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Asymmetric anisotropic fractional Sobolev norms

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Abstract. Bourgain, Brezis, and Mironescu showed that (with suitable scaling) the fractional Sobolev s-seminorm of a function $f \in W^{1,p}(\mathbb{R}^n)$ converges to the Sobolev seminorm of f as $s \to 1^-$. Ludwig introduced the anisotropic fractional Sobolev s-seminorms of f defined by a norm on $\mathbb{R}^n$ with unit ball K and showed that they converge to the anisotropic Sobolev seminorm of f defined by the norm whose unit ball is the polar L_p moment body of K, as $s \to 1^-$. The asymmetric anisotropic s-seminorms are shown to converge to the anisotropic Sobolev seminorm of f defined by the Minkowski functional of the polar asymmetric L_p moment body of K.

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1. Introduction. Let Ω be an open set in $\mathbb{R}^n$. For $p \geq 1$ and $0 < s < 1$, Gagliardo introduced the fractional Sobolev spaces

$$W^{s,p}(\Omega) = \left\{ f \in L^p(\Omega) : \frac{|f(x) - f(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\}$$

and the fractional Sobolev s-seminorm of a function $f \in L^p(\Omega)$

$$\|f\|^p_{W^{s,p}(\Omega)} = \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+ps}} dxdy$$

(see [8]). They have found many applications in pure and applied mathematics (see [3,5,24]).

Although $\|f\|_{W^{s,p}(\Omega)} \to \infty$ as $s \to 1^-$, Bourgain, Brezis, and Mironescu showed in [2] that

$$\lim_{s \to 1^-} (1-s) \|f\|^p_{W^{s,p}(\Omega)} = \frac{K_{n,p}}{p} \|f\|^p_{W^{1,p}(\Omega)} \tag{1.1}$$

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for $f \in W^{1,p}(\Omega)$ and $\Omega \subset \mathbb{R}^n$ a smooth bounded domain, where

$$K_{n,p} = \frac{2\Gamma((p+1)/2)\pi^{(n-1)/2}}{\Gamma((n+p)/2)}$$

is a constant depending on n and p,

$$\|f\|_{W^{1,p}}^p = \int_\Omega |\nabla f(x)|^p \, dx$$

is the Sobolev seminorm of f, and $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ denotes the L^p weak derivative of f.

If instead of the Euclidean norm $|\cdot|$, we consider an arbitrary norm $\|\cdot\|_K$ with unit ball K, we obtain the anisotropic Sobolev seminorm,

$$\|f\|_{W^{1,p},K}^p = \int_{\mathbb{R}^n} \|\nabla f(x)\|_{K^*}^p \, dx,$$

where $K^* = \{v \in \mathbb{R}^n : v \cdot x \leq 1 \text{ for all } x \in K\}$ is the polar body of K, and $v \cdot x$ denotes the inner product between v and x. Anisotropic Sobolev seminorms and the corresponding Sobolev inequalities attracted a lot of attention in recent years (see [1,4,7,10]).

Anisotropic s-seminorms, introduced very recently by Ludwig [17], reflect a fine structure of the anisotropic fractional Sobolev spaces. She established that

$$\lim_{s \to 1^-} (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_K^{n+ps}} dx dy = \frac{2}{p} \int_{\mathbb{R}^n} \|\nabla f(x)\|_{Z_p^* K}^p \, dx$$

for $f \in W^{1,p}(\mathbb{R}^n)$ with compact support, where the norm associated with $Z_p^* K$, the polar L_p moment body of K, is defined as

$$\|v\|_{Z_p^* K}^p = \frac{n+p}{2} \int_K |v \cdot x|^p \, dx$$

for $v \in \mathbb{R}^n$ and a convex body $K \subset \mathbb{R}^n$. Several different other cases were considered in [16,17,29].

In this paper, by replacing the absolute value $|\cdot|$ by the positive part $(\cdot)_+$, for $x \in \mathbb{R}$, where $(x)_+ = \max\{0, x\}$, we obtain the following generalization. Note that here it is no longer required that K is origin-symmetric. As a consequence, for $K \subset \mathbb{R}^n$ a convex body containing the origin in its interior and $x \in \mathbb{R}^n$,

$$\|x\|_K = \min\{\lambda \geq 0 : x \in \lambda K\}$$

just defines the Minkowski functional of K and no longer a norm.

