S^{n-1},$$

where $\xi^{\perp}$ is the central hyperplane perpendicular to ξ. Does it follow that

$$\mathrm{Vol}_n(K) \le \mathrm{Vol}_n(L)?$$

The answer is affirmative if $n \le 4$ and negative if $n \ge 5$. The solution was completed at the end of the 90's as the result of a sequence of papers [1,5,10,11,13,14,18,19,26,27,31, 36–38]; see [20, p. 3] or [12, p. 343] for details.

It is natural to ask what happens if hyperplane sections are replaced by sections of lower dimensions. Suppose that for every $(n-k)$-dimensional subspace $H \in \mathbb{R}^n$,

$$\mathrm{Vol}_{n-k}(K \cap H) \le \mathrm{Vol}_{n-k}(L \cap H).$$

Does it follow that

$$\mathrm{Vol}_n(K) \le \mathrm{Vol}_n(L)?$$

Zhang [39] proved that the answer is affirmative if and only if all origin-symmetric convex bodies in $\mathbb{R}^n$ are generalized k-intersection bodies (see definition in Sect. 2; this is similar to the connection between the original Busemann–Petty problem and intersection bodies established by Lutwak in [27]). Using this connection, Bourgain and Zhang [6] proved that the answer is negative if the dimension of sections $n-k > 3$ (see also [33] and different later proof in [21]). However, the cases of two- and three-dimensional sections remain open. Other results on the lower dimensional Busemann–Petty problem can be found in [28–30,32–35].

In this paper, we establish stability in the affirmative part of the lower dimensional Busemann–Petty problem. Stability problems in convex geometry have been considered for a long time; see [16] for numerous results and references. Stability in volume comparison problems was first studied in [22], where such results were proved for the Busemann–Petty and Shephard problems. We extend the result of [22, Theorem 1] to sections of lower dimensions in the following way.

Theorem 1 *Let K and L be origin-symmetric star bodies in $\mathbb{R}^n$, and $1 \le k < n$. Suppose K is a generalized k-intersection body and $\varepsilon > 0$. If for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$*

$$\mathrm{Vol}_{n-k}(K \cap H) \le \mathrm{Vol}_{n-k}(L \cap H) + \varepsilon, \tag{1}$$

then

$$\mathrm{Vol}_n(K)^{\frac{n-k}{n}} \le \mathrm{Vol}_n(L)^{\frac{n-k}{n}} + c_{n,k}\,\varepsilon, \tag{2}$$

where $c_{n,k} = |B_2^n|^{(n-k)/n}/|B_2^{n-k}|$ and $|B_2^n|$ is the volume of the unit Euclidean ball.

Note that $c_{n,k} < 1$, which immediately follows from the log-convexity of the Γ-function (see for example [24, Lemma 2.1]). Also, in the formulation of Theorem 1 in [22] the constant $c_{n,1}$ was replaced by 1, though the proof there gives the result with $c_{n,1}$.

Zvavitch [40] found a remarkable generalization of the Busemann–Petty problem to arbitrary measures. It appears that one can replace volume by any measure with even continuous density in $\mathbb{R}^n$. Let f be an even continuous non-negative function on $\mathbb{R}^n$, and denote by μ the measure on $\mathbb{R}^n$ with density f. For every closed bounded set $B \subset \mathbb{R}^n$ define

$$\mu(B) = \int_B f(x)\,dx.$$

It was proved in [40] that, for $n \leq 4$ and any origin-symmetric convex bodies K and L in $\mathbb{R}^n$, the inequalities

$$\mu(K \cap \xi^\perp) \leq \mu(L \cap \xi^\perp), \quad \forall \xi \in S^{n-1}$$

imply

$$\mu(K) \leq \mu(L).$$

Zvavitch also proved that this is generally not true if $n \geq 5$, namely, for any μ with strictly positive even continuous density there exist K and L providing a counterexample.

Stability in Zvavitch's result was established in [23, Theorem 2]. Here we extend this result to sections of lower dimensions, as follows.

