THE SHAPE OF UNLABELED ROOTED RANDOM TREES

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Abstract. We consider the number of nodes in the levels of unlabeled rooted random trees and show that the stochastic process given by the properly scaled level sizes weakly converges to the local time of a standard Brownian excursion. Furthermore we compute the average and the distribution of the height of such trees. These results extend existing results for conditioned Galton-Watson trees and forests to the case of unlabeled rooted trees and show that they behave in this respect essentially like a conditioned Galton-Watson process.

1. Introduction

We consider the profile and height of unlabeled rooted random trees. This kind of trees is also called Pólya trees, because the enumeration theory developed by Pĺya allows an analytical treatment of this class of trees by means of generating functions (see [40]). The profile of a rooted tree T is defined as follows. First we define the k-th level of T to be the set of all nodes having distance k from the root (where we use the usual shortest path graph metric). Let $L_k(T)$ denote the number of nodes of the k-th level. The *profile* of T is the sequence $(L_k(T))_{k\geq 0}$. For a random tree this sequence becomes a stochastic process.

The first investigations of the profile of random trees seem to go back to Stepanov [42] who derived explicit formulas for the distribution of the size of one level. Further papers deal mainly with simply generated trees as defined by Meir and Moon [34]. Note that simply generated trees are defined by a functional equation for their generating function but can also viewed as family trees of a Galton-Watson process conditioned on the total progeny. Kolchin (see [31, 32]) related the level size distributions to distributions occurring in particle allocation schemes. Later Takács [43] derived another expression for the level sizes by means of generating functions. Aldous [1] conjectured two functional limit theorems for the profile in two different ranges which were proved in [12, 22]. The first author [10] studied restrictions of the profile to nodes of fixed degree. An extension to random forests of simply generated trees is given by the second author [23].

Later other tree classes have been considered as well. The profile of random binary search trees has been first studied by [4] and later by [11] and [15]. Random recursive trees have been investigated recently by [16] and [45]. See also [5, 21, 28, 29, 33, 37] for related research. Extremal studies of the profile (called the width of trees) of simply generated trees have been started in [38]. The distribution including moment convergence has been presented independently in [3] and [14]. For other tree classes see [9, 16].

Whereas simply generated trees have an average height of order $\sqrt{n}$, the other tree classes mentioned above have height of order $\log n$. Pólya trees do not belong to the class of simply generated trees which can be seen as follows. The generating functions enumerating the number of Pólya tree and simply generated trees, respectively, have a fundamentally different singularity structure. Whereas the first one has one or a finite number of singularities (the latter occurs in the periodic case) on the circle of convergence and allows analytic continuation to a slit plane (or at least a slit disk), the generating function associated to Pólya trees is much more complicated.

Date: April 3, 2008.

Key words and phrases. unlabeled trees, profile, height, Brownian excursion, local time.

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This work has been supported by the Austrian Science Foundation FWF, grant S9604, as well as by the EU FP6-NEST-Adventure Programme, contract no. 028875 (NEST).

In fact, for the latter function the unit circle is a natural boundary (i.e., no analytic continuation beyond it is possible). There is exactly one singularity on the circle of convergence of the power series expansion at 0, but the analytic continuation has an infinite number of singularities inside the unit circle. Each point on the unit circle is an accumulation point of the set of singularities. These facts follow from the functional equation defining this generating function and the fact that the power series expansion around zero has radius of convergence strictly smaller than one (see next section). It also involves an analytically complicated structure like the cycle index of the symmetric group. Due to this difference with respect to the analytic behaviour of the generating function, Pólya trees are certainly not simply generated. Therefore they cannot be represented as branching processes.

Nevertheless Pólya trees behave in many respects similar to simply generated trees (compare with [41, 26, 35, 36, 13, 24]). Hence it is expected that the order of the height is $\sqrt{n}$ as well. In this paper we will give an affirmative answer to this question. This justifies the choice of $\sqrt{n}$ for the scaling of the level sizes in the subsequent theorems.

