DIGITAL EXPANSIONS WITH RESPECT TO DIFFERENT BASES[1]

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ABSTRACT. We consider the g-ary expansion $N = \sum_k b_k(N,g)g^k$ of non-negative integers N and prove various results on the distribution and the mean value of the k-th digit $b_k(N,g)$ if g varies in an interval of the form $2 \le g \le N^\eta$. As an application we also consider the average value of the sum-of-digits function $s(N,g) = \sum_k b_k(N,g)$.

1. Introduction

Let $g \ge 2$ be an given integer. Then every non-negative integer N can be uniquely represented in its g-ary expansion

$$N = \sum_{k\ge 0} b_k(N,g)g^k \tag{1.1}$$

with digits

$$b_k(N,g) \in \{0,1,\dots,g-1\}.$$

It is an easy exercise to show that

$$b_k(N,g) = \left[g \left\{ \frac{N}{g^{k+1}} \right\} \right], \tag{1.2}$$

where $[x]$ is the integer value of x, i.e. $[x] = \max\{k \in \mathbb{Z} : k \le x\}$, and $\{x\} = x - [x]$ denotes the fractional part of x. In other words, we have

$$b_k(N,g) = b \Longleftrightarrow \left\{ \frac{N}{g^{k+1}} \right\} \in \left[\frac{b}{g}, \frac{b+1}{g} \right). \tag{1.3}$$

The g-ary *sum-of-digits function* $s(N,g)$ is defined by

$$s(N,g) = \sum_{k\ge 0} b_k(n,g). \tag{1.4}$$

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It is well known that the *average value* of the k-th digit is given by

$$\frac{1}{N}\sum_{n<N} b_k(n,g) = \frac{g-1}{2} + \mathcal{O}\left(\frac{1}{N}\right) \tag{1.5}$$

as $N\to\infty$, and similarly for the sum-of-digits function:[1]

$$\frac{1}{N}\sum_{n<N} s(n,g) = \frac{g-1}{2}\frac{\log N}{\log g} + \mathcal{O}\left(1\right). \tag{1.6}$$

The purpose of this paper is to study the average value of the k-th digit and the sum-of-digits function from a different point of view. We want to consider the average values

$$\frac{1}{G}\sum_{g=2}^{G} b_k(N,g) \tag{1.7}$$

and

$$\frac{1}{G}\sum_{g=2}^{G} s(N,g), \tag{1.8}$$

where $G=G(N)\ge N^\eta$ for some $\eta>0$, and the distribution function

$$\frac{1}{G}\#\left\{2\le g\le G : \frac{b_k(N,g)}{g}\le x\right\}.$$

of the normalized digits $b_k(N,g)/g$. The main tools we use are exponential sums and ψ-sums (where $\psi(x)=x-[x]-\frac12$ denotes the first Bernoulli polynomial).

2. Results

2.1. Distribution of the k-th digit. We first consider the distribution of the k-th digit $b_k(N,g)$. For this purpose set

$$A_k(G,N;a,b) = \#\left\{2\le g\le G : b_k(N,g)/g\in[a,b)\right\}, \tag{2.1}$$

where $0\le a<b\le 1$. Note that $b_k(N,g)=0$ if $g>N^{\frac{1}{k}}$. Thus it is sufficient to consider the case $g\le N^{\frac1k}$. Furthermore, it turns out that there is another threshold, namely if $g\approx N^{\frac{1}{k+1}}$. The situation is especially easy if g is smaller than $N^{\frac{1}{k+1}}$. Here the (normalized) digits are uniformly distributed in $[0,1]$.

Theorem 1. *For every integer $k\ge 0$ and for every $\varepsilon>0$ there exists $\eta>0$ such that for G with $N^\varepsilon\le G\le N^{\frac{1}{k+1}-\varepsilon}$*

$$A_k(G,N;a,b) = (b-a)G + \mathcal{O}\left(G^{1-\eta}\right) \tag{2.2}$$

uniformly for $0\le a<b\le 1$.

Remark 1. It is possible to make η explicit in terms of ε and k.

The remaining case is covered by the following theorem.

[1]The $\mathcal{O}(1)$-term is in this formula is exactly given by a periodic continuous and nowhere differentiable function function $\Phi(\log N)$, see [1].

Theorem 2. *Suppose that* $N^{\frac{1}{k+1}} \le G \le N^{\frac{1}{k}}$. *Then we have*

$$A_k(G,N;a,b) = N^{\frac{1}{k+1}}\left(\zeta\left(\frac{1}{k+1},a\right) - \zeta\left(\frac{1}{k+1},b\right) - a^{-\frac{1}{k+1}} + b^{-\frac{1}{k+1}}\right) + \min\left\{G, \left(\frac{N}{a}\right)^{\frac{1}{k+1}}\right\} - \min\left\{G, \left(\frac{N}{b}\right)^{\frac{1}{k+1}}\right\} + \mathcal{O}\left(N^{\frac{1}{k+2}}\right), \tag{2.3}$$

where $\zeta(s,a)$ *denotes the (analytically continued) Hurwitz* ζ*-function Furthermore, for* $N^{\frac{1}{k+2}} \le G \le N^{\frac{1}{k+1}}$ *we have*

$$A_k(G,N;a,b) = (b-a)G + \mathcal{O}\left(\frac{G^{k+2}}{N}\right) + \mathcal{O}\left(N^{\frac{1}{k+2}}\right) \tag{2.4}$$

2.2. **The k-th digit.** We now turn to the average value of the k-th digit. The first theorem is directly implied by Theorem 1.

Theorem 3. *For every integer* $k \ge 0$ *and for every* $\varepsilon > 0$ *there exists an* $\eta > 0$ *such that for* G *with* $N^{\varepsilon} \le G \le N^{\frac{1}{k+1}-\varepsilon}$

$$\sum_{g=2}^{G} \frac{1}{g} b_k(N,g) = \frac{G}{2} + \mathcal{O}\left(G^{1-\eta}\right) \tag{2.5}$$

and

$$\sum_{g=2}^{G} b_k(N,g) = \frac{G^2}{4} + \mathcal{O}\left(G^{2-\eta}\right). \tag{2.6}$$

As above it is possible to make η explicit in terms of ε and k.

For the region of the *threshold* we have to be more precise.

The first range is $N^{\frac{1}{k+2}} \le G \le N^{\frac{1}{k+1}}$.

Theorem 4. *Suppose that* $k \ge 0$ *and* $N^{\frac{1}{k+2}} \le G \le N^{\frac{1}{k+1}}$

$$\sum_{g=2}^{G} \frac{1}{g} b_k(N,g) = \frac{G}{2} + \frac{G}{k+1}\int_1^{\infty} \psi\left(\frac{N}{G^{k+1}}x\right) x^{-1-\frac{1}{k+1}}\,dx + \mathcal{O}\left(N^{\frac{1}{k+3}}\right). \tag{2.7}$$

Theorem 5. *Suppose that* $N^{\frac{1}{2}} \le G \le N$. *Then*

$$\sum_{g=2}^{G} b_0(N,g) = \frac{G^2}{4} + G^2\int_1^{\infty} \psi\left(\frac{N}{G}x\right) x^{-3}\,dx + \mathcal{O}\left(N(\log N)^{\frac{2}{3}}\right). \tag{2.8}$$

For $N^{\frac{1}{3}} \le G \le N^{\frac{1}{2}}$ *we have*

$$\sum_{g=2}^{G} b_1(N,g) = \frac{G^2}{4} + \frac{G^2}{2}\int_1^{\infty} \psi\left(\frac{N}{G^2}x\right) x^{-2}\,dx + \mathcal{O}\left(N^{\frac{3}{5}}\right). \tag{2.9}$$

Finally, if $k \ge 2$ and $N^{\frac{1}{k+2}} \le G \le N^{\frac{1}{k+1}}$

$$\sum_{g=2}^{G} b_k(N,g) = \frac{G^2}{4} + \frac{G^2}{k+1}\int_1^\infty \psi\left(\frac{N}{G^{k+1}}x\right) x^{-1-\frac{2}{k+1}}\,dx + \mathcal{O}\left(N^{\frac{1}{k+2}+\frac{1}{k+3}}\right). \tag{2.10}$$

The second range is $N^{\frac{1}{k+1}} \le G \le N^{\frac{1}{k}}$.

Theorem 6. *Suppose that $G \ge N$. Then*

$$\sum_{g=2}^{G} \frac{1}{g} b_0(N,g) = (1-\gamma)N + N\log\frac{G}{N} + \mathcal{O}\left(N^{\frac{1}{3}}\log N\right), \tag{2.11}$$

where γ denotes Euler's constant. Furthermore for $k \ge 1$ and $N^{\frac{1}{k+1}} \le G \le N^{\frac{1}{k}}$

$$\sum_{g=2}^{G} \frac{1}{g} b_k(N,g) = -\zeta\left(\frac{1}{k+1}\right) N^{\frac{1}{k+1}} + \mathcal{O}\left(N^{\frac{1}{k+3}}\right). \tag{2.12}$$

Theorem 7. *Suppose that $G \ge N$. Then*

$$\sum_{g=2}^{G} b_0(N,g) = NG - \frac{\pi^2}{12}N^2 + \mathcal{O}\left(N(\log N)^{\frac{2}{3}}\right). \tag{2.13}$$

For $N^{\frac{1}{2}} \le G \le N$ we have

$$\sum_{g=2}^{G} b_1(N,g) = N\log\frac{G}{\sqrt{N}} - G\left(\frac{1}{2} + \int_1^\infty \psi\left(\frac{Nx}{G}\right) x^{-2}\,dx\right) + \frac{1-\gamma}{2}N + \mathcal{O}\left(N^{\frac{3}{5}}\right). \tag{2.14}$$

Finally, if $k \ge 2$ and $N^{\frac{1}{k+1}} \le G \le N^{\frac{1}{k}}$

$$\sum_{g=2}^{G} b_k(N,g) = -\frac{1}{2}\zeta\left(\frac{2}{k+1}\right) N^{\frac{2}{k+1}} - \frac{N}{(k-1)G^{k-1}} \tag{2.15}$$

$$+ \mathcal{O}\left(N^{\frac{1}{k+2}+\frac{1}{k+3}}\right) + \mathcal{O}\left(N^{\frac{1}{k}}\right). \tag{2.16}$$

Overall, we get the following picture. If $G/N^{\frac{1}{k+1}} \to \infty$ (and $G \le N^{\frac{1}{k}}$) then

$$\sum_{g=2}^{G} b_k(N,g) \sim -\frac{1}{2}\zeta\left(\frac{2}{k+1}\right) N^{\frac{2}{k+1}}. \tag{2.17}$$

whereas if $G/N^{\frac{1}{k+1}} \to 0$ (and $G \ge N^{\varepsilon}$) then

$$\sum_{g=2}^{G} b_k(N,g) \sim \frac{G^2}{4} \tag{2.18}$$

2.3. **The sum-of-digits function.** We now turn to the average of the sum-of-digits function. As usual $\psi(x) = x - [x] - \frac{1}{2}$ denotes the first Bernoulli polymial and $\zeta(s)$ the Riemann ζ-function.