Theorem 2 *Let K and L be origin-symmetric star bodies in $\mathbb{R}^n$, and $1 < k < n$. Suppose K is a generalized k-intersection body and $\varepsilon > 0$. If for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$*

$$\mu(K \cap H) \leq \mu(L \cap H) + \varepsilon, \tag{3}$$

then

$$\mu(K) \leq \mu(L) + \frac{n}{n-k} c_{n,k} \mathrm{Vol}_n(K)^{k/n} \varepsilon.$$

In the case $f \equiv 1$, we get another stability result for volume which is weaker than what is provided by Theorem 1. This is the reason why we state Theorem 1 separately. However, for arbitrary measures the constant in Theorem 2 is the best possible, as follows from the example after Corollary 5.

The stability results mentioned above were applied in [22,23] to the hyperplane (or slicing) problem of Bourgain [2,3] that can be formulated as follows. Does there exist an absolute constant C so that for any origin-symmetric convex body K in $\mathbb{R}^n$

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \leq C \max_{\xi \in S^{n-1}} \mathrm{Vol}_{n-1}(K \cap \xi^\perp)? \tag{4}$$

The best-to-date estimate $C \sim n^{1/4}$ is due to Klartag [17], who removed the logarithmic term from the previous estimate of Bourgain [4]. We refer the reader to recent papers [8,9] for the history and current state of the hyperplane problem.

In the case where K is an intersection body (see Sect. 2 for definitions and properties), the inequality (4) is known for sections of arbitrary dimension with the best possible constant. For any $1 \leq k < n$,

$$\mathrm{Vol}_n(K)^{\frac{n-k}{n}} \leq c_{n,k} \max_{H \in G(n,n-k)} \mathrm{Vol}_{n-k}(K \cap H), \tag{5}$$

where $G(n, n-k)$ is the Grassmanian of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$, and the equality is attained when $K = B_2^n$. In particular, if the dimension $n \leq 4$, then (5) is true for any origin-symmetric convex body K. The proof is an immediate consequence of Zhang's connection between generalized intersection bodies and the lower dimensional Busemann–Petty problem; apply this connection to any generalized k-intersection body K and $L = B_2^n$. Then use the fact that every intersection body is a generalized k-intersection body for every k (see [15] or [28]). For every fixed k, the inequality (5) holds for any generalized k-intersection body K.

We prove several generalizations of (5) using the stability results formulated above. First, interchanging K and L in Theorem 1, we get the following "difference" inequality, previously established in [22, Corollary 1] in the hyperplane case.

Corollary 3 *Let K and L be origin-symmetric star bodies in $\mathbb{R}^n$, and $1 \le k < n$. Suppose K and L are generalized k-intersection bodies, then*

$$\left|\mathrm{Vol}_n(K)^{\frac{n-k}{n}} - \mathrm{Vol}_n(L)^{\frac{n-k}{n}}\right| \le c_{n,k} \max_{H \in G(n,n-k)} |\mathrm{Vol}_{n-k}(K \cap H) - \mathrm{Vol}_{n-k}(L \cap H)| .$$

Putting $L = \varnothing$ in the latter inequality, we get (5) for any generalized k-intersection body K.

Interchanging K and L in Theorem 2, we get the following inequality, which was earlier proved for $k = 1$ in [23, Corollary 1].

Corollary 4 *Let K and L be origin-symmetric star bodies in $\mathbb{R}^n$, and $1 \le k < n$. Suppose that K and L are generalized k-intersection bodies. Then*

$$|\mu(K) - \mu(L)| \le \frac{n}{n-k} c_{n,k} \max_H |\mu(K \cap H) - \mu(L \cap H)| \max\left\{\mathrm{Vol}_n(K)^{k/n}, \mathrm{Vol}_n(L)^{k/n}\right\},$$

where maximum is taken over all $(n-k)$-dimensional subspaces H of $\mathbb{R}^n$.

Putting $L = \varnothing$, we generalize to lower dimensions the hyperplane inequality for arbitrary measures from [23, Theorem 1].