The plan of the paper is as follows. In the next section we present our main results. Then we will set up the generating functions for our counting problem of trees with nodes in certain levels marked. This function is given as solution of a recurrence relation which has to be analyzed in detail. Knowing the singular behaviour of the considered generating functions allows us to show that the finite dimensional distributions (fdd's) of the profile, i.e., the distributions of the sizes of several levels considered simultaneously, converge to the fdd's of Brownian excursion local time. The singularity analysis is carried out in Section 4. There we first prove that the limiting profile has the proposed shape and in a second step we show the proposed normalization. In order to complete the functional limit theorem we need to prove tightness. This means, roughly speaking, that the sample paths of the process do not have too strong fluctuations (see [2] for the general theory).

In the final section we turn to the height. Most of the work has already been done by Flajolet and Odlyzko [20] and Flajolet et al. [19] in their studies of simply generated trees where they completed the program started in [8]. In fact, what we have to do is to show that the generating function appearing in the analysis of the height has a local structure which is amenable to the steps carried out in [20] and [19]. This is done in the last section and leads to average and distribution of the height.

2. Preliminaries and Results

First we collect some results for unlabeled unrooted trees. Let $\mathcal{Y}_n$ denote the set of unlabeled rooted trees consisting of n vertices and y_n be the cardinality of this set. Pólya [40] already discussed the generating function

$$y(x) = \sum_{n\geq 1} y_n x^n$$

and showed that the radius of convergence ρ satisfies $0 < \rho < 1$ and that $x = \rho$ is the only singularity on the circle of convergence $|z| = \rho$. He also showed that $y(x)$ satisfies the functional equation

$$y(x) = x \exp\left(\sum_{i\geq 1} \frac{y(x^i)}{i}\right).$$

Later [39] showed that $y(\rho) = 1$ as well as the asymptotic expansion

$$y(x) = 1 - b(\rho - x)^{1/2} + c(\rho - x) + d(\rho - x)^{3/2} + \cdots \tag{1}$$

which he used to deduce that

$$y_n \sim \frac{b\sqrt{\rho}}{2\sqrt{\pi}} n^{-3/2} \rho^{-n}. \tag{2}$$

Furthermore he calculated the first constants appearing in this expansion: $\rho \approx 0.3383219$, $b \approx 2.6811266$, and $c = b^2/3 \approx 2.3961466$.

The height of a tree is the maximal number of edges on a path from the root to another vertex of the tree. It turns out that the average height is of order $\sqrt{n}$.

Theorem 1. *Let H_n denote the height of an unlabeled rooted random tree with n vertices. Then we have*

$$\mathbf{E}H_n \sim \frac{2\sqrt{\pi}}{b\sqrt{\rho}}\sqrt{n}. \tag{3}$$

Moreover, let $y_n^{(h)}$ denote the number of unlabeled rooted trees with n vertices and height equal to h and let $\delta > 0$ arbitrary but fixed. If we set $\beta = 2\sqrt{n}/hb\sqrt{\rho}$, then, as $n \to \infty$, we have

$$\mathbf{P}\left\{H_n = h\right\} = \frac{y_n^{(h)}}{y_n} \sim 4b\sqrt{\frac{\rho\pi^5}{n}}\beta^4 \sum_{m\geq 1} m^2(2(m^2\pi^2\beta^2 - 3)e^{-m^2\pi^2\beta^2} \tag{4}$$

uniformly for $\frac{1}{\delta\sqrt{\log n}} \leq \frac{h}{\sqrt{n}} \leq \delta\sqrt{\log n}$.

The proof of this theorem is deferred to the last section, since the proofs of the auxiliary lemmas which will eventually establish the assertion will build upon the results needed to prove the next three theorems.

Let $L_n(t)$ denote the number of nodes at distance t from the root of a randomly chosen unlabeled rooted tree of size n. If t is not an integer, then define $L_n(t)$ by linear interpolation:

$$L_n(t) = (\lfloor t\rfloor + 1 - t)L_n(\lfloor t\rfloor) + (t - \lfloor t\rfloor)L_n(\lfloor t\rfloor + 1), \quad t \geq 0.$$

We will show the following theorem.

