



# Estimates for the extremal sections of $\ell_p^n$ -balls <sup>☆</sup>

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## ABSTRACT

The problem of finding the maximal hyperplane section of  $B_p^n$ , where  $p > 2$ , has been open for a long time. It is known that the answer depends on both  $p$  and  $n$ . In this paper, using the well-known equivalence between hyperplane sections and the isotropic constant of a body, we give an upper bound estimate for the volume of hyperplane sections of normalized  $\ell_p^n$ -balls that does not depend on  $n$  and  $p$ . In addition, on the basis of results of Meyer, Pajor and Schmuckenschläger, we show further the corresponding extremal body and hyperplane section when this volume attains its minimum.

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## 1. Introduction

Let  $B_p^n$  denote the unit ball of  $\ell_p^n$ -space, that is,  $B_p^n = \{x \in \mathbb{R}^n \mid \|x\|_p \leq 1\}$ , if  $0 < p < +\infty$ , and  $B_\infty^n = \{x \in \mathbb{R}^n \mid \max_{1 \leq i \leq n} |x_i| \leq 1\}$ . A very concrete problem of finding the extremal sections of  $\ell_p^n$ -balls has motivated some beautiful mathematics; the problem seems to be quite difficult, for example, the maximal hyperplane section of  $\ell_p^n$ -balls with  $2 < p < \infty$  is still open (see [18]). However, there are many interesting partial results concerning this subject (see Refs. [4,5,19,18]). In particular, for the unit cube ( $p = \infty$ ), the following theorem is well known:

**Theorem A.** (See Hadwiger [9], Hensley [11], Ball [2].) For every  $\xi \in S^{n-1}$ ,

$$1 \leq \text{vol}_{n-1} \left( \frac{1}{2} B_\infty^n \cap \xi^\perp \right) \leq \sqrt{2}.$$

Equality on the left-hand side holds for  $\xi = (1, 0, \dots, 0)$ , and equality on the right-hand side holds for  $\xi = (1/\sqrt{2}, 1/\sqrt{2}, 0, \dots, 0)$ , respectively.

In fact, Hensley [11] proved that hyperplane sections of the cube are bounded from above by a constant not depending on the dimension in 1979, and the exact value  $\sqrt{2}$  was given by Ball [2] in 1986. The minimal hyperplane sections of the cube were first found by Hadwiger [9], other proofs were given by Vaaler [26] and Hensley [11].

In general case, Oleszkiewicz [24] showed that the solution to the problem of finding the maximal hyperplane sections of  $\ell_p^n$ -balls must depend on both  $p$  and  $n$ , for  $p > 2$  in 2003. In this paper, using the well-known equivalence between hyperplane sections and the isotropic constant of a body, we give an upper bound estimate for the volume of hyperplane sections of normalized  $\ell_p^n$ -balls that does not depend on  $n$  and  $p$ . In addition, the lower bound estimate for the volume

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of hyperplane sections of normalized  $\ell_p^n$ -balls, Meyer and Pajor [21] proved that for all  $p \geq 2$  and  $p = 1$ :  $\text{vol}_{n-1}(r_{n,p}B_p^n \cap \xi^\perp) \geq 1$  (in fact Meyer and Pajor in [21] proved that  $\text{vol}_{n-k}(r_{n,p}B_p^n \cap E) \geq 1$  for any subspace  $E \subset \mathbb{R}^n$  of codimension  $k$ ), and Schmuckenschläger [25] showed that  $\text{vol}_{n-1}(r_{n,p}B_p^n \cap \xi^\perp) \geq 1$  for all  $1 < p < 2$ . On the basis of results of Meyer, Pajor and Schmuckenschläger, we show further the corresponding extremal body and hyperplane section when this volume attains its minimum. Our main result may be formulated as follows:

**Theorem 1.1.** *Let  $p \geq 1$ . Then, for every  $\xi \in S^{n-1}$ ,*

$$1 \leq \text{vol}_{n-1}(r_{n,p}B_p^n \cap \xi^\perp) \leq \sqrt{\pi e},$$

where  $r_{n,p} = \text{vol}_n^{-1/n}(B_p^n)$ . In addition, the minimum occurs for the unit cube with  $\xi = (1, 0, \dots, 0)$ .

Otherwise, Meyer and Pajor [21] proved also that among central hyperplane sections of  $B_1^n$  the central section orthogonal to  $(1, 1, \dots, 1)$  had the smallest volume, and they conjectured that the results were still correct for  $0 < p \leq 2$ . Ten years later, Koldobsky [17] verified the conjecture.