Theorem 8. *Suppose that* $N^{\frac{1}{2}} \le G \le N$. *Then*

$$\sum_{g=2}^{G} \frac{1}{g} s(N,g) = G\left(\frac{1}{2} + \int_1^\infty x^{-2}\psi\left(\frac{Nx}{G}\right)\,dx\right) - \zeta\left(\frac{1}{2}\right)\sqrt{N} + \mathcal{O}\left(N^{1/3}\log N\right). \tag{2.19}$$

Furthermore, if $N^{\frac{1}{L+1}} \le G \le N^{\frac{1}{L}}$ *for some* $L \ge 2$ *then there exists* $\eta_L > 0$ *such that*

$$\sum_{g=2}^{G} \frac{1}{g} s(N,g) = L\frac{G}{2} + \frac{G}{L}\int_1^\infty \psi\left(\frac{N}{G^L}x\right) x^{-1-\frac{2}{L}}\,dx + \mathcal{O}\left(G^{1-\eta_L}\right) \tag{2.20}$$

Theorem 9. *Suppose that* $N^{\frac{1}{2}} \le G \le N$. *Then*

$$\sum_{g=2}^{G} s(N,g) = \frac{G^2}{4} + G^2\int_1^\infty \psi\left(\frac{N}{G}x\right) x^{-3}\,dx - \frac{1}{2}N\log N + N\log G + \mathcal{O}\left(N(\log N)^{\frac{2}{3}}\right). \tag{2.21}$$

Furthermore, if $N^{\frac{1}{L+1}} \le G \le N^{\frac{1}{L}}$ *for some* $L \ge 2$ *then there exists* $\eta_L > 0$ *such that*

$$\sum_{g=2}^{G} s(N,g) = L\frac{G^2}{4} + \frac{G^2}{L}\int_1^\infty \psi\left(\frac{N}{G}x\right) x^{-1-\frac{2}{L}}\,dx - \frac{1}{2}\zeta\left(\frac{2}{L+1}\right)N^{\frac{2}{L+1}} - \frac{N}{(L-1)G^{L-1}} + \mathcal{O}\left(G^{2-\eta_L}\right). \tag{2.22}$$

Remark 2. For $G = N$ and $G = N^{\frac{1}{2}}$ we especially have

$$\sum_{g=2}^{N} s(N,g) = \left(1 - \frac{\pi^2}{12}\right)N^2 - N\log N + \mathcal{O}\left(N(\log N)^{\frac{2}{3}}\right)$$

and

$$\sum_{g=2}^{N^{\frac{1}{2}}} s(N,g) = \frac{1}{2}\left(\gamma - \frac{1}{2}\right)N + \mathcal{O}\left(N^{\frac{9}{10}}\right),$$

where γ denotes Euler's constant.

3. Exponential Sums

The proof of Theorem 1 is based on estimates of exponential sums of the form $\sum e(N/g^{k+1})$ which are collected in this section. (As usual we use the notation $e(x) = e^{2\pi i x}$.)

We have to combine several methods. Lemma 1 relies on Van der Corput's method whereas Lemma 2 on exponential pairs. (Alternatively we can also use a method of Walfisz, see Lemma 3.)

Lemma 1. *Let $k \ge 0$ be given. Then we have*

$$\sum_{g\le G} e\left(\frac{N}{g^{k+1}}\right) \ll N^{\frac{1}{k+3}} + N^{-\frac12} G^{\frac{k+3}{2}}. \tag{3.1}$$

Proof. We apply Van der Corput's theorem [5, p. 31] saying that

$$\sum_{a<n\le b} e(f(n)) \ll \frac{|f'(b)-f'(a)|+1}{\min\limits_{a\le x\le b} |f''(x)|^{1/2}}$$

for every twice continuously differentiable function $f(x)$. For $f(x) = N/x^{k+1}$ we obtain

$$\sum_{G<g\le G'} e\left(\frac{N}{g^{k+1}}\right) \ll N^{\frac12} G^{-\frac{k+1}{2}} + N^{-\frac12} G^{\frac{k+3}{2}}$$

uniformly for all G' with $G < G' \le 2G$. Thus, if $G \ge N^{\frac{1}{k+3}}$ we get

$$\begin{aligned}\sum_{g\le G} e\left(\frac{N}{g^{k+1}}\right) &\ll N^{\frac{1}{k+3}} + \sum_{l=0}^{L}\left(N^{\frac12}\left(2^l N^{\frac{1}{k+3}}\right)^{-\frac{k+1}{2}} + N^{-\frac12}\left(2^l N^{\frac{1}{k+3}}\right)^{\frac{k+3}{2}}\right)\\ &\ll N^{\frac{1}{k+3}} + N^{\frac12-\frac{k+1}{2(k+3)}} + N^{-\frac12}\left(2^L N^{\frac{1}{k+3}}\right)^{\frac{k+3}{2}}\\ &\ll N^{\frac{1}{k+3}} + N^{-\frac12} G^{\frac{k+3}{2}},\end{aligned}$$

where L is the maximal such that $2^L N^{\frac{1}{k+3}} \le G$.

Finally, if $G < N^{\frac{1}{k+3}}$ then (3.1) is trivial. □

Lemma 2. *Let $k \ge 0$ be given, let $r \ge 1$ an arbitrary integer. Then we have*

$$\sum_{g\le G} e\left(\frac{N}{g^{k+1}}\right) \ll \begin{cases} N^{\frac{1}{2^{r+1}-2}} G^{1-\frac{r+k+2}{2^{r+1}-2}} & \text{if } k < 2^{r+1}-r-4,\\ N^{\frac{1}{r+k+2}} \log N & \text{if } k = 2^{r+1}-r-4,\\ N^{\frac{1}{r+k+2}} & \text{if } k > 2^{r+1}-r-4\end{cases} \tag{3.2}$$

uniformly for

$$N^{\frac{1}{r+k+2}} \le G \le N^{\frac{1}{k+2}}. \tag{3.3}$$

Proof. We apply the method of exponential pairs (see [5, p. 52]) again with the function $f(x) = N/x^{k+1}$ and directly obtain (with $z = N/G^{k+2} \ge 1$ and $a = G$, compare also with the proof of Lemma 5.11 [5, p. 223])

$$\sum_{G<g\le G'} e\left(\frac{N}{g^{k+1}}\right) \ll z^{\kappa} a^{\lambda} = N^{\kappa} G^{\lambda-\kappa(k+2)},$$

where (κ,λ) is any exponential pair and $G \le G' \le 2G$. Especially, if we use the pair (compare with [5, p. 59])

$$(\kappa,\lambda) = \left(\frac{1}{2^{r+1}-2}, 1-\frac{r}{2^{r+1}-2}\right)$$

we obtain

$$\sum_{G<g\le G'} e\left(\frac{N}{g^{k+1}}\right) \ll N^{\frac{1}{2^{r+1}-2}} G^{1-\frac{r+k+2}{2^{r+1}-2}},$$

which is non-trivial if $G \geq N^{\frac{1}{r+k+2}}$. First assume that $k < 2^{r+1} - r - 4$ or equivalently

$$1 - \frac{r+k+2}{2^{r+1}-2} > 0.$$

Then we have

$$\begin{aligned}
\sum_{g \leq G} e\left(\frac{N}{g^{k+1}}\right) &\ll N^{\frac{1}{r+k+2}} + \sum_{l=0}^{L} \left(N^{\frac{1}{2^{r+1}-2}} \left(2^l N^{\frac{1}{r+k+2}}\right)^{1-\frac{r+k+2}{2^{r+1}-2}} \right) \\
&\ll N^{\frac{1}{r+k+2}} + N^{\frac{1}{r+k+2}} \left(2^L N^{\frac{1}{r+k+2}}\right)^{1-\frac{r+k+2}{2^{r+1}-2}} \\
&\ll N^{\frac{1}{r+k+2}} + N^{\frac{1}{2^{r+1}-2}} G^{1-\frac{r+k+2}{2^{r+1}-2}} \\
&\ll N^{\frac{1}{2^{r+1}-2}} G^{1-\frac{r+k+2}{2^{r+1}-2}},
\end{aligned}$$

where L is maximal such that $2^L N^{\frac{1}{r+k+2}} \leq G$.

The cases $k = 2^{r+1} - r - 4$ and $k > 2^{r+1} - r - 4$ can be worked out in the same way. □

The method of exponential pairs is thus quite easy to apply. The disadvantage is that the constant implied by $\ll$ depends on k and r in a non-explicit way. We therefore also present a result which can be obtained by a method of Walfisz [6] and provides estimates which are uniform in k and r.

Lemma 3. *There exists a real constant $c > 0$ such that for all integers $k \geq 0, r \geq 1$ with $k < r$ and $(k, r) \neq (0, 1)$ we have*

$$\sum_{g \leq G} e\left(\frac{N}{g^{k+1}}\right) \leq c N^{\frac{1}{(r+1)2^r - 1}} G^{1-\frac{r+k+2}{(r+1)2^r-1}} \log N \tag{3.4}$$

uniformly for

$$8kN^{\frac{1}{r+k+2}} \leq G \leq N^{\frac{2}{r+2k+3}}. \tag{3.5}$$

Proof. *(Sketch)* The major step is to prove that

$$\sum_{G < g \leq G'} e\left(\frac{N}{g^{k+1}}\right) \leq c N^{\frac{1}{(r+1)2^r - 1}} G^{1-\frac{r+k+2}{(r+1)2^r-1}} \log N \tag{3.6}$$

uniformly for

$$8kN^{\frac{1}{r+k+2}} \leq G \leq G' \leq 2G \leq 2N^{\frac{2}{r+2k+3}}.$$

Is is clear that (3.4) follows from (3.6) as in the previous two proofs.

In order to prove (3.6) one just has to repeat (and generalize) the proof of Satz 1 [6, p. 22] for $k \geq 0$ instead of $k = 0$. □

4. Proof of Theorem 1

We first consider the case $N^{\frac{1}{k+3}} \leq G \leq N^{\frac{1}{k+1}}$.

Lemma 4. *Suppose that $N^{\frac{1}{k+3}} \leq G \leq N^{\frac{1}{k+1}}$. Then*

$$\frac{A_k(G, N; a, b)}{G} = b - a + \mathcal{O}\left(N^{\frac{1}{k+4}} G^{-\frac{k+3}{k+4}} + N^{-\frac{1}{2}} G^{\frac{k+1}{2}}\right) \tag{4.1}$$

uniformly for $0 \leq a < b \leq 1$.

Proof. Let us consider the numbers

$$\tilde{A}_k(G,N;a,b) = \#\left\{2 \le g \le G : \left\{\frac{N}{g^{k+1}}\right\} \in [a,b)\right\}$$

By Erdős-Turan's inequality (see [3] or [2]) and Lemma 1 we obtain for every integer $H > 0$

$$\begin{aligned}\Delta := \sup_{0\le a<b\le 1}\left|\frac{\tilde{A}_k(G,N;a,b)}{G} - (b-a)\right| &\ll \frac{1}{H} + \sum_{h=1}^{H}\frac{1}{h}\left|\frac{1}{G}\sum_{g\le G} e\left(\frac{N}{g^{k+1}}\right)\right| \\ &\ll \frac{1}{H} + \sum_{h=1}^{H}\frac{1}{h}\left((hN)^{\frac{1}{k+3}}G^{-1} + (hN)^{-\frac{1}{2}}G^{\frac{k+1}{2}}\right) \\ &\ll \frac{1}{H} + H^{\frac{1}{k+3}}N^{\frac{1}{k+3}}G^{-1} + N^{-\frac{1}{2}}G^{\frac{k+1}{2}}.\end{aligned}$$

Choosing

$$H = \left[N^{-\frac{1}{k+4}}G^{\frac{k+3}{k+4}}\right]$$

we get

$$\Delta \ll N^{\frac{1}{k+4}}G^{-\frac{k+3}{k+4}} + N^{-\frac{1}{2}}G^{\frac{k+1}{2}}. \tag{4.2}$$

Note that

$$0 \le \frac{b_k(N,g)}{g} - \left\{\frac{N}{g^{k+1}}\right\} \le \frac{1}{g}.$$

Thus, by considering the cases $2 \le g \le G_1$ and $G_1 < g \le G$ it follows that

$$\tilde{A}_k(G,N;a,b-G_1^{-1}) - 2G_1 \le A_k(G,N;a,b) \le \tilde{A}_k(G,N;a,b+G_1^{-1}) + 2G_1. \tag{4.3}$$

Consequently, by using (4.3) with $G_1 = \sqrt{G}$, (4.1) follows from (4.2). □

Similarly we can treat the case $N^{\frac{1}{k+r+2}} \le G \le N^{\frac{1}{k+2}}$.

Lemma 5. *Let $r \ge 1$ be a given integer and $k < r$. If $N^{\frac{1}{k+r+2}} \le G \le N^{\frac{1}{k+2}}$ then*

$$\frac{A_k(G,N;a,b)}{G} = b - a + \mathcal{O}\left(N^{-\frac{1}{2^{r+1}-1}}G^{\frac{r+k+2}{2^{r+1}-1}}\right) \tag{4.4}$$

uniformly for $0 \le a < b \le 1$.