Corollary 5 *Let $1 \le k < n$, and suppose that K is a generalized k-intersection body in $\mathbb{R}^n$. Then*

$$\mu(K) \le \frac{n}{n-k} c_{n,k} \max_{H \in G(n,n-k)} \mu(K \cap H)\, \mathrm{Vol}_n(K)^{k/n}. \tag{6}$$

The constant in the right-hand side is the best possible. In fact, let $K = B_2^n$ and, for every $j \in N$, let f_j be a non-negative continuous function on $[0, 1]$ supported in $(1 - \frac{1}{j}, 1)$ and such that $\int_0^1 f_j(t)dt = 1$. Let μ_j be the measure on $\mathbb{R}^n$ with density $f_j(|x|_2)$, where $|x|_2$ is the Euclidean norm. We have

$$\mu_j(B_2^n) = |S^{n-1}| \int_0^1 r^{n-1} f_j(r)dr,$$

where $|S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit sphere in $\mathbb{R}^n$. For every $H \in G(n, n-k)$,

$$\mu_j(B_2^n \cap H) = |S^{n-k-1}| \int_0^1 r^{n-k-1} f_j(r)dr.$$

Clearly,

$$\lim_{j\to\infty} \frac{\int_0^1 r^{n-1} f_j(r)dr}{\int_0^1 r^{n-k-1} f_j(r)dr} = 1.$$

Using $|S^{n-1}| = n|B_2^n|$, we get

$$\lim_{j\to\infty} \frac{\mu_j(B_2^n)}{\max_H \mu_j(B_2^n \cap H)\ \mathrm{Vol}_n(B_2^n)^{k/n}} = \frac{|S^{n-1}|}{|S^{n-k-1}||B_2^n|^{k/n}} = \frac{n}{n-k} c_{n,k},$$

which shows that the constant is asymptotically optimal.

2 Stability

We say that a closed bounded set K in $\mathbb{R}^n$ is a *star body* if every straight line passing through the origin crosses the boundary of K at exactly two points different from the origin, the origin is an interior point of K, and the *Minkowski functional* of K defined by

$$\|x\|_K = \min\{a \ge 0 : \ x \in aK\}$$

is a continuous function on $\mathbb{R}^n$.

The *radial function* of a star body K is defined by

$$\rho_K(x) = \|x\|_K^{-1}, \quad x \in \mathbb{R}^n.$$

If $x \in S^{n-1}$ then $\rho_K(x)$ is the radius of K in the direction of x.

Writing the volume of K in polar coordinates, one gets

$$\mathrm{Vol}_n(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^n(\theta) d\theta = \frac{1}{n} \int_{S^{n-1}} \|\theta\|_K^{-n} d\theta. \tag{7}$$

The *spherical Radon transform* $R : C(S^{n-1}) \mapsto C(S^{n-1})$ is a linear operator defined by

$$Rf(\xi) = \int_{S^{n-1}\cap\xi^\perp} f(x)\, dx, \quad \xi \in S^{n-1}$$

for every function $f \in C(S^{n-1})$.

The polar formula (7) for the volume of a hyperplane section expresses this volume in terms of the spherical Radon transform (see for example [20, p.15]):

$$S_K(\xi) = \mathrm{Vol}_{n-1}(K \cap \xi^\perp) = \frac{1}{n-1} R(\|\cdot\|_K^{-n+1})(\xi). \tag{8}$$

The spherical Radon transform is self-dual (see [16, Lemma 1.3.3]): for any functions $f, g \in C(S^{n-1})$

$$\int_{S^{n-1}} Rf(\xi)\, g(\xi)\, d\xi = \int_{S^{n-1}} f(\xi)\, Rg(\xi)\, d\xi. \tag{9}$$

Using self-duality, one can extend the spherical Radon transform to measures. Let μ be a finite Borel measure on S^{n-1}. We define the spherical Radon transform of μ as a functional $R\mu$ on the space $C(S^{n-1})$ acting by

$$(R\mu, f) = (\mu, Rf) = \int_{S^{n-1}} Rf(x)d\mu(x).$$

By Riesz's characterization of continuous linear functionals on the space $C(S^{n-1})$, $R\mu$ is also a finite Borel measure on S^{n-1}. If μ has continuous density g, then by (9) the Radon transform of μ has density Rg.