**Theorem B.** (See Koldobsky [17].) *For  $0 < p \leq 2$ , and every  $\xi \in S^{n-1}$ ,*

$$\frac{p}{\pi(n-1)\Gamma((n-1)/p)} \int_0^\infty \gamma_p^n(t/\sqrt{n}) dt \leq \text{vol}_{n-1}(B_p^n \cap \xi^\perp) \leq \frac{(2\Gamma(1+1/p))^{n-1}}{\Gamma(1+(n-1)/p)},$$

where  $\gamma_p$  denotes the Fourier transform of the function  $z \mapsto \exp(-|z|^p)$  ( $z \in \mathbb{R}$ ), equality on the left-hand side holds for  $\xi = (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})$ , equality on the right-hand side holds for  $\xi = (1, \dots, 0)$ , respectively.

By Theorem 3 in Huang et al. [13], the inequalities for the volume of  $B_p^n$  yield the following consequence.

**Theorem 1.2.** *For  $0 < p \leq 2$ ,  $n \geq 2$  and every  $\xi \in S^{n-1}$ ,*

$$0 < \text{vol}_{n-1}(r_{n,p}B_p^n \cap \xi^\perp) < \sqrt[p]{e},$$

where  $r_{n,p} = \text{vol}_n^{-1/n}(B_p^n)$ .

As  $p$  tends to zero,  $\sqrt[p]{e}$  tends to infinity, and  $B_p^n$  degenerates to the axes of coordinates. The situation is really bad.

## 2. Extremum of the isotropic constant of $B_p^n$

Let  $K$  be a convex body of volume 1 (a compact, convex subset with nonempty interior) in  $\mathbb{R}^n$ , whose barycenter is at the origin (i.e.,  $b(K) = \int_K \mathbf{x} dx = 0$ ). It is well known (see Ref. [23]) that there exists a unique positive definite linear transformation  $\phi$  with  $\det(\phi) = 1$ , such that for any unit vector  $u \in S^{n-1}$ ,

$$\int_{\phi K} \langle x, u \rangle^2 dx = L_K^2$$

independently of  $u$ , where  $S^{n-1}$  denotes the Euclidean unit sphere in  $\mathbb{R}^n$ . The number  $L_K$  is referred to as the *isotropic constant* of the convex body  $K$ ; if the transformation  $\phi$  is the identity map, we say that  $K$  is *isotropic*, or that it is in *isotropic position*. It is known that the slicing problem is equivalent to the question of whether there is a uniform upper bound, independent of dimension, on the isotropic constants of isotropic bodies of volume 1.

To date, the slicing problem is solved for several classes of convex sets: unconditional convex bodies [23], zonoids, duals to zonoids [22], bodies with a bounded out volume ratio [23], random bodies [16], unit ball of Schatten norms [20], and others (e.g., [6] and [14]). However, the best estimate of the upper bound available for arbitrary bodies is  $n^{\frac{1}{4}}$  by Klartag [15], which slightly improves Bourgain's estimate,  $n^{\frac{1}{4}} \log(n+1)$  (see Ref. [8]). Interesting results are also established in [7] and [10].

For the isotropic constant of  $B_p^n$ , we have the following lemma.

**Lemma 2.1.** *Let  $1 \leq p \leq \infty$ , then  $B_p^n$  is an isotropic convex body in  $\mathbb{R}^n$ . Furthermore, its isotropic constant is*

$$L_{B_p^n} = \left( \frac{\Gamma(1+3/p)\Gamma(1+n/p)^{1+\frac{2}{n}}}{12\Gamma(1+(n+2)/p)\Gamma(1+1/p)^3} \right)^{\frac{1}{2}}. \quad (2.1)$$

**Proof.** Since  $B_p^n$  is convex, when  $p \geq 1$ , the lemma is followed from direct calculations (see Ref. [25] or [27]).  $\square$

Although to minimize the expression (2.1) seems to be difficult, in the following, we utilize some skills of the analysis and get the result as required. To this end, we first optimize the expression (2.1) for a given positive integer  $n$ , and then optimize the expression (2.1) for some given  $p \geq 1$ .

**Lemma 2.2.** *Let  $p > 0$ . Then, for each given positive integer  $n$ ,*

$$F(p) = \frac{\Gamma(1 + 3/p)\Gamma(1 + n/p)^{1+\frac{2}{n}}}{\Gamma(1 + (n + 2)/p)\Gamma(1 + 1/p)^3}$$

attains its minimum at  $p = 2$ .