Proof. By using Lemma 2 instead of Lemma 1 we get

$$\begin{aligned}\Delta &\ll \frac{1}{H} + \sum_{h=1}^{H}\frac{1}{h}\left((hN)^{\frac{1}{2^{r+1}-2}}G^{-\frac{r+k+2}{2^{r+1}-2}}\right) \\ &\ll \frac{1}{H} + H^{\frac{1}{2^{r+1}-2}}N^{\frac{1}{2^{r+1}-2}}G^{-\frac{r+k+2}{2^{r+1}-2}}.\end{aligned}$$

Choosing

$$H = \left[N^{-\frac{1}{2^{r+1}-1}}G^{\frac{r+k+2}{2^{r+1}-1}}\right]$$

one directly obtains

$$\Delta \ll N^{\frac{1}{2^{r+1}-1}}G^{-\frac{r+k+2}{2^{r+1}-1}}.$$

Now we can proceed as in the proof of Lemma 4 and complete the proof of Lemma 5. □

Obviously, a combination of Lemma 4 and Lemma 5 proves Theorem 1.

5. Proof of Theorem 2

For notational convenience we set (as above)

$$\tilde{A}_k(G,N;a,b) = \#\left\{2 \le g \le G : \left\{\frac{N}{g^{k+1}}\right\} \in [a,b)\right\}$$

and start with the following observation.

Lemma 6. *Suppose that $N^{\frac{1}{k+2}} \le G \le N^{\frac{1}{k}}$. Then we have uniformly for $0 \le a < b \le 1$*

$$\tilde{A}_k(G,N;a,b) = N^{\frac{1}{k+1}} \sum_{l>l_0} \left((a+l)^{-\frac{1}{k+1}} - (b+l)^{-\frac{1}{k+1}}\right) + \min\left\{G, \left(\frac{N}{a+l_0}\right)^{\frac{1}{k+1}}\right\} - \min\left\{G, \left(\frac{N}{b+l_0}\right)^{\frac{1}{k+1}}\right\} + \mathcal{O}\left(N^{\frac{1}{k+2}}\right),$$

where $l_0 = \left[NG^{-k-1}\right]$.

Proof. Set $G_1 = N^{\frac{1}{k+2}}$. Assume first that $G_1 \le N/G^{k+1}$. Then $G = \mathcal{O}(G_1)$ and hence $\tilde{A}_k(G,N;a,b) = \mathcal{O}(G_1)$. On the other hand the right hand side above is given by

$$\begin{aligned}
&\mathcal{O}\left(N^{\frac{1}{k+1}} \sum_{l \ge G_1} \left((a+l)^{-\frac{1}{k+1}} - (b+l)^{-\frac{1}{k+1}}\right) + G_1\right) \\
&= \mathcal{O}\left(N^{\frac{1}{k+1}} \sum_{l \ge G_1} l^{-1-\frac{1}{k+1}} + G_1\right) \\
&= \mathcal{O}\left(N^{\frac{1}{k+1}} G_1^{-\frac{1}{k+1}} + G_1\right) = \mathcal{O}(G_1).
\end{aligned}$$

Next assume that $G_1 > N/G^{k+1}$. Then

$$\begin{aligned}
&\tilde{A}_k(G,N;a,b) \\
&= \#\left\{G_1 < g \le G : \left\{\frac{N}{g^{k+1}}\right\} \in [a,b)\right\} + \mathcal{O}(G_1) \\
&= \sum_{l \ge 0} \#\left\{G_1 < g \le G : \left(\frac{N}{b+l}\right)^{\frac{1}{k+1}} < g \le \left(\frac{N}{a+l}\right)^{\frac{1}{k+1}}\right\} + \mathcal{O}(G_1).
\end{aligned}$$

If $l < l_0$ or if $l > [G_1] + 1$ then there is no contribution. Next observe that for $l = [G_1]$ and for $l = [G_1] + 1$ the corresponding summand is bounded by

$$\mathcal{O}\left(\left(\frac{N}{G_1}\right)^{\frac{1}{k+1}}\right) = \mathcal{O}(G_1).$$

For $l = l_0$ we have

$$\min\left\{G, \left(\frac{N}{a+l_0}\right)^{\frac{1}{k+1}}\right\} - \min\left\{G, \left(\frac{N}{b+l_0}\right)^{\frac{1}{k+1}}\right\} + \mathcal{O}\left(G_1\right).$$

Finally, the remaining sum for $l_0 < l < G_1$ is given by

$$\begin{aligned}
&\sum_{l_0<l<G_1} \left(\left(\frac{N}{a+l}\right)^{\frac{1}{k+1}} - \left(\frac{N}{b+l}\right)^{\frac{1}{k+1}}\right) \\
&= N^{\frac{1}{k+1}} \sum_{l>l_0} \left((a+l)^{-\frac{1}{k+1}} - (b+l)^{-\frac{1}{k+1}}\right) + \mathcal{O}\left(G_1\right),
\end{aligned}$$

which completes the proof of the lemma. □

Now it is easy to complete the proof of Theorem 2.

Proof. (*Theorem 2*) First suppose that $N^{\frac{1}{k+1}} < G \le N^{\frac{1}{k}}$. Here we have $l_0 = 0$

$$\begin{aligned}
\tilde{A}_k(G,N;a,b) &= N^{\frac{1}{k+1}} \sum_{l\ge 1} \left((a+l)^{-\frac{1}{k+1}} - (b+l)^{-\frac{1}{k+1}}\right) \\
&\quad + \min\left\{G, \left(\frac{N}{a}\right)^{\frac{1}{k+1}}\right\} - \min\left\{G, \left(\frac{N}{b}\right)^{\frac{1}{k+1}}\right\} + \mathcal{O}\left(N^{\frac{1}{k+2}}\right) \\
&= N^{\frac{1}{k+1}} \left(\zeta\left(\frac{1}{k+1}, a\right) - \zeta\left(\frac{1}{k+1}, b\right) - a^{-\frac{1}{k+1}} + b^{-\frac{1}{k+1}}\right) \\
&\quad + \min\left\{G, \left(\frac{N}{a}\right)^{\frac{1}{k+1}}\right\} - \min\left\{G, \left(\frac{N}{b}\right)^{\frac{1}{k+1}}\right\} + \mathcal{O}\left(N^{\frac{1}{k+2}}\right).
\end{aligned}$$

By using (4.3) with $G_1 = N^{\frac{1}{k+2}}$ the first part of Theorem 2, i.e. (2.3), follows.

Finally for $N^{\frac{1}{k+2}} \le G \le N^{\frac{1}{k+1}}$ we observe that

$$\begin{aligned}
&\sum_{l\ge NG^{-k-1}} \left((a+l)^{-\frac{1}{k+1}} - (b+l)^{-\frac{1}{k+1}}\right) \\
&= \sum_{l\ge NG^{-k-1}} \frac{b-a}{(k+1)l^{\frac{k+2}{k+1}}} + \mathcal{O}\left(N^{-\frac{k+2}{k+1}} G^{k+2}\right) \\
&= \frac{b-a}{k+1} \int_{N/G^{k+1}}^{\infty} x^{-\frac{k+2}{k+1}}\, dx + \mathcal{O}\left(N^{-\frac{k+2}{k+1}} G^{k+2}\right) \\
&= (b-a) G N^{-\frac{1}{k+1}} + \mathcal{O}\left(N^{-\frac{k+2}{k+1}} G^{k+2}\right).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\tilde{A}_k(G,N;a,b) &= N^{\frac{1}{k+1}} \sum_{l\ge NG^{-k-1}} \left((a+l)^{-\frac{1}{k+1}} - (b+l)^{-\frac{1}{k+1}}\right) \\
&\quad + \mathcal{O}\left(N^{\frac{1}{k+2}}\right) + \mathcal{O}\left(G^{k+2}N^{-1}\right) \\
&= (b-a)G + \mathcal{O}\left(N^{\frac{1}{k+2}}\right) + \mathcal{O}\left(G^{k+2}N^{-1}\right).
\end{aligned}$$

By another application of (4.3) with $G_1 = N^{\frac{1}{k+2}}$ we directly get (2.4) and the proof of Theorem 2 is completed. □

6. ψ-Sums

We now use another representation for the k-digit:

$$b_k(N,g) = \left[\frac{N}{g^k}\right] - g\left[\frac{N}{g^{k+1}}\right].$$

The advantage of this representation is that sums over $b_k(N,g)$ can be represented with help of the ψ-function $\psi(x) = \{x\} - \frac{1}{2} = x - [x] - \frac{1}{2}$ which are very well studied in the literature (see for example [5, 6]).

We start with an easy observation.

Lemma 7. *Suppose that $N^{\frac{1}{k+1}} \le G \le N^{\frac{1}{k}}$ and that $(f(g))_{1\le g\le G}$ is a sequence of complex numbers. Then we have*

$$\sum_{g=1}^{G} f(g)\left[\frac{N}{g^k}\right] = \left[\frac{N}{G^k}\right]\sum_{g=1}^{G} f(g) + \sum_{\frac{N}{G^k} < d \le N^{\frac{1}{k+1}}} \sum_{1\le g\le (\frac{N}{d})^{1/k}} f(g)$$
$$+ \sum_{g\le N^{\frac{1}{k+1}}} f(g)\left(\left[\frac{N}{g^k}\right] - \left[N^{\frac{1}{k+1}}\right]\right).$$

Proof. The left hand side is given by

$$\sum_{g=1}^{G} f(g)\left[\frac{N}{g^k}\right] = \sum_{dg^k\le N, g\le G} f(g)$$
$$= \sum\nolimits'_{dg^k\le N, d\le g\le G} f(g) + \sum\nolimits'_{dg^k\le N, g\le d, g\le G} f(g),$$

where Σ' means that terms with $d = k$ are counted with a factor $\frac{1}{2}$. We further have

$$\sum_{d^{k+1}\le N} \sum\nolimits'_{d\le g\le G, g\le(\frac{N}{d})^{1/k}} f(g) + \sum_{g^{k+1}\le N} f(g) \sum\nolimits'_{g\le d\le \frac{N}{g^k}} 1$$
$$= \sum_{d\le \frac{N}{G^k}} \sum\nolimits'_{d\le g\le G} f(g) + \sum_{\frac{N}{G^k}<d\le N^{\frac{1}{k+1}}} \sum\nolimits'_{d\le g\le(\frac{N}{d})^{1/k}} f(g) + \sum_{g^{k+1}\le N} f(g)\left(\left[\frac{N}{g^k}\right] - g + \frac{1}{2}\right)$$
$$= \sum_{d\le \frac{N}{G^k}} \left(\sum_{g\le G} f(g) - \sum_{g=1}^{d} f(g) + \frac{f(d)}{2}\right)$$
$$+ \sum_{\frac{N}{G^k}<d\le N^{\frac{1}{k+1}}} \left(\sum_{g\le(\frac{N}{d})^{1/k}} f(g) - \sum_{g=1}^{d} f(g) + \frac{f(d)}{2}\right)$$
$$+ \sum_{g^{k+1}\le N} f(g)\left(\left[\frac{N}{g^k}\right] - g + \frac{1}{2}\right)$$

$$= \left[\frac{N}{G^k}\right] \sum_{g=1}^{G} f(g) + \sum_{\frac{N}{G^k} < d \le N^{\frac{1}{k+1}}} \sum_{g \le (\frac{N}{d})^{1/k}} f(g) - \sum_{d^{k+1} \le N} \sum_{g=1}^{d} f(g)$$

$$+ \sum_{d^{k+1} \le N} f(d) \left(\left[\frac{N}{d^k}\right] - d + 1 \right)$$

$$= \left[\frac{N}{G^k}\right] \sum_{g=1}^{G} f(g) + \sum_{\frac{N}{G^k} < d \le N^{\frac{1}{k+1}}} \sum_{g \le (\frac{N}{d})^{1/k}} f(g)$$

$$+ \sum_{d^{k+1} \le N} f(d) \left(\left[\frac{N}{d^k}\right] - d + 1 - \sum_{d \le g \le N^{\frac{1}{k+1}}} 1 \right)$$

which proves the lemma. □

In what follows we will make use of the following abbreviations:

$$R(k,N) := \sum_{g^{k+1} \le N} \psi\left(\frac{N}{g^k}\right),$$

$$R_0(k,N) := \sum_{g^{k+1} \le N} \frac{1}{g} \psi\left(\frac{N}{g^k}\right),$$

$$R^0(k,N) := \sum_{g^{k+1} \le N} g\psi\left(\frac{N}{g^k}\right),$$

$$S(k,N,G) := \sum_{\frac{N}{G^k} < d \le N^{\frac{1}{k+1}}} \psi\left(\left(\frac{N}{d}\right)^{1/k}\right),$$

$$S^0(k,N,G) := \sum_{\frac{N}{G^k} < d \le N^{\frac{1}{k+1}}} \left(\frac{N}{d}\right)^{1/k} \psi\left(\left(\frac{N}{d}\right)^{1/k}\right).$$

The next lemma lists some properties which will be needed.