Theorem 1. *If $f \in W^{1,p}(\mathbb{R}^n)$ has compact support, then*

$$\lim_{s \to 1^-} (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x)-f(y))_+^p}{\|x-y\|_K^{n+sp}} dx dy = \frac{1}{p} \int_{\mathbb{R}^n} \|\nabla f(x)\|_{Z_p^{+,*} K}^p \, dx,$$

where $Z_p^{+,*} K$ is the polar asymmetric L_p moment body of K.

For a convex body $K \subset \mathbb{R}^n$, the polar asymmetric L_p moment body is the unit ball of the Minkowski functional defined by

$$\|v\|^p_{Z_p^{+,*}K} = (n+p) \int\limits_K (v \cdot x)_+^p \, dx$$

for $v \in \mathbb{R}^n$, $Z_p^- K = Z_p^+(-K)$. For $p > 1$, in [14], Ludwig introduced and characterized the two-parameter family

$$c_1 \cdot Z_p^+ K +_p c_2 \cdot Z_p^- K$$

as all possible L_p analogs of moment bodies, including the symmetric case

$$Z_p K = \frac{1}{2} \cdot Z_p^+ K +_p \frac{1}{2} \cdot Z_p^- K,$$

where $\|\cdot\|^p_{(\alpha \cdot K +_p \beta \cdot L)^*} = \alpha \|\cdot\|^p_{K^*} + \beta \|\cdot\|^p_{L^*}$, for $\alpha, \beta \geq 0$, defines the L_p Minkowski combination. In recent years, this family of convex bodies has found important applications within convex geometry, probability theory, and the local theory of Banach spaces (see [9,11–15,18–23,25–28,31]).

The proof given in this paper makes use of an asymmetric version of the one-dimensional case of result (1.1) by Bourgain, Brezis, and Mironescu and an asymmetric decomposition of Blaschke-Petkantschin type.

2. Proof of the main result. First, we need the asymmetric one-dimensional analogue of (1.1). For its proof we require the following result from [2].

Lemma 2. *Let $\rho \in L^1(\mathbb{R}^n)$ and $\rho \geq 0$. If $f \in W^{1,p}(\mathbb{R}^n)$ is compactly supported and $1 \leq p < \infty$, then*

$$\int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho(x-y) dx dy \leq C \|f\|^p_{W^{1,p}} \|\rho\|_{L^1},$$

where C depends only on p and the support of f.

Let $\Omega \subset \mathbb{R}$ be a bounded domain.

Proposition 3. *If $f \in W^{1,p}(\Omega)$, then*

$$\lim_{s \to 1^-} (1-s) \int\limits_{\Omega} \int\limits_{\Omega \cap \{x > y\}} \frac{(f(x) - f(y))_+^p}{|x-y|^{1+ps}} dx dy = \frac{1}{p} \int\limits_{\Omega} (f'(x))_+^p \, dx. \quad (2.1)$$

Proof. Take a sequence (ρ_ε) of radial mollifiers, i.e. $\rho_\varepsilon(x) = \rho_\varepsilon(|x|)$; $\rho_\varepsilon \geq 0$; $\int_0^\infty \rho_\varepsilon(x) dx = 1$; $\lim\limits_{\varepsilon \to 0} \int_\delta^\infty \rho_\varepsilon(r) dr = 0$ for every $\delta > 0$. Let $F_\varepsilon(x,y) = \frac{(f(x)-f(y))_+}{|x-y|} \rho_\varepsilon^{1/p}(x-y)$ for $x > y$. It suffices to prove that

$$\lim_{\varepsilon \to 0} \int\limits_{\Omega} \int\limits_{\Omega \cap \{x > y\}} {F_\varepsilon}^p(x,y) dx dy = \int\limits_{\Omega} (f'(x))_+^p \, dx. \quad (2.2)$$

Indeed, as in [30], let $R > \max\{|x-y| : x, y \in \Omega\}$, $\varepsilon = 1-s$, and

$$\rho_\varepsilon(x) = \frac{\chi_{[0,R]}(|x|)}{R^{\varepsilon p}} \frac{p\varepsilon}{|x|^{1-p\varepsilon}},$$

where χ_A is the indicator function of A. Then one obtains (2.1) from (2.2) as desired.

By Lemma 2 we have, for any $\varepsilon > 0$ and $f, g \in W^{1,p}(\Omega)$,

$$\left| \|F_\varepsilon\|_{L^p(\Omega\times\Omega)} - \|G_\varepsilon\|_{L^p(\Omega\times\Omega)} \right| \le \|F_\varepsilon - G_\varepsilon\|_{L^p(\Omega\times\Omega)} \le C \|f - g\|_{W^{1,p}}$$

for some constant C dependent on ε, f, and g. Therefore it suffices to establish (2.2) for f in some dense subset of $W^{1,p}(\Omega)$, e.g., for $f \in C^2(\bar{\Omega})$, where $\bar{\Omega}$ is the closure of Ω.