Theorem 2. *Let b be the constant of Equation (1),*

$$l_n(t) = \frac{1}{\sqrt{n}}L_n\left(t\sqrt{n}\right),$$

and $l(t)$ denote the local time of a standard scaled Brownian excursion. Then $l_n(t)$ converges weakly to the local time of a Brownian excursion, i.e., we have

$$(l_n(t))_{t\geq 0} \xrightarrow{w} \left(\frac{b\sqrt{\rho}}{2\sqrt{2}} \cdot l\left(\frac{b\sqrt{\rho}}{2\sqrt{2}} \cdot t\right)\right)_{t\geq 0}$$

in $C[0,\infty)$, as $n \to \infty$.

In order to prove this result we have to show the following two theorems

Theorem 3. *Let b and $l_n(t)$ be as in Theorem 2, then for any d and any choice of fixed numbers $t_1, \ldots, t_d$ the following limit theorem holds:*

$$(l_n(t_1), \ldots, l_n(t_d)) \xrightarrow{w} \frac{b\sqrt{\rho}}{2\sqrt{2}}\left(l\left(\frac{b\sqrt{\rho}}{2\sqrt{2}} \cdot t_1\right), \ldots, l\left(\frac{b\sqrt{\rho}}{2\sqrt{2}} \cdot t_d\right)\right),$$

as $n \to \infty$.

Theorem 4. *With the notation of Theorem 2 we have*

$$\mathbf{E}\left(L_n(r) - L_n(r+h)\right)^4 \leq C\,h^2 n \tag{5}$$

for all non-negative integers n, r, h and some fixed constant $C > 0$. Consequently, the process $l_n(t)$ is tight.

3. Combinatorial Setup

In order to compute the distribution of the number of nodes in some given levels in a tree of size n we have to calculate the number $y_{k_1m_1k_2m_2\cdots k_dm_dn}$ of trees of size n with m_i nodes in level k_i, $i = 1, \ldots, d$ and normalize by y_n.

Therefore we introduce the generating functions $y_k(x,u)$ defined by the recurrence relation

$$\begin{aligned} y_0(x,u) &= uy(x) \\ y_{k+1}(x,u) &= x\exp\left(\sum_{i\ge 1}\frac{y_k(x^i,u^i)}{i}\right), \quad k\ge 0. \end{aligned} \tag{6}$$

The function $y_k(x,u)$ represents trees where the nodes in level k are marked (and counted by u). If we want to look at two levels at once, say k and ℓ, then we have to take trees with height at most k and substitute the leaves in level k by trees with all nodes at level $\ell-k$ marked (counted by v) and marking their roots as well (counted by u). This leads to the generating function $y_{k,\ell}(x,u,v)=\tilde{y}_{k,\ell-k}(x,u,v)$ satisfying the recurrence relation

$$\begin{aligned} \tilde{y}_{0,m}(x,u,v) &= uy_m(x,v) \\ \tilde{y}_{k+1,m}(x,u,v) &= x\exp\left(\sum_{i\ge 1}\frac{\tilde{y}_{k,m}(x^i,u^i,v^i)}{i}\right), \quad k\ge 0. \end{aligned} \tag{7}$$

In general we get therefore

$$\begin{aligned} y_{k_1,\ldots,k_d}(x,u_1,\ldots,u_d) &= \sum_{m_1,\ldots,m_d,n\ge 0} y_{k_1m_1k_2m_2\cdots k_dm_dn}u_1^{m_1}\cdots u_d^{m_d}x^n \\ &= \tilde{y}_{k_1,k_2-k_1,k_3-k_2,\ldots,k_d-k_{d-1}}(x,u_1,\ldots,u_d) \end{aligned}$$

where

$$\begin{aligned} \tilde{y}_{0,m_2,\ldots,m_d}(x,u_1,\ldots,u_d) &= u_1\tilde{y}_{m_2,\ldots,m_d}(x,u_2,\ldots,u_d) \\ \tilde{y}_{0,m_2,\ldots,m_d}(x,u_1,\ldots,u_d) &= x\exp\left(\sum_{i\ge 1}\frac{\tilde{y}_{m_2,\ldots,m_d}(x^i,u_2^i,\ldots,u_d^i)}{i}\right), \quad k\ge 0. \end{aligned}$$