Lemma 8. *Suppose that* $N^{\frac{1}{k+1}} \le G \le N^{\frac{1}{k}}$. *Then*

$$\begin{aligned}\sum_{g\le G} g\left[\frac{N}{g^k}\right] &= N\zeta(k-1)+\frac{1}{2}NG^{1-k}-\frac{N}{k-2}G^{2-k}\\ &- G^2\left(\frac{1}{4}+\frac{1}{k}\int_1^\infty x^{-1-\frac{2}{k}}\psi\left(\frac{Nx}{G^k}\right)dx\right)\\ &- S^0(k,N,G)-R^0(k,N)+\mathcal{O}\left(N^{\frac{1}{k}}\right) \qquad \text{for } k>2,\end{aligned} \tag{6.1}$$

$$\begin{aligned}\sum_{g\le G} g\left[\frac{N}{g^2}\right] &= N(\log G+\gamma)+\frac{1}{2}NG^{-1}-G^2\left(\frac{1}{4}+\frac{1}{2}\int_1^\infty x^{-2}\psi\left(\frac{Nx}{G^2}\right)dx\right)\\ &- S^0(2,N,G)-R^0(2,N)+\mathcal{O}\left(\sqrt{N}\right).\end{aligned} \tag{6.2}$$

$$\sum_{g\le G}\left[\frac{N}{g}\right] = NG-G^2\left(\frac{1}{4}+\int_1^\infty x^{-3}\psi\left(\frac{Nx}{G}\right)dx\right)-S^0(1,N,G)+\mathcal{O}(N) \tag{6.3}$$

$$\begin{aligned}\sum_{g\le G}\left[\frac{N}{g^k}\right] &= N\zeta(k)+\frac{1}{2}NG^{-k}-\frac{N}{k-1}G^{1-k}\\ &- G\left(\frac{1}{2}+\frac{1}{k}\int_1^\infty x^{-1-\frac{1}{k}}\psi\left(\frac{Nx}{G^k}\right)dx\right)\\ &- S(k,N,G)-R(k,N)+\mathcal{O}(k) \qquad \text{for } k>1,\end{aligned} \tag{6.4}$$

$$\begin{aligned}\sum_{g\le G}\left[\frac{N}{g}\right] &= N(\log G+\gamma)+\frac{1}{2}NG^{-1}-G\left(\frac{1}{2}+\int_1^\infty x^{-2}\psi\left(\frac{Nx}{G}\right)dx\right)\\ &- S(1,N,G)+\mathcal{O}(1),\end{aligned} \tag{6.5}$$

$$\sum_{g\le G}\frac{1}{g}\left[\frac{N}{g^k}\right] = N\zeta(k+1)-\frac{1}{2}\log G-\frac{1}{k}NG^{-k}-R_0(k,N)+\mathcal{O}(1). \tag{6.6}$$

Proof. We concentrate on the cas $k>2$. The remaining cases can be proved by obvious modifications. We first prove (6.1) and start with the following calculation:

$$\begin{aligned}\sum_{\frac{N}{G^k}<d\le N^{\frac{1}{k+1}}}\ \sum_{g\le (\frac{N}{d})^{1/k}} g &= \sum_{\frac{N}{G^k}<d\le N^{\frac{1}{k+1}}}\frac{1}{2}\left(\left(\left(\frac{N}{d}\right)^{1/k}-\psi\left(\left(\frac{N}{d}\right)^{1/k}\right)\right)^2-\frac{1}{4}\right)\\ &= \frac{1}{2}\sum_{\frac{N}{G^k}<d\le N^{\frac{1}{k+1}}}\left(\frac{N}{d}\right)^{2/k}-S^0(k,N,G)+\mathcal{O}\left(N^{\frac{1}{k+1}}\right).\end{aligned}$$

Here the first sum can be replaced by

$$\frac{1}{2}\sum_{\frac{N}{G^k}<d\le N^{\frac{1}{k+1}}}\left(\frac{N}{d}\right)^{2/k} = \frac{1}{2}G^2\psi\left(\frac{N}{G^k}\right) - \frac{1}{2}N^{\frac{2}{k+1}}\psi\left(N^{\frac{1}{k+1}}\right)$$

$$+\frac{1}{2}N^{2/k}\int_{NG^{-k}}^{N^{\frac{1}{k+1}}} x^{-2/k}\,dx - \frac{1}{k}N^{2/k}\int_{NG^{-k}}^{N^{\frac{1}{k+1}}} x^{-1-\frac{2}{k}}\psi(x)\,dx.$$

Note that the appearing integrals are given by

$$\int_{NG^{-k}}^{N^{\frac{1}{k+1}}} x^{-2/k}\,dx = \frac{k}{k-2}N^{-2/k}\left(N^{\frac{3}{k+1}} - NG^{2-k}\right)$$

and by

$$\int_{NG^{-k}}^{N^{\frac{1}{k+1}}} x^{-1-\frac{2}{k}}\psi(x)\,dx = \int_{NG^{-k}}^{\infty} x^{-1-\frac{2}{k}}\psi(x)\,dx + \mathcal{O}\left(N^{-\frac{k+2}{k(k+1)}}\right)$$

$$= NG^{-k}\int_1^{\infty}\left(\frac{Nx}{G^k}\right)^{-1-\frac{2}{k}}\psi\left(\frac{Nx}{G^k}\right)\,dx + \mathcal{O}\left(N^{\frac{1}{k+1}-\frac{2}{k}}\right)$$

$$= G^2N^{-2/k}\int_1^{\infty} x^{-1-\frac{2}{k}}\psi\left(\frac{Nx}{G^k}\right)\,dx + \mathcal{O}\left(N^{\frac{1}{k+1}-\frac{2}{k}}\right).$$

Hence we get

$$\sum_{\frac{N}{G^k}<d\le N^{\frac{1}{k+1}}}\ \sum_{g\le(\frac{N}{d})^{1/k}} g = \frac{1}{2}G^2\psi\left(\frac{N}{G^k}\right) - \frac{1}{2}N^{\frac{2}{k+1}}\psi\left(N^{\frac{1}{k+1}}\right)$$

$$+\frac{k}{2(k-2)}\left(N^{\frac{3}{k+1}} - NG^{2-k}\right)$$

$$-\frac{1}{k}G^2\int_1^{\infty} x^{-1-\frac{2}{k}}\psi\left(\frac{Nx}{G^k}\right)\,dx$$

$$-S^0(k,N,G) + \mathcal{O}\left(N^{\frac{1}{k+1}}\right).$$

We further have

$$\sum_{g^{k+1}\le N} g\left[\frac{N}{g^k}\right] = \sum_{g^{k+1}\le N} Ng^{1-k} - \sum_{g^{k+1}\le N} g\psi\left(\frac{N}{g^k}\right) - \frac{1}{2}\sum_{g^{k+1}\le N} g$$

in which the first sum is given by

$$\sum_{g^{k+1}\le N} Ng^{1-k} = N\zeta(k-1) - N\sum_{g^{k+1}>N} g^{1-k}.$$

Now consider the sum

$$
\begin{aligned}
N \sum_{g^{k+1}>N} g^{1-k} &= N^{1-\frac{k-1}{k+1}} \psi\left(N^{\frac{1}{k+1}}\right) + N \int_{N^{\frac{1}{k+1}}}^{\infty} x^{1-k}\, dx + (1-k) N \int_{N^{\frac{1}{k+1}}}^{\infty} x^{-k} \psi(x)\, dx \\
&= N^{\frac{2}{k+1}} \psi\left(N^{\frac{1}{k+1}}\right) - \frac{N}{k-2} x^{2-k} \Big|_{N^{\frac{1}{k+1}}}^{\infty} + \mathcal{O}\left(k N^{\frac{1}{k+1}}\right) \\
&= N^{\frac{2}{k+1}} \psi(N^{\frac{1}{k+1}}) + \frac{1}{k-2} N^{\frac{3}{k+1}} + \mathcal{O}\left(k N^{\frac{1}{k+1}}\right),
\end{aligned}
$$

which implies

$$
\begin{aligned}
\sum_{g^{k+1}\le N} g \left[\frac{N}{g^k}\right] = N\zeta(k-1) - N^{\frac{2}{k+1}} \psi\left(N^{\frac{1}{k+1}}\right) - \frac{1}{k-2} N^{\frac{3}{k+1}} \\
- \frac{1}{4} N^{\frac{2}{k+1}} - R^0(k,N) + \mathcal{O}\left(k N^{\frac{1}{k+1}}\right).
\end{aligned}
$$

Moreover

$$
\begin{aligned}
\left[N^{\frac{1}{k+1}}\right] \sum_{g^{k+1}\le N} g &= \frac{1}{2}\left[N^{\frac{1}{k+1}}\right]^2 \left(\left[N^{\frac{1}{k+1}}\right]+1\right) \\
&= \frac{1}{2}\left(\left(N^{\frac{1}{k+1}} - \psi\left(N^{\frac{1}{k+1}}\right)\right)^2 - \frac{1}{4}\right)\left(N^{\frac{1}{k+1}} - \psi\left(N^{\frac{1}{k+1}}\right) - \frac{1}{2}\right) \\
&= \frac{1}{2}\left(N^{\frac{1}{k+1}} - \psi\left(N^{\frac{1}{k+1}}\right)\right)^3 - \frac{1}{4}\left(N^{\frac{1}{k+1}} - \psi\left(N^{\frac{1}{k+1}}\right)\right)^2 + \mathcal{O}\left(N^{\frac{1}{k+1}}\right) \\
&= \frac{1}{2} N^{\frac{3}{k+1}} - \frac{3}{2} N^{\frac{2}{k+1}} \psi\left(N^{\frac{1}{k+1}}\right) - \frac{1}{4} N^{\frac{2}{k+1}} + \mathcal{O}\left(N^{\frac{1}{k+1}}\right).
\end{aligned}
$$

Since $G = \mathcal{O}\left(N^{1/k}\right)$ we also have

$$
\begin{aligned}
\left[\frac{N}{G^k}\right] \sum_{g\le G} g &= \frac{1}{2}(G^2+G)\left(\frac{N}{G^k} - \psi\left(\frac{N}{G^k}\right) - \frac{1}{2}\right) \\
&= \frac{1}{2} N G^{2-k} + \frac{1}{2} N G^{1-k} - \frac{1}{2} G^2 \psi\left(\frac{N}{G^k}\right) - \frac{1}{4} G^2 + \mathcal{O}\left(N^{1/k}\right).
\end{aligned}
$$

Thus, by using Lemma 7 we get

$$\begin{aligned}
\sum_{g\le G} g\left[\frac{N}{g^k}\right] &= \frac{1}{2}NG^{2-k}+\frac{1}{2}NG^{1-k}-\frac{1}{2}G^2\psi\left(\frac{N}{G^k}\right)-\frac{1}{4}G^2\\
&+\frac{1}{2}G^2\psi\left(\frac{N}{G^k}\right)-\frac{1}{2}N^{\frac{2}{k+1}}\psi\left(N^{\frac{1}{k+1}}\right)\\
&+\frac{k}{2(k-2)}\left(N^{\frac{3}{k+1}}-NG^{2-k}\right)-\frac{1}{k}G^2\int_1^\infty x^{-1-\frac{2}{k}}\psi\left(\frac{Nx}{G^k}\right)\,dx\\
&+N\zeta(k-1)-N^{\frac{2}{k+1}}\psi\left(N^{\frac{1}{k+1}}\right)-\frac{1}{k-2}N^{\frac{3}{k+1}}\\
&-\frac{1}{4}N^{\frac{2}{k+1}}-\frac{1}{2}N^{\frac{3}{k+1}}+\frac{3}{2}N^{\frac{2}{k+1}}\psi(N^{\frac{1}{k+1}})+\frac{1}{4}N^{\frac{2}{k+1}}\\
&-S^0(k,N,G)-R^0(k,N)+\mathcal{O}\left(N^{1/k}\right)\\
&=-\frac{1}{k-2}NG^{2-k}-G^2\left(\frac{1}{4}+\frac{1}{k}\int_1^\infty x^{-1-\frac{2}{k}}\psi\left(\frac{Nx}{G^k}\right)\,dx\right)+\frac{1}{2}NG^{1-k}\\
&+N\zeta(k-1)-S^0(k,N,G)-R^0(k,N)+\mathcal{O}\left(N^{1/k}\right).
\end{aligned}$$

This completes the proof of (6.1).