The class of intersection bodies was introduced by Lutwak [27]. Let K, L be origin-symmetric star bodies in $\mathbb{R}^n$. We say that K is the intersection body of L if the radius of K in every direction is equal to the $(n-1)$-dimensional volume of the section of L by the central hyperplane orthogonal to this direction, i.e. for every $\xi \in S^{n-1}$,

$$\rho_K(\xi) = \|\xi\|_K^{-1} = \mathrm{Vol}_{n-1}(L \cap \xi^\perp). \tag{10}$$

All the bodies K that appear as intersection bodies of different star bodies form *the class of intersection bodies of star bodies*.

Note that the right-hand side of (10) can be written in terms of the spherical Radon transform using (8):

$$\|\xi\|_K^{-1} = \frac{1}{n-1} \int_{S^{n-1}\cap\xi^\perp} \|\theta\|_L^{-n+1} d\theta = \frac{1}{n-1} R(\|\cdot\|_L^{-n+1})(\xi).$$

It means that a star body K is the intersection body of a star body if and only if the function $\|\cdot\|_K^{-1}$ is the spherical Radon transform of a continuous positive function on S^{n-1}. This allows to introduce a more general class of bodies. A star body K in $\mathbb{R}^n$ is called an *intersection body* if there exists a finite Borel measure μ on the sphere S^{n-1} so that $\|\cdot\|_K^{-1} = R\mu$ as functionals on $C(S^{n-1})$, i.e. for every continuous function f on S^{n-1},

$$\int_{S^{n-1}} \|x\|_K^{-1} f(x)\, dx = \int_{S^{n-1}} Rf(x)\, d\mu(x). \tag{11}$$

Intersection bodies played the crucial role in the solution of the original Busemann–Petty problem due to the following connection found by Lutwak [27]. If K in an origin-symmetric intersection body in $\mathbb{R}^n$ and L is any origin-symmetric star body in $\mathbb{R}^n$, then the inequalities $S_K(\xi) \le S_L(\xi)$ for all $\xi \in S^{n-1}$ imply that $\mathrm{Vol}_n(K) \le \mathrm{Vol}_n(L)$, i.e. the answer to the Busemann–Petty problem in this situation is affirmative. For more information about intersection bodies, see [20,25, Chapter 4], [12, Chapter 8] and references there. In particular, every origin-symmetric convex body in $\mathbb{R}^n$, $n \le 4$ is an intersection body; see [11,38,13]. Also the unit ball of any finite dimensional subspace of L_p, $0 < p \le 2$ is an intersection body; see [18].

Zhang in [39] introduced a generalization of intersection bodies. For $1 \le k \le n-1$, the *$(n-k)$-dimensional spherical Radon transform* is an operator $\mathcal{R}_{n-k} : C(S^{n-1}) \mapsto C(G(n, n-k))$ defined by

$$\mathcal{R}_{n-k}(f)(H) = \int_{S^{n-1}\cap H} f(x)dx, \quad H \in G(n, n-k).$$

Denote the image of the operator $\mathcal{R}_{n-k}$ by X:

$$\mathcal{R}_{n-k}\left(C(S^{n-1})\right) = X \subset C(G(n, n-k)).$$

Let $M^+(X)$ be the space of linear positive continuous functionals on X, i.e. for every $\nu \in M^+(X)$ and non-negative function $f \in X$, we have $\nu(f) \ge 0$.

An origin-symmetric star body K in $\mathbb{R}^n$ is called a *generalized k-intersection body* if there exists a functional $\nu \in M^+(X)$, so that for every $f \in C(S^{n-1})$,

$$\int_{S^{n-1}} \|x\|_K^{-k} f(x)dx = \nu(\mathcal{R}_{n-k}(f)).$$

When $k = 1$ we get the class of intersection bodies. It was proved by Grinberg and Zhang [15, Lemma 6.1] that every intersection body in $\mathbb{R}^n$ is a generalized k-intersection body for every $k < n$. More generally, as proved later by Milman [28], if m divides k, then every generalized m-intersection body is a generalized k-intersection body. Zhang [39] showed that the answer to the lower dimensional Busemann–Petty problem is affirmative if and only if every origin-symmetric convex body in $\mathbb{R}^n$ is a generalized k-intersection body.