**Proof.** It is sufficient to show that  $\ln F$  attains its minimum at  $p = 2$ . Take the logarithmic derivative of  $F$ , and this gives

$$\frac{d \ln F(p)}{dp} = \frac{n + 2}{p^2} \left( \psi \left( 1 + \frac{n + 2}{p} \right) - \psi \left( 1 + \frac{n}{p} \right) \right) - \frac{3}{p^2} \left( \psi \left( 1 + \frac{3}{p} \right) - \psi \left( 1 + \frac{1}{p} \right) \right), \tag{2.2}$$

where  $\psi(x) = \Gamma'(x)/\Gamma(x)$ . Consider

$$(n + 2) \left( \psi \left( 1 + \frac{n + 2}{p} \right) - \psi \left( 1 + \frac{n}{p} \right) \right), \tag{2.3}$$

and in (2.3), we get

$$(n + 2) \int_0^\infty \frac{1}{z} \left( \frac{1}{(1 + z)^{1+\frac{n}{p}}} - \frac{1}{(1 + z)^{1+\frac{n+2}{p}}} \right) dz \tag{2.4}$$

from the following integral representation for the function  $\psi$  (see Ref. [1])

$$\psi(x) = \int_0^\infty \frac{1}{z} \left( e^{-z} - \frac{1}{(1 + z)^x} \right) dz.$$

Introduce a change of variable,  $t = \sqrt[p]{1 + z}$ , then (2.4) is equal to

$$-p \int_1^\infty \frac{t^2 - 1}{t^p - 1} d \frac{1}{t^{n+2}}. \tag{2.5}$$

Let  $n = 1$  in (2.5), then

$$3 \left( \psi \left( 1 + \frac{3}{p} \right) - \psi \left( 1 + \frac{1}{p} \right) \right) = -p \int_1^\infty \frac{t^2 - 1}{t^p - 1} d \frac{1}{t^3}.$$

Thus,

$$\frac{d \ln F(p)}{dp} = \frac{1}{p} \int_1^\infty \frac{t^2 - 1}{t^p - 1} d \left( \frac{1}{t^3} - \frac{1}{t^{n+2}} \right).$$

It is not hard to show that

$$\left. \frac{d \ln F}{dp} \right|_{p=2} = \frac{1}{2} \int_1^\infty d \left( \frac{1}{t^3} - \frac{1}{t^{n+2}} \right) = 0, \tag{2.6}$$

and also

$$\begin{aligned} \frac{d \ln F(p)}{dp} &= \frac{1}{p} \int_1^\infty \frac{t^2 - 1}{t^p - 1} d \left( \frac{1}{t^3} - \frac{1}{t^{n+2}} \right) \\ &= -\frac{1}{p} \int_1^\infty \frac{2t \left[ \left(1 - \frac{p}{2}\right)t^p + \frac{p}{2} \cdot t^{p-2} - 1 \right]}{(t^p - 1)^2} \left( \frac{1}{t^3} - \frac{1}{t^{n+2}} \right) dt \\ &< 0, \end{aligned} \tag{2.7}$$

if  $0 < p < 2$ . In fact, it follows from the arithmetic–geometric means inequality, i.e.  $(1 - \lambda)x + \lambda y \geq x^{1-\lambda}y^\lambda$ , where  $x, y \geq 0$  and  $0 < \lambda < 1$ . Similarly,

$$\begin{aligned} \frac{d \ln F(p)}{dp} &= \frac{1}{p} \int_1^\infty \frac{t^2 - 1}{t^p - 1} d\left(\frac{1}{t^3} - \frac{1}{t^{n+2}}\right) \\ &= \int_1^\infty \frac{t[(1 - \frac{2}{p})t^p + \frac{2}{p}t^0 - t^{p-2}]}{(t^p - 1)^2} \left(\frac{1}{t^3} - \frac{1}{t^{n+2}}\right) dt \\ &> 0, \end{aligned} \tag{2.8}$$

if  $p > 2$ . The theorem directly follows from (2.6), (2.7), (2.8).  $\square$

For a given positive integer  $n$ , it follows from Lemmas 2.1 and 2.2 that  $L_{B_p^n}$  is minimal when  $p = 2$  and the minimum is equal to

$$\left(\frac{\Gamma(1 + n/2)^{1 + \frac{2}{n}}}{2\pi\Gamma(2 + n/2)}\right)^{\frac{1}{2}}. \tag{2.9}$$

To find the minimum of  $L_{B_p^n}$ , it remains to find the minimum of (2.9). Noting that  $\Gamma(2 + n/2) = (1 + n/2)\Gamma(1 + n/2)$  by  $\Gamma(x + 1) = x\Gamma(x)$ , (2.9) becomes

$$\left(\frac{1}{2\pi} \frac{1}{1 + n/2} \Gamma(1 + n/2)^{2/n}\right)^{\frac{1}{2}}.$$

**Lemma 2.3.** *Let  $n$  be a positive integer. Then*

$$G(n) = \frac{1}{1 + n/2} \Gamma(1 + n/2)^{2/n} \tag{2.10}$$

is decreasing.