In the next step we prove (6.4). We first observe that

$$\begin{aligned}
\sum_{\frac{N}{G^k}<d\le N^{\frac{1}{k+1}}}\left[\left(\frac{N}{d}\right)^{1/k}\right] &= \sum_{\frac{N}{G^k}<d\le N^{\frac{1}{k+1}}}\left(\left(\frac{N}{d}\right)^{1/k}-\psi\left(\left(\frac{N}{d}\right)^{1/k}\right)-\frac{1}{2}\right)\\
&=\sum_{\frac{N}{G^k}<d\le N^{\frac{1}{k+1}}}\left(\frac{N}{d}\right)^{1/k}-S(k,N,G)-\frac{1}{2}\left(N^{\frac{1}{k+1}}-\frac{N}{G^k}\right)\\
&\quad+\mathcal{O}(1)
\end{aligned}$$

and that

$$\begin{aligned}
\sum_{\frac{N}{G^k}<d\le N^{\frac{1}{k+1}}}\left(\frac{N}{d}\right)^{1/k} &= G\psi\left(\frac{N}{G^k}\right)-N^{\frac{1}{k+1}}\psi\left(N^{\frac{1}{k+1}}\right)\\
&+N^{1/k}\int_{NG^{-k}}^{N^{\frac{1}{k+1}}}x^{-1/k}\,dx-\frac{1}{k}N^{1/k}\int_{NG^{-k}}^{N^{\frac{1}{k+1}}}x^{-1-\frac{1}{k}}\psi(x)\,dx,
\end{aligned}$$

in which

$$\int_{NG^{-k}}^{N^{\frac{1}{k+1}}}x^{-1/k}\,dx=\frac{k}{k-1}N^{-1/k}\left(N^{\frac{2}{k+1}}-NG^{1-k}\right)$$

and

$$\int_{NG^{-k}}^{N^{\frac{1}{k+1}}} x^{-1-\frac{1}{k}}\psi(x)\,dx = \int_{NG^{-k}}^{\infty} x^{-1-\frac{1}{k}}\psi(x)\,dx + \mathcal{O}\left(N^{-1/k}\right)$$
$$= \frac{N}{G^k}\int_1^{\infty}\left(\frac{Nx}{G^k}\right)^{-1-\frac{1}{k}}\psi\left(\frac{Nx}{G^k}\right)\,dx + \mathcal{O}\left(N^{-1/k}\right)$$
$$= GN^{-1/k}\int_1^{\infty} x^{-1-\frac{1}{k}}\psi\left(\frac{Nx}{G^k}\right)\,dx + \mathcal{O}\left(N^{-1/k}\right).$$

Thus

$$\sum_{\frac{N}{G^k}<d\le N^{\frac{1}{k+1}}}\left[\left(\frac{N}{d}\right)^{1/k}\right] = G\psi\left(\frac{N}{G^k}\right) - N^{\frac{1}{k+1}}\psi\left(N^{\frac{1}{k+1}}\right)$$
$$+\frac{k}{k-1}\left(N^{\frac{2}{k+1}} - NG^{1-k}\right) - \frac{1}{k}G\int_1^{\infty} x^{-1-\frac{1}{k}}\psi\left(\frac{Nx}{G^k}\right)\,dx$$
$$-\frac{1}{2}N^{\frac{1}{k+1}} + \frac{1}{2}NG^{-k} - S(k,N,G) + \mathcal{O}\left(1\right).$$

Furthermore

$$\sum_{g\le N^{\frac{1}{k+1}}}\left[\frac{N}{g^k}\right] = \sum_{g\le N^{\frac{1}{k+1}}}\left(\frac{N}{g^k}-\frac{1}{2}\right) - R(k,N).$$

Now from

$$N\sum_{g\le N^{\frac{1}{k+1}}} g^{-k} = \zeta(k)N - N\sum_{g>N^{\frac{1}{k+1}}} g^{-k}$$
$$= \zeta(k)N - N^{\frac{1}{k+1}}\psi(N^{\frac{1}{k+1}}) - N\int_{N^{\frac{1}{k+1}}}^{\infty} x^{-k}\,dx + kN\int_{N^{\frac{1}{k+1}}}^{\infty} x^{-1-k}\psi(x)\,dx$$
$$= \zeta(k)N - N^{\frac{1}{k+1}}\psi\left(N^{\frac{1}{k+1}}\right) + \frac{N}{k-1}x^{1-k}\Big|_{N^{\frac{1}{k+1}}}^{\infty} + \mathcal{O}\left(k\right)$$

we get

$$\sum_{g\le N^{\frac{1}{k+1}}}\left[\frac{N}{g^k}\right] = \zeta(k)N - N^{\frac{1}{k+1}}\psi(N^{\frac{1}{k+1}}) - \frac{1}{k-1}N^{\frac{2}{k+1}}$$
$$-\frac{1}{2}N^{\frac{1}{k+1}} - R(k,N) + \mathcal{O}\left(k\right).$$

Thus, using

$$\left[N^{\frac{1}{k+1}}\right]^2 = (N^{\frac{1}{k+1}} - \psi(N^{\frac{1}{k+1}}) - \frac{1}{2})^2 = N^{\frac{2}{k+1}} - 2N^{\frac{1}{k+1}}\psi\left(N^{\frac{1}{k+1}}\right) - N^{\frac{1}{k+1}} + \mathcal{O}\left(1\right)$$

and Lemma 7 we obtain

$$\begin{aligned}
\sum_{g\le G}\left[\frac{N}{g^k}\right] &= G\left(\frac{N}{G^k}-\psi\left(\frac{N}{G^k}\right)-\frac{1}{2}\right)+G\psi\left(\frac{N}{G^k}\right)-N^{\frac{1}{k+1}}\psi\left(N^{\frac{1}{k+1}}\right)\\
&+\frac{k}{k-1}(N^{\frac{2}{k+1}}-NG^{1-k})-\frac{1}{k}G\int_1^\infty x^{-1-\frac{1}{k}}\psi\left(\frac{Nx}{G^k}\right)\,dx\\
&-\frac{1}{2}N^{\frac{1}{k+1}}+\frac{1}{2}NG^{-k}+\zeta(k)N-N^{\frac{1}{k+1}}\psi\left(N^{\frac{1}{k+1}}\right)-\frac{1}{k-1}N^{\frac{2}{k+1}}\\
&-\frac{1}{2}N^{\frac{1}{k+1}}-N^{\frac{2}{k+1}}+2N^{\frac{1}{k+1}}\psi\left(N^{\frac{1}{k+1}}\right)+N^{\frac{1}{k+1}}\\
&-S(k,N,G)-R(k,N)+\mathcal{O}\left(k\right)\\
&=\zeta(k)N-\frac{N}{k-1}G^{1-k}-G\left(\frac{1}{2}+\frac{1}{k}\int_1^\infty x^{-1-\frac{1}{k}}\psi\left(\frac{Nx}{G^k}\right)\,dx\right)\\
&+\frac{1}{2}NG^{-k}-S(k,N,G)-R(k,N)+\mathcal{O}\left(k\right).
\end{aligned}$$

This completes the proof of (6.4).

Finally, in order to prove (6.6) we have

$$\begin{aligned}
\left[\frac{N}{G^k}\right]\sum_{g=1}^{G}\frac{1}{g} &= \left(\frac{N}{G^k}-\psi\left(\frac{N}{G^k}\right)-\frac{1}{2}\right)\left(\log G+\gamma+\mathcal{O}\left(\frac{1}{G}\right)\right)\\
&= NG^{-k}\log G-\psi\left(\frac{N}{G^k}\right)\log G-\frac{1}{2}\log G+\gamma NG^{-k}+\mathcal{O}\left(1\right)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\frac{N}{G^k}<d\le N^{\frac{1}{k+1}}}\ \sum_{g\le(\frac{N}{d})^{1/k}}\frac{1}{g} &= \sum_{\frac{N}{G^k}<d\le N^{\frac{1}{k+1}}}\left(\frac{1}{k}\log\frac{N}{d}+\gamma+\mathcal{O}\left(\left(\frac{d}{N}\right)^{1/k}\right)\right)\\
&=\frac{1}{k}\left(k\psi\left(\frac{N}{G^k}\right)\log G-\psi\left(N^{\frac{1}{k+1}}\right)\log N^{\frac{k}{k+1}}\right)\\
&+\frac{1}{k}\int_{NG^{-k}}^{N^{\frac{1}{k+1}}}\log\frac{N}{x}\,dx+\gamma\left(N^{\frac{1}{k+1}}-\frac{N}{G^k}\right)+\mathcal{O}\left(1\right)\\
&=\psi\left(\frac{N}{G^k}\right)\log G-\frac{1}{k+1}\psi\left(N^{\frac{1}{k+1}}\right)\log N\\
&-\frac{N}{k}\int_{G^{-k}}^{N^{-\frac{k}{k+1}}}\log u\,du+\gamma\left(N^{\frac{1}{k+1}}-\frac{N}{G^k}\right)+\mathcal{O}\left(1\right)\\
&=\psi\left(\frac{N}{G^k}\right)\log G-\frac{1}{k+1}\psi\left(N^{\frac{1}{k+1}}\right)\log N\\
&+\frac{1}{k+1}N^{\frac{1}{k+1}}\log N-NG^{-k}\log G+\frac{1}{k}N^{\frac{1}{k+1}}\\
&-\frac{1}{k}NG^{-k}+\gamma\left(N^{\frac{1}{k+1}}-NG^{-k}\right)+\mathcal{O}\left(1\right).
\end{aligned}$$

Furthermore

$$\begin{aligned}
\sum_{g\le N^{\frac{1}{k+1}}} \frac{1}{g}\left[\frac{N}{g^k}\right] &= \sum_{g\le N^{\frac{1}{k+1}}} \frac{1}{g}\left(\frac{N}{g^k} - \psi\left(\frac{N}{g^k}\right) - \frac{1}{2}\right)\\
&= N\zeta(k+1) - \sum_{g>N^{\frac{1}{k+1}}} \frac{N}{g^{k+1}} - R_0(k,N) - \frac{1}{2(k+1)}\log N + O(1)\\
&= \zeta(k+1)N - \psi\left(N^{\frac{1}{k+1}}\right) - N\int_{N^{\frac{1}{k+1}}}^{\infty} x^{-1-k}\,dx\\
&\quad - N(2+k)\int_{N^{\frac{1}{k+1}}}^{\infty} x^{-2-k}\psi(x)\,dx - R_0(k,N) - \frac{1}{2(k+1)}\log N + \mathcal{O}(1)\\
&= \zeta(k+1)N + \frac{N}{k}x^{-k}\Big|_{N^{\frac{1}{k+1}}}^{\infty} - R_0(k,N) - \frac{1}{2(k+1)}\log N + \mathcal{O}(1)\\
&= \zeta(k+1)N - \frac{1}{k}N^{\frac{1}{k+1}} - \frac{1}{2(k+1)}\log N - R_0(k,N) + \mathcal{O}(1)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{g\le N^{\frac{1}{k+1}}} \frac{1}{g}\left[N^{\frac{1}{k+1}}\right] &= \left(N^{\frac{1}{k+1}} - \psi\left(N^{\frac{1}{k+1}}\right) - \frac{1}{2}\right)\left(\frac{1}{k+1}\log N + \gamma + \mathcal{O}\left(N^{-\frac{1}{k+1}}\right)\right)\\
&= \frac{1}{k+1}N^{\frac{1}{k+1}}\log N - \frac{1}{k+1}\psi\left(N^{\frac{1}{k+1}}\right)\log N - \frac{1}{2(k+1)}\log N\\
&\quad + \gamma N^{\frac{1}{k+1}} + \mathcal{O}(1).
\end{aligned}$$