Denote by $1_S \equiv 1$ and $1_G \equiv 1$ the functions which are equal to 1 everywhere on the unit sphere S^{n-1} and the Grassmanian $G(n, n-k)$, correspondingly. Then, $\mathcal{R}_{n-k}(1_S) = |S^{n-k-1}|\, 1_G$.

We are now ready to prove the stability in the lower dimensional Busemann–Petty problem.

Proof of Theorem 1 By the polar formula for volume (7), for each $H \in G(n, n-k)$ we have

$$\mathrm{Vol}_{n-k}(K \cap H) = \frac{1}{n-k}\mathcal{R}_{n-k}\left(\|\cdot\|_K^{-n+k}\right)(H), \tag{12}$$

Then the inequality (1) can be written as

$$\mathcal{R}_{n-k}\left(\|\cdot\|_K^{-n+k}\right)(H) \le \mathcal{R}_{n-k}\left(\|\cdot\|_L^{-n+k}\right)(H) + (n-k)\varepsilon. \tag{13}$$

Since K is a generalized k-intersection body, there exists $\mu_0 \in M^+$, such that for each $\psi \in C\left(S^{n-1}\right)$,

$$\int_{S^{n-1}} \|x\|_K^{-k} \psi(x)dx = \mu_0(\mathcal{R}_{n-k}(\psi)). \tag{14}$$

Since μ_0 is a positive functional, by (13) and (14), we have

$$\begin{aligned} n\mathrm{Vol}_n(K) &= \int_{S^{n-1}} \|x\|_K^{-k} \|x\|_K^{-n+k} dx \\ &= \mu_0\left(\mathcal{R}_{n-k}\left(\|\cdot\|_K^{-n+k}\right)\right) \\ &\le \mu_0\left(\mathcal{R}_{n-k}\left(\|\cdot\|_L^{-n+k}\right)\right) + (n-k)\varepsilon\mu_0(1_G) \\ &:= \mathrm{I} + \mathrm{II}. \end{aligned} \tag{15}$$

Using (14), Hölder's inequality and polar formula for the volume, we get

$$\begin{aligned} \mathrm{I} &= \int_{S^{n-1}} \|x\|_K^{-k} \|x\|_L^{-n+k} dx \\ &\le \left(\int_{S^{n-1}} \|x\|_K^{-n} dx\right)^{k/n} \left(\int_{S^{n-1}} \|x\|_L^{-n} dx\right)^{(n-k)/n} \\ &= n\mathrm{Vol}_n(K)^{k/n}\mathrm{Vol}_n(L)^{(n-k)/n}. \end{aligned} \tag{16}$$

Now, by (14), the well-known formula $|S^{n-1}| = n|B_2^n|$ (see [20, p. 33]) and Hölder's inequality,

$$\begin{aligned}
\mathrm{II} = (n-k)\varepsilon\mu_0(1_G) &= \frac{(n-k)\varepsilon}{|S^{n-k-1}|}\int\limits_{S^{n-1}} \|x\|_K^{-k}\, 1_S(x)\, dx \\
&\le \frac{(n-k)\varepsilon}{|S^{n-k-1}|}\left(\int\limits_{S^{n-1}} \|x\|_K^{-n} dx\right)^{k/n} |S^{n-1}|^{\frac{n-k}{n}} \\
&= \frac{n^{k/n}(n-k)|S^{n-1}|^{\frac{n-k}{n}}}{|S^{n-k-1}|}\mathrm{Vol}_n(K)^{k/n}\varepsilon \\
&= \frac{n|B_2^n|^{\frac{n-k}{n}}}{|B_2^{n-k-1}|}\mathrm{Vol}_n(K)^{k/n}\varepsilon.
\end{aligned}$$

Combining this with (15) and (16), we get the result. □

We now pass to stability for arbitrary measures. Let μ be a measure on $\mathbb{R}^n$ with even continuous density f. Let χ be the indicator function of the interval $[0,\ 1]$. The measure μ of a star body K can be expressed in polar coordinates as follows:

$$\begin{aligned}
\mu(K) &= \int\limits_K f(x)\, dx = \int\limits_{\mathbb{R}^n} \chi(\|x\|_K) f(x)\, dx \\
&= \int\limits_{S^{n-1}} \left(\int\limits_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta)\, dr\right) d\theta. \qquad (17)
\end{aligned}$$