**Proof.** It is not hard to calculate that  $G(1) > G(2) > G(3)$ . And, for the proof of the theorem, it is sufficient to show that  $G(x)$  ( $x \in \mathbb{R}$ ,  $x > 1$ ) is decreasing.

Write

$$g(x) = \ln G(x) = \frac{1}{x} \ln \Gamma(1 + x) - \ln(1 + x),$$

for  $x \in \mathbb{R}$ ,  $x > 1$ . Using the representation (see Ref. [1])

$$\frac{d\psi(x)}{dx} = \sum_{k=0}^{\infty} \frac{1}{(x+k)^2},$$

we have

$$\begin{aligned} \frac{1}{x}(x^2 g'(x))' &= \psi'(1+x) - \frac{x+2}{(x+1)^2} \\ &= \sum_{k=1}^{\infty} \frac{1}{(x+k)^2} - \frac{x+2}{(x+1)^2} \\ &< \frac{1}{(x+1)^2} + \int_1^{\infty} \frac{dt}{(x+t)^2} - \frac{x+2}{(x+1)^2} = 0. \end{aligned}$$

Thus, for each  $x > 1$ ,

$$x^2 g'(x) < g'(1) = -\gamma + \frac{1}{2} < 0,$$

where  $\gamma$  stands for the Euler constant and it is equal to 0.5772.... Therefore  $G(x)$  is decreasing for  $x > 1$ .  $\square$

We derive from Lemmas 2.2, 2.3 and Stirling’s asymptotic formula (see Ref. [1])

$$\Gamma(x) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \text{ as } x \rightarrow \infty$$

that  $L_{B_p^n}$  must not be less than  $1/\sqrt{2\pi e}$  for  $p \geq 1$ .

For the upper bound of  $L_{B_p^n}$ , it is not hard to show that

$$L_{B_p^n} \leq \max\{L_{B_1^n}, L_{B_\infty^n}\},$$

since  $d \ln F(p)/dp < 0$ , when  $0 < p < 2$ , and  $d \ln F(p)/dp > 0$ , when  $p > 2$ . Noting that

$$L_{B_1^n} < L_{B_1^1} = L_{B_\infty^n} = 1/(2\sqrt{3}),$$

for each positive integer  $n$ , therefore,  $L_{B_p^n}$  must not be greater than  $(2\sqrt{3})^{-1}$  for  $p \geq 1$ .

We summarize the discussion above in the following propositions.

**Proposition 2.4.** *Let  $1 \leq p \leq \infty$ ,  $L_{B_p^n}$  denotes the isotropic constant of  $B_p^n$ . Then,*

$$1/\sqrt{2\pi e} \leq L_{B_p^n} \leq 1/(2\sqrt{3}),$$

where the minimum occurs when  $p = 2$  and  $n$  tends to infinity, and the maximum occurs when  $n = 1$  or  $p = +\infty$ .

For the sake of finding isotropic constants of  $B_p^n$ , for  $0 < p < 1$ , we extend the definition of isotropic body to compact sets, since they are no longer convex (see Ref. [18, p. 21]).

**Remark 2.5.** Noting that formula (2.1) also holds for  $0 < p < 1$ ,

$$\int_{r_{n,p} B_p^n} \langle x, \xi \rangle^2 dx = L_{r_{n,p} B_p^n}^2 = \frac{\Gamma(1 + 3/p)\Gamma(1 + n/p)^{1 + \frac{2}{n}}}{12\Gamma(1 + (n + 2)/p)\Gamma(1 + 1/p)^3},$$

for every  $\xi \in S^{n-1}$ , where  $r_{n,p} = \text{vol}_n^{-1/n}(B_p^n)$ , although  $B_p^n$  ( $0 < p < 1$ ) is really not convex. Furthermore, for each positive integer  $n$ ,  $L_{r_{n,p} B_p^n}$ , say, tends to infinity when  $p$  tends to zero; and due to Lemma 2.2 and direct calculation, we have

**Proposition 2.6.** *Let  $0 < p < 1$ . Then,*

$$L_{B_p^n} \geq 1/(\sqrt{2}e),$$

where the equality occurs when  $p = 1$  and  $n$  tends to infinity.