Combining all this and using Lemma 7 we finally obtain

$$\begin{aligned}
\sum_{g\le G} \frac{1}{g}\left[\frac{N}{g^k}\right] &= NG^{-k}\log N - \psi\left(\frac{N}{G^k}\right)\log G - \frac{1}{2}\log G + \gamma NG^{-k}\\
&\quad + \psi\left(\frac{N}{G^k}\right)\log G - \frac{1}{k+1}\psi\left(N^{\frac{1}{k+1}}\right)\log N + \frac{1}{k+1}N^{\frac{1}{k+1}}\log N\\
&\quad - NG^{-k}\log G + \frac{1}{k}N^{\frac{1}{k+1}} - \frac{1}{k}NG^{-k} + \gamma(N^{\frac{1}{k+1}} - NG^{-k}) + \zeta(k+1)N\\
&\quad - \frac{1}{k}N^{\frac{1}{k+1}} - \frac{1}{2(k+1)}\log N - R_0(k,N) - \frac{1}{k+1}N^{\frac{1}{k+1}}\log N\\
&\quad + \frac{1}{k+1}\psi\left(N^{\frac{1}{k+1}}\right)\log N + \frac{1}{2(k+1)}\log N - \gamma N^{\frac{1}{k+1}} + \mathcal{O}(1)\\
&= -\frac{1}{2}\log G - \frac{1}{k}NG^{-k} + \zeta(k+1)N - R_0(k,N) + \mathcal{O}(1)
\end{aligned}$$

as proposed. □

Our next aim it to prove Lemma 12 which is a generalization of a deep result of Walfisz [6] (where the case $T = \sqrt{x}$ was considered). Fortunately the proof of this generalization runs along the same ideas as in [6]. For the sake of the reader's convenience we give a detailed proof.

Lemma 9. *Let $r \ge 1$ be an integer and set $R := 2^{r-1}$. Then we have, as $x \to \infty$,*

$$\sum_{x^{\frac{2}{r+4}} < m \le T} \frac{1}{m}\psi\left(\frac{x}{m}\right) = \mathcal{O}\left(x^{-\frac{1}{20Rr}}(\log x)^3\right) \tag{6.7}$$

uniformly for $T \le x^{\frac{2}{r+3}}$. Furthermore, the $\mathcal{O}$-constant is independent of r.

Proof. If $x \le 2^{r+3}$, i.e. $x^{\frac{1}{r+3}} \le 2$, then $T \le 4$ and there is nothing to show since

$$(\log x)^3 x^{-\frac{1}{20Rr}} \ge 2^{-\frac{r+3}{20Rr}} \ge 2^{-\frac{1}{5R}} \ge \frac{1}{2}.$$

Now suppose that $2^{r+3} \le x$. If $x^{\frac{2}{r+4}} < M \le M' \le 2M \le 2x^{\frac{2}{r+3}}$ then we can use (18) of [6, p. 92] (with $R_1 = R(r+1)$)

$$\sum_{m=M}^{M'} \frac{1}{m}\psi\left(\frac{x}{m}\right) \ll \left(M^{-\left(\frac{1}{R}+\frac{1}{R_1}\right)} x^{\frac{11}{10R_1}} + M^{-r-2}x^{\frac{11}{10}} + x^{-\frac{1}{10}}\right)(\log x)^2.$$

Set $X = \left[x^{\frac{2}{r+4}}\right]$ and choose an integer h with $X2^h < T \le X2^{h+1}$. Further, for $0 \le j \le h$ set $M_j = X2^j$ und $M_{h+1} = T$. Since

$$h = \left[\frac{\log T/X}{\log 2}\right] \ll \log x$$

we obtain

$$\begin{aligned}\sum_{x^{\frac{2}{r+4}} < m \le T} \frac{1}{m}\psi\left(\frac{x}{m}\right) &= \sum_{j=0}^{h} \sum_{m=M_j+1}^{M_{j+1}} \frac{1}{m}\psi\left(\frac{x}{m}\right) \\ &\ll \left(x^{-\frac{2}{r+4}\left(\frac{1}{R}+\frac{1}{R_1}\right)+\frac{11}{10R_1}} + x^{-\frac{2(r+2)}{r+4}+\frac{11}{10}} + x^{-1/10}\right)(\log x)^3\end{aligned}$$

Since

$$-\frac{2}{r+4}\left(\frac{1}{R}+\frac{1}{R_1}\right) + \frac{11}{10R_1} = \frac{1}{R_1}\left(-\frac{r}{r+4}+\frac{1}{10}\right) \le \frac{1}{2Rr}\left(-\frac{1}{5}+\frac{1}{10}\right) = -\frac{1}{20Rr}$$

and

$$-\frac{2(r+2)}{r+4} + \frac{11}{10} = -\frac{r}{r+4} + \frac{1}{10} \le -\frac{1}{5} + \frac{1}{10} = -\frac{1}{10} \le -\frac{1}{20Rr}$$

we immediately obtain (6.7). □

Lemma 10. *Suppose that $X = 4\left[\frac{1}{4}\log\log x\right] \ge 4$. Then we have, as $x \to \infty$,*

$$\sum_{x^{\frac{2}{X+4}} < m \le T} \frac{1}{m}\psi\left(\frac{x}{m}\right) = \mathcal{O}(1) \tag{6.8}$$

uniformly for $x^{\frac{2}{X+2}} \le T \le \sqrt{x}$.

Proof. There exists an $r_0 \geq 1$, such that $x^{\frac{2}{r_0+4}} < T \leq x^{\frac{2}{r_0+3}}$ and $r_0 \leq X$. For $r_0 < r \leq X$ we set $M_r = x^{\frac{2}{r+3}}$ and $M_{r_0} = T$. Hence, with $R_r = 2^{r-1}$ we obtain

$$\begin{aligned} x^{-\frac{1}{20R_r r}} &= e^{-\log x/(20R_r r)} \leq e^{-\log x/(20\cdot 2^{X-1}X)} \\ &\leq e^{-\log x/(10\cdot 2^{\log\log x}\log\log x)} \\ &= e^{-(\log x)^{1-\log 2}/(10\log\log x)} \\ &= \mathcal{O}\left(\frac{1}{(\log x)^3\log\log x}\right). \end{aligned}$$

Consequently, by using Lemma 7 we get

$$\begin{aligned} \sum_{x^{\frac{2}{X+4}} < m \leq T} \frac{1}{m}\psi\left(\frac{x}{m}\right) &= \sum_{r=r_0}^{X-1} \sum_{M_{r+1} < m \leq M_r} \frac{1}{m}\psi\left(\frac{x}{m}\right) \\ &= \mathcal{O}\left(\sum_{r=r_0}^{X-1} x^{-\frac{1}{20R_r r}} \log^3 x\right) \\ &= \mathcal{O}\left(X/\log\log x\right) = \mathcal{O}(1). \end{aligned}$$

□

Lemma 11. *Suppose that* $95 \leq r \leq 10^{-2}(\log x)^{1/3}$ *and set* $R = 10^{-6}r^{-3}$. *Then we have, as* $x \to \infty$,

$$\sum_{x^{\frac{3}{2r}} < m \leq T} \frac{1}{m}\psi\left(\frac{x}{m}\right) = \mathcal{O}\left(r^{-5}x^{-R}(\log x)^2\right). \tag{6.9}$$

uniformly for $x^{\frac{3}{2r}} \leq T \leq x^{\frac{3}{2(r-1)}}$. *Furthermore, the* $\mathcal{O}$*-constant is independent of* r.

Proof. Let $x^{\frac{3}{2r}} \leq M \leq M' \leq 2M \leq x^{\frac{3}{2(r-1)}}$. Then by (39) of [6, p. 97] we get

$$\sum_{M < m \leq M'} \frac{1}{m}\psi\left(\frac{x}{m}\right) = \mathcal{O}\left(r^{-3}x^{-R}\log x\right).$$

We set $X = \left[x^{\frac{3}{2r}}\right] + 1$ and choose an integer $h \geq 0$ such that $2^h X \leq T < 2^{h+1}X$, i.e. $h = \left[\frac{\log T/X}{\log 2}\right]$. Furthermore, for $1 \leq j \leq h$ set $M_j = 2^{j-1}X$ and $M_{h+1} = T$. Then we obtain

$$\begin{aligned} \sum_{x^{\frac{3}{2r}} < m \leq T} \frac{1}{m}\psi\left(\frac{x}{m}\right) &= \sum_{j=1}^{h} \sum_{M_j < m \leq M_{j+1}} \frac{1}{m}\psi\left(\frac{x}{m}\right) \\ &= \mathcal{O}\left(\sum_{j=1}^{h} r^{-3}x^{-R}\log x\right) \\ &= \mathcal{O}\left(r^{-3}x^{-R}\log x\log T/X\right) \\ &= \mathcal{O}\left(r^{-3}x^{-R}\left(\frac{3}{2(r-1)} - \frac{3}{2r}\right)(\log x)^2\right) \\ &= \mathcal{O}\left(r^{-5}x^{-R}(\log x)^2\right). \end{aligned}$$

□

Lemma 12. *We have*

$$\sum_{1\le m\le T} \frac{1}{m}\psi\left(\frac{x}{m}\right) = \mathcal{O}\left((\log x)^{2/3}\right) \tag{6.10}$$

uniformly for $T \le \sqrt{x}$.

Proof. Set $h := 3\,[\log\log x] + 4$, $k := \left[\frac{1}{1000}(\log x)^{1/3}\right]$ (where we assume that x is sufficiently large that $95 \le h < k$) and $X := 4\left[\frac{1}{4}\log\log x\right]$. We consider three cases.

First suppose that $T \le x^{\frac{3}{2k}}$. Then

$$\sum_{m\le T} \frac{1}{m}\psi\left(\frac{x}{m}\right) = \mathcal{O}(\log T) = \mathcal{O}\left(\frac{1}{k}\log x\right) = \mathcal{O}\left((\log x)^{2/3}\right). \tag{6.11}$$

Second, we consider the case $x^{\frac{3}{2k}} < T \le x^{\frac{3}{2(h-1)}}$. There exists an r_0 such that $x^{\frac{3}{2r_0}} < T \le x^{\frac{3}{2(r_0-1)}}$ and $h \le r_0 \le k$. It is now an easy exercise to show that that mapping $f : [h,k] \to \mathbb{R}$, $y \mapsto y^{-5}x^{-10^{-6}y^{-3}}$ is monotonically increasing: Since

$$f'(y) = \frac{f(y)}{y}\left(\frac{3\log x}{10^6 y^3} - 5\right)$$

and $\frac{3\log x}{10^6y^3} \ge \frac{3}{10^6}k^{-3}\log x > \frac{10^9}{10^6} > 5$ it is clear that $f'(y) > 0$.

Next, for $r_0 < r \le k$ set $M_r = x^{\frac{3}{2r}}$ and $M_{r_0} = T$. Then, by Lemma 10 we obtain

$$\begin{aligned}
\sum_{x^{\frac{3}{2k}} < m \le T} \frac{1}{m}\psi\left(\frac{x}{m}\right) &= \sum_{r=r_0}^{k-1} \sum_{M_{r+1} < m \le M_r} \frac{1}{m}\psi\left(\frac{x}{m}\right) \\
&\ll \sum_{r=r_0}^{k-1} f(r)(\log x)^2 \\
&\ll kf(k)(\log x)^2 \\
&\ll k^{-4}(\log x)^2 \\
&\ll (\log x)^{2/3}.
\end{aligned}$$

Combining this with (6.11) completes the proof of the second case.