As claimed in Theorem 2, the process $l_n(t)=\frac{1}{\sqrt{n}}L_n(t)$ converges weakly to Brownian excursion local time. From [27] (cf. [6, 12] as well) we know that the characteristic function $\phi(t)$ of the total local time of a standard Brownian excursion at level κ is

$$\phi(t) = 1+\frac{\sqrt{2}}{\sqrt{\pi}}\int_\gamma \frac{t\sqrt{-s}\exp(-\kappa\sqrt{-2s}\,)}{\sqrt{-s}\exp(\kappa\sqrt{-2s}\,)-it\sqrt{2}\sinh\left(\kappa\sqrt{-2s}\,\right)}e^{-s}\,ds \tag{8}$$

where $\gamma=(c-i\infty,c+i\infty)$ with some arbitrary $c>0$. The characteristic function of the joint distribution of the local time at several levels $\kappa_1,\ldots,\kappa_d$ was computed in [12] (for $d=2$ already in [6] albeit written down in a form which does not exhibit the recursive structure) and is given by

$$\phi_{\kappa_1\ldots\kappa_d}(t_1,\ldots,t_d) = 1+\frac{\sqrt{2}}{i\sqrt{\pi}}\int_\gamma f_{\kappa_1,\ldots,\kappa_d}(x,t_1,\ldots,t_d)e^{-x}\,dx, \tag{9}$$

where

$$f_{\kappa_1,\ldots,\kappa_p}(x,t_1,\ldots,t_d) = \Psi_{\kappa_1}(x,t_1+\Psi_{\kappa_2-\kappa_1}(\ldots\Psi_{\kappa_{d-1}-\kappa_{d-2}}(x,t_{d-1}+\Psi_{\kappa_d-\kappa_{d-1}}(x,t_d))\cdots))$$

with

$$\Psi_\kappa(x,t) = \frac{it\sqrt{-x}\exp(-\kappa\sqrt{-2x}\,)}{\sqrt{-x}\exp(\kappa\sqrt{-2x}\,)-it\sqrt{2}\sinh\left(\kappa\sqrt{-2x}\right)}.$$

In order to show the weak limit theorem we have to show pointwise convergence of the characteristic function $\phi_{k_1\cdots k_dn}(t_1,\ldots,t_d)$ of the joint distribution of $\frac{1}{\sqrt{n}}L_n(k_1),\ldots,\frac{1}{\sqrt{n}}L_n(k_d)$ to the corresponding characteristic function of the local time in some interval containing zero. We have

$$\phi_{k_1\cdots k_dn}(t_1,\ldots,t_d) = \frac{1}{y_n}[x^n]y_{k_1,\ldots,k_d}\left(x,e^{it_1/\sqrt{n}},\ldots e^{it_d/\sqrt{n}}\right).$$

This coefficient will be calculated asymptotically by singularity analysis (see [18]). Thus knowing the local behaviour of $y_k(x,u)$ near its dominant singularity is the crucial step in proving Theorem 2. Hence the following theorem will be the crucial step of the proof.

Theorem 5. *Set $w_k(x,u) = y_k(x,u) - y(x)$. Let $x = \rho\left(1+\frac{s}{n}\right)$, $u = e^{it/\sqrt{n}}$, and $k = \kappa\sqrt{n}$. Moreover, assume that $|\arg s| \ge \theta > 0$ and, as $n \to \infty$, we have $s = \mathcal{O}\left(\log^2 n\right)$ whereas t and κ are fixed. Then $w_k(x,u)$ admits the local representation*

$$w_k(x,u) \sim \frac{b^2\rho}{2\sqrt{n}} \cdot \frac{it\sqrt{-s}\exp\left(-\kappa b\sqrt{-\rho s}\,\right)}{\sqrt{-s} - \frac{itb\sqrt{\rho}}{4}\left(1-\exp\left(-\kappa b\sqrt{-\rho s}\,\right)\right)} \tag{10}$$

uniformly for $k = \mathcal{O}\left(\sqrt{n}\right)$.