Similarly, we can express the volume of a section of K by an $(n-k)$-dimensional subspace H of $\mathbb{R}^n$ as

$$\begin{aligned}
\mu(K\cap H) &= \int\limits_H \chi\left(\|x\|_K\right) f(x) dx \\
&= \int\limits_{S^{n-1}\cap H} \left(\int\limits_0^{\|\theta\|_K^{-1}} t^{n-k-1} f\left(t\theta\right) dt\right) d\theta \\
&= \mathcal{R}_{n-k}\left(\int\limits_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta)\, dr\right)(H), \qquad (18)
\end{aligned}$$

where the Radon transform is applied to a function of the variable $\theta \in S^{n-1}$.

We need the following lemma, which was also used by Zvavitch in his proof.

Lemma 6 *Let $a, b, k \in \mathbb{R}^+$, and α be a non-negative function on* $(0, \max\{a, b\})$*, such that the integral below converges. Then*

$$\int_0^a r^{n-1}\alpha(r)\,dt - a^k \int_0^a r^{n-k-1}\alpha(r)\,dr$$
$$\leq \int_0^b r^{n-1}\alpha(r)\,dr - a^k \int_0^b r^{n-k-1}\alpha(r)\,dr$$

Proof The result follows from

$$a^k \int_a^b r^{n-k-1}\alpha(r)\,dr \leq \int_a^b r^{n-1}\alpha(r)\,dr.$$

□

Proof of Theorem 2 Using (18), inequality (3) can be written as

$$\mathcal{R}_{n-k}\left(\int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta)\,dr\right)(H)$$
$$\leq \mathcal{R}_{n-k}\left(\int_0^{\|\theta\|_L^{-1}} r^{n-k-1} f(r\theta)\,dr\right)(H) + \varepsilon, \quad \forall H \in G(n, n-k). \tag{19}$$

As in the proof of Theorem 1, let μ_0 be the positive functional associated with the generalized k-intersection body K. Applying μ_0 to both sides of (19) and then using (14), we get

$$\int_{S^{n-1}} \|\theta\|_K^{-k}\left(\int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta)\,dr\right)d\theta$$
$$\leq \int_{S^{n-1}} \|\theta\|_K^{-k}\left(\int_0^{\|\theta\|_L^{-1}} r^{n-k-1} f(r\theta)\,dr\right)d\theta + \varepsilon\mu_0(1_G). \tag{20}$$

Applying Lemma 6 with $a = \|\theta\|_K^{-1}$, $b = \|\theta\|_L^{-1}$ and $\alpha(r) = f(r\theta)$ and then integrating over the sphere, we get

$$\int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta)\,dr - \|\theta\|_K^{-k} \int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta)\,dr$$
$$\leq \int_0^{\|\theta\|_L^{-1}} r^{n-1} f(r\theta)\,dr - \|\theta\|_K^{-k} \int_0^{\|\theta\|_L^{-1}} r^{n-k-1} f(r\theta)\,dr,$$

and

$$\int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta)\, dr \right) d\theta$$
$$- \int_{S^{n-1}} \|\theta\|_K^{-k} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta)\, dr \right) d\theta$$
$$\leq \int_{S^{n-1}} \left(\int_0^{\|\theta\|_L^{-1}} r^{n-1} f(r\theta)\, dr \right) d\theta$$
$$- \int_{S^{n-1}} \|\theta\|_K^{-k} \left(\int_0^{\|\theta\|_L^{-1}} r^{n-k-1} f(r\theta)\, dr \right) d\theta. \tag{21}$$

Adding (20) and (21) and using (17) we get

$$\mu(K) \leq \mu(L) + \varepsilon \mu_0(1_G).$$

As shown in the proof of Theorem 1,

$$\mu_0(1_G) \leq \frac{n}{n-k} c_{n,k} \mathrm{Vol}_n(K)^{k/n},$$

which completes the proof. □

As mentioned earlier, every intersection body is a generalized k-intersection body for every k, so if K is an intersection body, the results of Theorems 1 and 2 hold for all k at the same time, as well as the results of Corollaries 3, 4, 5.

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