Finally suppose that $x^{\frac{3}{2(h-1)}} < T \le \sqrt{x}$. Since $\frac{2}{X+4} = \frac{3}{2(h-1)}$ we just have to combine Lemma 9 and the second case. $\square$

Lemma 13. *Suppose that* $k > 1$. *Then we have uniformly for* $1 \le x \le N^{\frac{1}{k+1}}$

$$\sum_{d\le x} \psi\left(\left(\frac{N}{d}\right)^{1/k}\right) = \mathcal{O}\left(N^{\frac{1}{3k}} x^{\frac{1}{3}(1-\frac{1}{k})}\right) = \mathcal{O}\left(N^{\frac{2}{3(k+1)}}\right) \tag{6.12}$$

and

$$\sum_{d\le x} \left(\frac{N}{d}\right)^{1/k} \psi\left(\left(\frac{N}{d}\right)^{1/k}\right) = \begin{cases} N^{\frac{3}{2k+1}} & \text{for } k=2 \text{ and } k=3, \\ N^{1/3}\log N & \text{for } k=4, \\ N^{\frac{5}{3(k+1)}} & \text{for } k>4, \end{cases} \tag{6.13}$$

Proof. We will use Van der Corput's estimate (see [5, p. 32])

$$\sum_{a\le n\le b} \psi(f(n)) \ll \int_a^b |f''(t)|^{1/3}\,dt + |f''(a)|^{-1/2} + |f''(b)|^{-1/2}.$$

with $f(z) = (\frac{N}{z})^{1/k}$, $(z \in [1,x])$. Since $f''(z) = \frac{1}{k}(1+\frac{1}{k})N^{\frac{1}{k}} z^{-2-\frac{1}{k}}$, we have

$$\int_1^x |f''(z)|^{1/3}\,dz = \mathcal{O}\left(N^{\frac{1}{3k}} \int_1^x z^{-\frac{2}{3}-\frac{1}{3k}}\,dz\right) = \mathcal{O}\left(N^{\frac{1}{3k}} x^{\frac{1}{3}(1-\frac{1}{k})}\right)$$

and

$$|f''(x)|^{-1/2} = \mathcal{O}\left(N^{-\frac{1}{2k}} x^{1+\frac{1}{2k}}\right)$$

which proves (6.12).

Now suppose that $k \ge 4$. Then

$$\begin{aligned}\sum_{1\le d\le x} (\frac{N}{d})^{1/k}\psi\left(\left(\frac{N}{d}\right)^{1/k}\right) &= \left(\frac{N}{x}\right)^{1/k} \sum_{d\le x} \psi\left(\left(\frac{N}{d}\right)^{1/k}\right)\\ &\quad + \int_1^x N^{\frac{1}{k}} \frac{1}{k} u^{-1-\frac{1}{k}} \sum_{d\le u} \psi\left(\left(\frac{N}{d}\right)^{1/k}\right) du\\ &\ll \left(\frac{N}{x}\right)^{1/k} N^{\frac{1}{3k}} x^{\frac{1}{3}(1-\frac{1}{k})} + N^{\frac{4}{3k}} \int_1^x u^{-\frac{2}{3}-\frac{4}{3k}}\,du\\ &\ll N^{\frac{4}{3k}} x^{\frac{1}{3}-\frac{4}{3k}} + N^{\frac{4}{3k}} x^{\frac{1}{3}-\frac{4}{3k}},\end{aligned}$$

in which the last factor has to be replaced by $\log N$ if $k = 4$. Since

$$\frac{4}{3k} + \frac{1}{3(k+1)} - \frac{4}{3k(k+1)} = \frac{5}{3(k+1)}$$

this proves (6.13) for $k \ge 4$.

Finally suppose that $k \in \{2,3\}$. If $x \le N^{\frac{1}{2k+1}}$, the above sum can be estimated by

$$\ll N^{1/k} x^{1-\frac{1}{k}} \ll N^{\frac{3}{2k+1}}.$$

Thus, we may assume that $y := N^{\frac{1}{2k+1}} \le x$. Here we have

$$\sum_{d\le y} \left(\frac{N}{d}\right)^{1/k} \psi\left(\left(\frac{N}{d}\right)^{1/k}\right) \ll N^{\frac{3}{2k+1}}$$

and

$$\begin{aligned}\sum_{y<d\le x}\left(\frac{N}{d}\right)^{1/k}\psi\left(\left(\frac{N}{d}\right)^{1/k}\right) &= \left(\frac{N}{x}\right)^{1/k}\sum_{y<d\le x}\psi\left(\left(\frac{N}{d}\right)^{1/k}\right)\\ &\quad+\int_y^x N^{\frac{1}{k}}\frac{1}{k}u^{-1-\frac{1}{k}}\sum_{y<d\le u}\psi\left(\left(\frac{N}{d}\right)^{1/k}\right)du\\ &\ll \left(\frac{N}{x}\right)^{1/k}N^{\frac{1}{3k}}x^{\frac{1}{3}(1-\frac{1}{k})}+N^{\frac{4}{3k}}\int_y^x u^{-\frac{2}{3}-\frac{4}{3k}}\,du\\ &\ll N^{\frac{4}{3k}}x^{\frac{1}{3}(1-\frac{4}{k})}+N^{\frac{4}{3k}}y^{\frac{1}{3}-\frac{4}{3k}}\\ &\ll N^{\frac{3}{2k+1}}\end{aligned}$$

which completes the proof of (6.13). □

Lemma 14. *For* $1\le x\le N^{\frac{1}{k+1}}$ *we have*

$$\sum_{g\le x}\psi\left(\frac{N}{g^k}\right)=\mathcal{O}\left(N^{\frac{1}{k+2}}\right). \tag{6.14}$$

Proof. First suppose that $k>1$. If $x\le N^{\frac{1}{k+2}}$ there is nothing to show. If $x>N^{\frac{1}{k+2}}$ set $w=N^{\frac{1}{k+2}}<x$, which is greater than 1. We (again) use Van der Corput's estimate [5, p. 32] with $f:[w,x]\to\mathbb{R},\ f(z)=Nz^{-k}$. Since $f''(z)=Nk(k+1)z^{-k-2}$ we get

$$\begin{aligned}\sum_{g\le x}\psi(f(n)) &\ll w+\int_w^x N^{1/3}z^{-\frac{k+2}{3}}\,dz+\frac{1}{\sqrt{N}}x^{\frac{k+2}{2}}\\ &\ll w+N^{1/3}w^{-\frac{k-1}{3}}+N^{-\frac{1}{2}+\frac{k+2}{2(k+1)}}\\ &\ll N^{\frac{1}{k+2}}.\end{aligned}$$

For $k=1$ much more is known. The above sum can be estimated by $\mathcal{O}\left(N^{\theta}\right)$, where θ is best exponent of the divisor problem, which is surely $\le\frac{1}{3}$ (e.g. see Kolesnik [4]). □

7. Proof of Theorems 3–7

We start with the proof of Theorem 3

Proof. (*Theorem 3*) By using Theorem 1 we first obtain

$$\begin{aligned}\sum_{g\le G}\frac{1}{g}\,b_k(N,g) &= \int_0^1 A_k(G,N,x,1)\,dx\\ &= G\int_0^1(1-x)\,dx+\mathcal{O}\left(G^{1-\eta}\right)\\ &= \frac{1}{2}G+\mathcal{O}\left(G^{1-\eta}\right).\end{aligned}$$

Consequently

$$\begin{aligned}\sum_{g\le G} b_k(N,g) &= \sum_{g\le G} g\frac{1}{g}\, b_k(N,g)\\ &= G\sum_{g\le G}\frac{1}{g}\, b_k(N,g) - \sum_{g<G}\sum_{h\le g}\frac{1}{h}\, b_k(N,h)\\ &= G\left(\frac{G}{2} + \mathcal{O}\left(G^{1-\eta}\right)\right) - \sum_{g<G}\left(\frac{g}{2} + \mathcal{O}\left(g^{1-\eta}\right)\right)\\ &= \frac{1}{4}G^2 + \mathcal{O}\left(G^{2-\eta}\right),\end{aligned}$$

which completes the proof of Theorem 3. □

We now turn to the proof of Theorem 4

Proof. (*Theorem 4*) We start with the representation

$$\begin{aligned}\sum_{g=2}^{G}\frac{1}{g} b_k(N,g) &= \sum_{g=1}^{G}\frac{1}{g}\left[\frac{N}{g^k}\right] - \sum_{g=1}^{G}\left[\frac{N}{g^{k+1}}\right]\\ &= N\sum_{g=1}^{G} g^{-k-1} - \zeta(k+1)N - \frac{1}{2}NG^{-k-1}\\ &+ \frac{N}{k}G^{-k} + G\left(\frac{1}{2} + \frac{1}{k+1}\int_1^\infty x^{-1-\frac{1}{k+1}}\psi\left(\frac{Nx}{G^{k+1}}\right)dx\right)\\ &+ S(k+1,N,G) + R(k+1,N) + \mathcal{O}\left(\log G\right).\end{aligned}$$

By Lemma 13 we have

$$S(k+1,N,G) = \mathcal{O}\left(N^{\frac{2}{3(k+1)}}\right)$$

and by Lemma 14

$$R(k+1,N) = \mathcal{O}\left(N^{\frac{1}{k+3}}\right).$$

Furthermore, since

$$\begin{aligned}N\sum_{g=1}^{G} g^{-k-1} &= N\left(\zeta(k+1) - \sum_{G<g} g^{-k-1}\right)\\ &= N\left(\zeta(k+1) - \psi(G)G^{-k-1} - \int_G^\infty x^{-k-1}\,dx\right)\\ &+ N(k+1)\int_G^\infty x^{-k-2}\psi(x)\,dx\\ &= \zeta(k+1)N + \frac{1}{2}NG^{-k-1} - \frac{N}{k}G^{-k} + \mathcal{O}\left(NG^{-k-2}\right),\end{aligned}$$

we thus get (2.7). The proof is similar in the case $k=0$. □

Proof. (*Theorem 5*) By Lemma 14 we have for $k \ge 0$

$$
\begin{aligned}
R^0(k+1,N) &= \sum_{g \le N^{\frac{1}{k+2}}} g\,\psi\left(\frac{N}{g^{k+1}}\right) \\
&= N^{\frac{1}{k+2}} \sum_{g \le N^{\frac{1}{k+2}}} \psi\left(\frac{N}{g^{k+1}}\right) + \int_1^{N^{\frac{1}{k+2}}} \sum_{g \le x} \psi\left(\frac{N}{g^{k+1}}\right)\,dx \\
&\ll N^{\frac{1}{k+2}+\frac{1}{k+3}} + \int_1^{N^{\frac{1}{k+2}}} N^{\frac{1}{k+3}}\,dx \\
&\ll N^{\frac{1}{k+2}+\frac{1}{k+3}}.
\end{aligned}
$$

By Lemma 8 we obtain for $k > 1$

$$
\begin{aligned}
\sum_{g=2}^{G} b_k(N,g) &= \sum_{g=1}^{G} \left(\left[\frac{N}{g^k}\right] - g\left[\frac{N}{g^{k+1}}\right]\right) \\
&= N\sum_{g=1}^{G} g^{-k} - \zeta(k)N - \frac{1}{2}NG^{-k} + \frac{N}{k-1}G^{1-k} \\
&\quad + G^2\left(\frac{1}{4} + \frac{1}{k+1}\int_1^{\infty} x^{-1-\frac{2}{k+1}}\psi\left(\frac{Nx}{G^{k+1}}\right)\,dx\right) \\
&\quad + S^0(k+1,N,G) + R^0(k+1,N) + \mathcal{O}\left(N^{\frac{1}{k+1}}\right).
\end{aligned}
$$

The first sum equals

$$
\begin{aligned}
N\sum_{g=1}^{G} g^{-k} &= \zeta(k)N - N\sum_{g>G} g^{-k} \\
&= \zeta(k)N - N\left(\psi(G)G^{-k} + \int_G^{\infty} x^{-k}\,dx - k\int_G^{\infty} x^{-k-1}\psi(x)\,dx\right) \\
&= \zeta(k)N + \frac{1}{2}NG^{-k} - \frac{N}{k-1}x^{1-k}\Big|_G^{\infty} + \mathcal{O}\left(NG^{-k-1}\right) \\
&= \zeta(k)N + \frac{1}{2}NG^{-k} - \frac{N}{k-1}G^{1-k} + \mathcal{O}\left(N^{\frac{1}{k+1}}\right).
\end{aligned}
$$