Fix $f \in C^2(\bar{\Omega})$. Since for $t \in \mathbb{R}$ and $\lambda > 0$, $(\lambda t)_+ = \lambda (t)_+$, there exists $\delta > 0$ such that for $y < x < y + \delta$ and a constant c,

$$\left| \frac{(f(x) - f(y))_+^p}{|x - y|^p} - (f'(y))_+^p \right| \le c(x - y).$$

We have

$$\begin{aligned}
&\int\limits_{\Omega\cap\{x>y\}} \frac{(f(x) - f(y))_+^p}{|x - y|^p} \rho_\varepsilon(x - y) dx \\
&\quad = \int\limits_{\Omega\cap\{y<x<y+\delta\}} \frac{(f(x) - f(y))_+^p}{|x - y|^p} \rho_\varepsilon(x - y) dx \\
&\qquad + \int\limits_{\Omega\cap\{x\ge y+\delta\}} \frac{(f(x) - f(y))_+^p}{|x - y|^p} \rho_\varepsilon(x - y) dx,
\end{aligned}$$

yet, only the former integral on the right hand side needs to be considered as the latter vanishes. In fact, for each fixed $y \in \Omega$, since

$$\begin{aligned}
&\left| \int\limits_y^{y+\delta} \left(\frac{(f(x) - f(y))_+^p}{|x - y|^p} - (f'(y))_+^p \right) \rho_\varepsilon(x - y) dx \right| \\
&\quad \le \int\limits_y^{y+\delta} \left| \frac{(f(x) - f(y))_+^p}{|x - y|^p} - (f'(y))_+^p \right| \rho_\varepsilon(x - y) dx \\
&\quad \le c \int\limits_y^{y+\delta} (x - y) \rho_\varepsilon(x - y) dx \\
&\quad = c \int\limits_0^{\delta} r \rho_\varepsilon(r) dr \to 0 \quad \text{as } \varepsilon \to 0,
\end{aligned}$$

we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_y^{y+\delta}\frac{(f(x)-f(y))_+^p}{|x-y|^p}\rho_\varepsilon(x-y)dx\\
&= (f'(y))_+^p\lim_{\varepsilon\to 0}\int_y^{y+\delta}\rho_\varepsilon(x-y)dx\\
&= (f'(y))_+^p\lim_{\varepsilon\to 0}\int_0^{\delta}\rho_\varepsilon(r)dr\\
&= (f'(y))_+^p\,.
\end{aligned}$$

Therefore,

$$\lim_{\varepsilon\to 0}\int_{\Omega\cap\{x>y\}}\frac{(f(x)-f(y))_+^p}{|x-y|^p}\rho_\varepsilon(x-y)dx = (f'(y))_+^p\,. \tag{2.3}$$

Since $f\in C^2(\bar\Omega)$, there exists $L>0$ such that $|f(x)-f(y)|<L\,|x-y|$ for every $x,y\in\Omega$, then

$$\int_\Omega\frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_\varepsilon(x-y)dx\le L^p\quad\text{for each } y\in\Omega. \tag{2.4}$$

Hence, for $f\in C^2(\Omega)$, (2.2) follows by dominated convergence theorem from (2.3) and (2.4). □

Now, for $u\in S^{n-1}$, the Euclidean unit sphere, let $[u]=\{\lambda u:\lambda\in\mathbb{R}\}$ and $[u]^+=\{\lambda u:\lambda>0\}$. Denote the $k-$dimensional Hausdorff measure on $\mathbb{R}^n$ by H^k. For $f\in W^{1,p}(\mathbb{R}^n)$, we denote by $\bar f$ its precise representative (see [6, Section 1.7.1]). We require the following result. For every $u\in S^{n-1}$, the precise representative $\bar f$ is absolutely continuous on the lines $L=\{x+\lambda u:\lambda\in\mathbb{R}\}$ for $H^{n-1}-$a.e. $x\in u^\perp$ and its first-order (classical) partial derivatives belong to $L^p(\mathbb{R}^n)$ (see [6, Section 4.9.2]). Hence, we have for the restriction of $\bar f$ to L

$$\bar f\big|_L\in W^{1,p}(L) \tag{2.5}$$

for a.e. line L parallel to u.