### 3. Estimates for the extremal sections of $\ell_p^n$ -balls

We refer back to [23], for the aim of formulating the equivalence.

**Lemma 3.1.** (See [23].) *Let  $K$  be a symmetric convex body in  $\mathbb{R}^n$ ,  $p > 0$ . Then*

$$\left(\frac{1}{\text{vol}_n(K)} \int_K |\langle x, \xi \rangle|^p dx\right)^{1/p} \geq \frac{\text{vol}_n(K)}{\text{vol}_{n-1}(K \cap \xi^\perp)} \frac{1}{2(p + 1)^{1/p}}.$$

The equality occurs for a cylinder in the direction of  $\xi$ .

**Lemma 3.2.** (See [23].) *Let  $K$  be a symmetric convex body in  $\mathbb{R}^n$ ,  $p > 0$  and  $\xi \in S^{n-1}$ . Then*

$$\left(\frac{1}{\text{vol}_n(K)} \int_K |\langle x, \xi \rangle|^p dx\right)^{1/p} \leq \frac{\text{vol}_n(K)}{\text{vol}_{n-1}(K \cap \xi^\perp)} \left(\frac{n}{2}\right) \left(\frac{n!}{(p + 1)(p + 2) \cdots (p + n)}\right)^{1/p}.$$

The equality occurs for a cone based on  $\xi^\perp$  with  $\xi$  as summit.

Lemma 3.2 for  $p = 2$  is due to Hensley [12]. The following can be summarized from the lemmas.

**Lemma 3.3.** Let  $K$  be a symmetric convex body of volume 1 in  $\mathbb{R}^n$  and  $\xi \in S^{n-1}$ ,

$$\frac{1}{2\sqrt{3}} \left( \int_K \langle x, \xi \rangle^2 dx \right)^{-1/2} \leq \text{vol}_{n-1}(K \cap \xi^\perp) \leq \frac{1}{\sqrt{2}} \left( \int_K \langle x, \xi \rangle^2 dx \right)^{-1/2},$$

where equality in the left-hand side holds for a cylinder in the direction of  $\xi$ , and equality in the right-hand side holds for a cone based on  $\xi^\perp$  with  $\xi$  as summit when  $n$  tends to infinity.

**Proof.** It is sufficient to consider the case  $p = 2$ , in Lemmas 3.1 and 3.2. Since

$$\frac{n}{\sqrt{(n+1)(n+2)}} \tag{3.1}$$

is increasing when the positive integer  $n$  is increasing, the maximum of (3.1) occurs as  $n$  tends to infinity. (The lemma also can be derived from Proposition 10 in [3].)  $\square$

Now we complete the proof of our main result.

**Proof of Theorem 1.1.** Write  $r_{n,p} = \text{vol}_n^{-1/n}(B_p^n)$ . Then,  $r_{n,p} B_p^n$  is isotropic, since its volume is 1. Hence,

$$\left( \int_{r_{n,p} B_p^n} \langle x, \xi \rangle^2 dx \right)^{1/2} = L_{r_{n,p} B_p^n} = \left( \frac{\Gamma(1+3/p)\Gamma(1+n/p)^{1+\frac{2}{n}}}{12\Gamma(1+(n+2)/p)\Gamma(1+1/p)^3} \right)^{\frac{1}{2}}$$

followed from Lemma 2.1 and formula (2.1). Therefore, Proposition 2.4 and Lemma 3.3 lead to the inequalities in the theorem. The situation where the equalities occur is determined by Proposition 2.4, Lemma 3.3 and Theorem A.  $\square$

**Lemma 3.4.** (See [13].) For all integers  $n \geq p - 1$ , we have

$$a \text{vol}_{n+1}^{\frac{n}{n+1}}(B_p^{n+1}) \leq \text{vol}_n(B_p^n) < b \text{vol}_{n+1}^{\frac{n}{n+1}}(B_p^{n+1}),$$

where  $a = \frac{p-1}{p} \sqrt{\Gamma(2)}/\Gamma(\frac{p-1}{p} + 1)$ , and  $b = \sqrt[p]{e}$ .

**Proof of Theorem 1.2.** Referring back to Theorem B, we have

$$\text{vol}_{n-1}(B_p^n \cap \xi^\perp) \leq \frac{(2\Gamma(1+1/p))^{n-1}}{\Gamma(1+(n-1)/p)} = \text{vol}_{n-1}(B_p^{n-1}), \tag{3.2}$$

for  $0 < p \leq 2$ , and every  $\xi \in S^{n-1}$ . Then, the theorem is followed from (3.2) and Lemma 3.4.  $\square$

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