Since $\frac{5}{3(k+2)} < \frac{1}{k+2} + \frac{1}{k+3}$ and $\frac{3}{7} < \frac{1}{4} + \frac{1}{5}$ we obtain by Lemma 13

$$
\begin{aligned}
\sum_{g=2}^{G} b_k(N,g) &= G^2\left(\frac{1}{4} + \frac{1}{k+1}\int_1^{\infty} x^{-1-\frac{2}{k+1}}\psi\left(\frac{Nx}{G^{k+1}}\right)\,dx\right) \\
&\quad + S^0(k+1,N,G) + R^0(k+1,N) + \mathcal{O}\left(N^{\frac{1}{k+1}}\right) \\
&= G^2\left(\frac{1}{4} + \frac{1}{k+1}\int_1^{\infty} x^{-1-\frac{2}{k+1}}\psi\left(\frac{Nx}{G^{k+1}}\right)\,dx\right) + \mathcal{O}\left(N^{\frac{1}{k+2}+\frac{1}{k+3}}\right).
\end{aligned}
$$

The proof is similar in the cases $k = 1$ and $k = 0$. □

Before starting with the next two proofs we recall the following formula:

$$\frac{1}{4} + \frac{1}{k+1} \int_1^\infty x^{-1-\frac{2}{k+1}} \psi(x)\, dx = \begin{cases} -\frac{1}{k-1} - \frac{1}{2}\zeta\left(\frac{2}{k+1}\right) & \text{for } k > 1. \\ \frac{1-\gamma}{2} & \text{for } k = 1. \end{cases}$$

Proof. (*Theorem 6*) By Theorem 4 we know that

$$\begin{aligned} \sum_{g=2}^{G} \frac{1}{g} b_k(N,g) &= \sum_{1<g\le N^{\frac{1}{k+1}}} \frac{1}{g} b_k(N,g) + \sum_{N^{\frac{1}{k+1}}<g\le G} \frac{1}{g}\left[\frac{N}{g^k}\right] \\ &= N^{\frac{1}{k+1}}\left(\frac{1}{2} + \frac{1}{k+1}\int_1^\infty x^{-1-\frac{1}{k+1}}\psi(x)\, dx\right) \\ &\quad + \sum_{N^{\frac{1}{k+1}}<g\le G} \frac{N}{g^{k+1}} + \mathcal{O}\left(N^{\frac{3}{k+1}}\right). \end{aligned}$$

Observe that

$$\frac{1}{2} + \frac{1}{k+1}\int_1^\infty x^{-1-\frac{1}{k+1}}\psi(x)\, dx = -\frac{1}{k} - \zeta\left(\frac{1}{k+1}\right)$$

and that

$$\begin{aligned} \sum_{N^{\frac{1}{k+1}}<g\le G} \frac{N}{g^{k+1}} &= N \int_{N^{\frac{1}{k+1}}}^{G} x^{-k-1}\, dx - (k+1)N \int_{N^{\frac{1}{k+1}}}^{G} x^{-k-2}\psi(x)\, dx + \mathcal{O}(1) \\ &= \frac{1}{k}N^{\frac{1}{k+1}} - \frac{1}{k}NG^{-k-1} + \mathcal{O}(1) = \frac{1}{k}N^{\frac{1}{k+1}} + \mathcal{O}(1), \end{aligned}$$

hence, we arrive at (2.9).

For $k = 0$ we proceed in a similar way. □

Proof. (*Theorem 7*) We first observe that for all $k \ge 0$

$$\sum_{g=2}^{G} b_k(N,g) = \sum_{1<g\le N^{\frac{1}{k+1}}} b_k(N,g) + \sum_{N^{\frac{1}{k+1}}<g\le G} \left[\frac{N}{g^k}\right].$$

By Theorem 5 the first sum is given by

$$\begin{aligned} \sum_{1<g\le N^{\frac{1}{k+1}}} b_k(N,g) &= N^{\frac{2}{k+1}}\left(\frac{1}{4} + \frac{1}{k+1}\int_1^\infty x^{-1-\frac{2}{k+1}}\psi(x)\, dx\right) \\ &\quad + \mathcal{O}\left(N^{\frac{1}{k+2}+\frac{1}{k+3}}\right). \end{aligned}$$

The second sum can be represented as

$$\begin{aligned}
\sum_{N^{\frac{1}{k+1}}<g\le G}\left[\frac{N}{g^k}\right] &= -\psi(G)NG^{-k}+N^{\frac{1}{k+1}}\psi\left(N^{\frac{1}{k+1}}\right)\\
&\quad + N\int_{N^{\frac{1}{k+1}}}^{G} x^{-k}\,dx - kN\int_{N^{\frac{1}{k+1}}}^{G} x^{-k-1}\psi(x)\,dx\\
&= \frac{N}{2}G^{-k}+\frac{N}{1-k}x^{1-k}\Big|_{N^{\frac{1}{k+1}}}^{G} + \mathcal{O}\left(N^{\frac{1}{k+1}}\right)\\
&= \frac{N^{\frac{2}{k+1}}}{k-1}-\frac{NG^{1-k}}{k-1}+\mathcal{O}\left(N^{\frac{1}{k}}\right).
\end{aligned}$$

Consequently

$$\sum_{g=2}^{G} b_k(N,g) = -\frac{1}{2}\zeta(\frac{2}{k+1})N^{\frac{2}{k+1}} - \frac{NG^{1-k}}{k-1} + \mathcal{O}\left(N^{\frac{1}{k+2}+\frac{1}{k+3}}+N^{\frac{1}{k}}\right),$$

which proves (2.15).

For the proof of (2.14) we again apply Theorem 5 and get

$$\begin{aligned}
\sum_{1<g\le\sqrt{N}} b_1(N,g) &= N\left(\frac{1}{4}+\frac{1}{2}\int_1^\infty x^{-2}\psi(x)\,dx\right)+\mathcal{O}\left(N^{3/5}\right)\\
&= N\frac{1-\gamma}{2}+\mathcal{O}\left(N^{3/5}\right).
\end{aligned}$$

Next, by Lemma 7

$$\begin{aligned}
\sum_{\sqrt{N}<g\le G}\left[\frac{N}{g}\right] &= \left[\frac{N}{G}\right]\sum_{\sqrt{N}<g\le G}1+\sum_{\frac{N}{G}<d\le\sqrt{N}}\ \sum_{\sqrt{N}<g\le\frac{N}{d}}1\\
&= \left[\frac{N}{G}\right]\left(G-\left[\sqrt{N}\right]\right)+\sum_{\frac{N}{G}<d\le\sqrt{N}}\left(\left[\frac{N}{d}\right]-\left[\sqrt{N}\right]\right)\\
&= \left[\frac{N}{G}\right]\left(G-\left[\sqrt{N}\right]\right)-\left[\sqrt{N}\right]^2+\left[\sqrt{N}\right]\left[\frac{N}{G}\right]+G\psi\left(\frac{N}{G}\right)\\
&\quad -\sqrt{N}\psi\left(\sqrt{N}\right)+N\log\sqrt{N}-N\log\frac{N}{G}-N\int_{N/G}^{\sqrt{N}}x^{-2}\psi(x)\,dx\\
&= \left(\frac{N}{G}-\psi\left(\frac{N}{G}\right)-\frac{1}{2}\right)G-N+\mathcal{O}\left(\sqrt{N}\right)\\
&\quad +G\psi\left(\frac{N}{G}\right)+N\log\frac{G}{\sqrt{N}}-N\int_{N/G}^{\infty}x^{-2}\psi(x)\,dx\\
&= -\frac{G}{2}+N\log\frac{G}{\sqrt{N}}-G\int_1^\infty u^{-2}\psi\left(\frac{Nx}{G}\right)dx+\mathcal{O}\left(\sqrt{N}\right)
\end{aligned}$$

which proves (2.14). The proof of (2.13) is similar and even simpler. □

8. Proof of Theorems 8 and 9

Proof. (*Theorem 8*) For $\sqrt{N} \leq G \leq N$ we have

$$\sum_{g=2}^{G} \frac{1}{g} b_0(N,g) = G\left(\frac{1}{2} + \int_1^\infty x^{-2}\psi\left(\frac{Nx}{G}\right)\,dx\right) + \mathcal{O}\left(N^{1/3}\log N\right)$$

and

$$\sum_{g=2}^{G} \frac{1}{g} b_1(N,g) = -\zeta\left(\frac{1}{2}\right)\sqrt{N} + \mathcal{O}\left(N^{1/4}\right)$$

whereas for $k > 1$

$$\begin{aligned}\sum_{g=2}^{G} \frac{1}{g} b_k(N,g) &= \sum_{1<g\leq N^{1/k}} \frac{1}{g} b_k(N,g) \\ &= -\zeta\left(\frac{1}{k+1}\right) N^{\frac{1}{k+1}} + \mathcal{O}\left(N^{\frac{3}{k+1}}\right) = \mathcal{O}\left(N^{1/3}\right),\end{aligned}$$

which proves (2.19).

If $N^{\frac{1}{L+1}} \leq G \leq N^{\frac{1}{L}}$ for some integer $L \geq 2$ we proceed in a similiar way. By using Theorems 3, 4 and 6 we get

$$\begin{aligned}\sum_{g\leq G} \frac{1}{g} s_g(N) &= \sum_{g\leq G} \frac{1}{g} \sum_{k\leq \log_2 N} b_k(N,g) \\ &= \sum_{k\leq \log_2 N} \sum_{g\leq G} \frac{1}{g} b_k(N,g) \\ &= \sum_{k\leq L-2} \sum_{g\leq G} \frac{1}{g} b_k(N,g) + \sum_{g\leq G} \frac{1}{g} b_{L-1}(N,g) \\ &+ \sum_{g\leq G} \frac{1}{g} b_L(N,g) + \sum_{L+1\leq k\leq \log_2 N} \sum_{g\leq G} \frac{1}{g} b_k(N,g) \\ &= (L-1)\frac{G}{2} + \mathcal{O}\left(G^{1-\eta}\right) + \frac{G}{2} \\ &+ \frac{G}{L}\int_1^\infty \psi\left(\frac{N}{G^L}x\right) x^{-1-\frac{2}{L}}\,dx + \mathcal{O}\left(N^{\frac{1}{L+2}}\right) \\ &- \zeta\left(\frac{1}{L+1}\right) N^{\frac{1}{L+1}} \\ &+ \mathcal{O}\left(N^{\frac{1}{L+1}}\right) + \mathcal{O}\left(\log N N^{\frac{1}{L+2}}\right) \\ &= L\frac{G}{2} + \frac{G}{L}\int_1^\infty \psi\left(\frac{N}{G^L}x\right) x^{-1-\frac{2}{L}}\,dx + \mathcal{O}\left(G^{1-\eta}\right)\end{aligned}$$

as proposed. ∎

Proof. (*Theorem 9*) Since $\sqrt{N} \le G \le N$ we have

$$\sum_{g=2}^{G} b_0(N,g) = G^2 \left(\frac{1}{4} + \int_1^\infty x^{-3}\psi\left(\frac{Nx}{G}\right)\,dx \right) + \mathcal{O}\left(N(\log N)^{1/3}\right)$$

and

$$\sum_{g=2}^{G} b_1(N,g) = \left(\frac{1-\gamma}{2} + \log\frac{G}{\sqrt{N}}\right)N - G\left(\frac{1}{2} + \int_1^\infty x^{-2}\psi\left(\frac{Nx}{g}\right)\,dx\right) + \mathcal{O}\left(N^{3/5}\right)$$
$$= N\log\frac{G}{\sqrt{N}} + \mathcal{O}\left(N\right)$$

whereas for $k > 1$

$$\sum_{g=2}^{G} b_k(N,g) = \mathcal{O}\left(\sum_{g\le N^{1/k}} g\right) = \mathcal{O}\left(N^{2/k}\right).$$

This proves (2.19).

For the case $N^{\frac{1}{L+1}} \le G \le N^{\frac{1}{L}}$ (where $L \ge 2$) we can proceed similarly to the proof of Theorem 8. □

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