Proof of Theorem 1. By the polar coordinate formula and Fubini's theorem, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(x)-f(y))_+^p}{\|x-y\|_K^{n+sp}} dH^n(x)dH^n(y)\\
&= \int_{\mathbb{R}^n}\int_{S^{n-1}} \|u\|_K^{-(n+ps)} \int_0^\infty \frac{(f(y+ru)-f(y))_+^p}{r^{1+sp}} dH^1(r)d\sigma(u)dH^n(y)\\
&= \int_{S^{n-1}} \|u\|_K^{-(n+ps)} \int_0^\infty \int_{u^\perp}\int_{[u]+z} \frac{(f(w+ru)-f(w))_+^p}{r^{1+sp}} dH^1(w)dH^{n-1}(z)dH^1(r)d\sigma(u)\\
&= \int_{S^{n-1}} \|u\|_K^{-(n+ps)} \int_{u^\perp}\int_{[u]+z}\int_0^\infty \frac{(f(w+ru)-f(w))_+^p}{r^{1+sp}} dH^1(r)dH^1(w)dH^{n-1}(z)d\sigma(u)\\
&= \int_{S^{n-1}} \|u\|_K^{-(n+ps)} \int_{u^\perp}\int_{[u]+z}\int_{[u]^++w} \frac{(f(t)-f(w))_+^p}{|t-w|^{1+sp}} dH^1(t)dH^1(w)dH^{n-1}(z)d\sigma(u),
\end{aligned}
\tag{2.6}
$$

where σ denotes the standard surface area measure on S^{n-1}. By Proposition 3 and (2.5), we obtain

$$
\begin{aligned}
&\lim_{s\to 1^-}(1-s)\int_{[u]+z}\int_{[u]^++w} \frac{(f(t)-f(w))_+^p}{|t-w|^{1+sp}} dH^1(t)dH^1(w)\\
&= \frac{1}{p}\int_{[u]+z} (\nabla f(t)\cdot u)_+^p \, dH^1(t).
\end{aligned}
\tag{2.7}
$$

By Fubini's theorem and the polar coordinate formula, we get

$$
\begin{aligned}
&\frac{1}{p}\int_{S^{n-1}} \|u\|_K^{-(n+p)} \int_{u^\perp}\int_{[u]+z} (\nabla f(t)\cdot u)_+^p \, dH^1(t)dH^{n-1}(z)d\sigma(u)\\
&= \frac{1}{p}\int_{S^{n-1}}\int_{\mathbb{R}^n} \|u\|_K^{-(n+p)} (\nabla f(x)\cdot u)_+^p \, dH^n(x)d\sigma(u)\\
&= \frac{n+p}{p}\int_K\int_{\mathbb{R}^n} (\nabla f(x)\cdot y)_+^p \, dH^n(x)dH^n(y).
\end{aligned}
$$

Using Fubini's theorem and the definition of the asymmetric L_p moment body of K, we obtain

$$
\begin{aligned}
&\int_{S^{n-1}} \|u\|_K^{-(n+p)} \int_{u^\perp}\int_{[u]+z} (\nabla f(t)\cdot u)_+^p \, dH^1(t)dH^{n-1}(z)d\sigma(u)\\
&= \int_{\mathbb{R}^n} \|\nabla f(x)\|_{Z_p^{+,*}K}^p \, dH^n(x).
\end{aligned}
\tag{2.8}
$$

So, in particular, we have

$$\int_{S^{n-1}} \int_{u^\perp} \int_{[u]+z} (\nabla f(t) \cdot u)_+^p \, dH^1(t) dH^{n-1}(z) d\sigma(u)$$
$$= \frac{n+p}{4} K_{n,p} \int_{\mathbb{R}^n} |\nabla f(x)|^p \, dH^n(x) < +\infty. \tag{2.9}$$

Using the dominated convergence theorem with Lemma 2 and (2.9), we obtain from (2.6), (2.7), and (2.8) that

$$\lim_{s \to 1^-} (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x)-f(y))_+^p}{\|x-y\|_K^{n+sp}} dx dy = \frac{1}{p} \int_{\mathbb{R}^n} \|\nabla f(x)\|_{Z_p^{+,*}K}^p \, dx.$$

□

Remark 4. In Theorem 1, let $g = -f$ and $(x)_- = -\min\{0, x\} = (-x)_+$, for $x \in \mathbb{R}$. Then, we get

$$\lim_{s \to 1^-} (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x)-f(y))_-^p}{\|x-y\|_K^{n+sp}} dx dy = \frac{1}{p} \int_{\mathbb{R}^n} \|\nabla f(x)\|_{Z_p^{-,*}K}^p \, dx.$$

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