The proof is deferred to the next section.

Proof of Theorem 3: For $d=1$ we have

$$\begin{aligned}\phi_{k,n}(t) &= \frac{1}{y_n}[x^n]y_k\left(x, e^{it/\sqrt{n}}\right)\\ &= \frac{1}{2\pi i y_n}\int_\Gamma y_k\left(x, e^{is/\sqrt{n}}\right)\frac{dz}{x^{n+1}}\end{aligned} \tag{11}$$

where the contour $\Gamma = \gamma \cup \Gamma'$ consists of a line

$$\gamma = \{x = \rho\left(1 - \frac{1+it}{n}\right) \mid -C\log^2 n \le t \le C\log^2 n\}$$

with an arbitrarily chosen fixed constant $C>0$ and Γ' is a circular arc centered at the origin and making Γ a closed curve.

The contribution of Γ_4 is exponentially small since for $x \in \Gamma'$ we have $|x^{-n-1}| \sim e^{-\log^2 n}$ whereas $\left|y_k\left(x, e^{is/\sqrt{n}}\right)\right|$ is bounded.

If $x \in \gamma$, then the local expansion (10) is valid. Insertion into (11), using (2), and taking the limit for $n \to \infty$ yields the characteristic function of the distribution of $l(b\kappa/2\sqrt{2}\,)/b\sqrt{\rho}$ as desired.

Now we can proceed by induction. For instance, for $d=2$ we have

$$\phi_{k,k+h,n}(t_1,t_2) = \frac{1}{y_n}[x^n]\tilde{y}_{k,h}\left(x, e^{it_1/\sqrt{n}}, e^{it_2/\sqrt{n}}\right)$$

and

$$\tilde{y}_{k,h}\left(x, e^{it_1/\sqrt{n}}, e^{it_2/\sqrt{n}}\right) = y(x) + \tilde{w}_k$$

where $\tilde{w}_k$ can be estimated similarly by application of Theorem 5. This step can be repeated easily for $d>2$ and in this way we get the characteristic functions of the fdd's of $l((b/2\sqrt{2})\cdot t)/b\sqrt{\rho}$ as desired. □

4. The Local Behaviour of y_k – Proof of Theorem 5

4.1. **The shape of the limiting process.** We will study the local behaviour of y_k by analyzing the quantity

$$w_k(x,u) = y_k(x,u) - y(x).$$

Obviously, $w_k(x,1) \equiv 0$. Since $y_k(x,u)$ represents the set of trees where the vertices of level k are marked, we expect that $\lim_{k\to\infty} y_k(x,u) = y(x)$ inside the domain of convergence. This is not obvious, but follows from what we derive in the sequel. We start with an *a priori* estimate for a smaller domain.

Lemma 1. *Let $|x| \le \rho^2 + \varepsilon$ for sufficiently small ε and $|u| \le 1$. Then there exist a constant L with $0 < L < 1$ and a positive constant C such that*

$$|w_k(x,u)| \le C|u-1|\cdot|x|\cdot L^k$$

for all non-negative integers k.

Proof: We first note that by using the recurrence relation (6) we obtain

$$\begin{aligned} w_{k+1}(x,u) &= y_{k+1}(x,u) - y(x) \\ &= x\exp\left(\sum_{i\ge 1}\frac{1}{i}y_k(x^i,u^i)\right) - y(x) \\ &= y(x)\left(\exp\left(w_k(x,u) + \sum_{i\ge 2}\frac{w_k(x^i,u^i)}{i}\right) - 1\right) \end{aligned} \tag{12}$$

For $k=0$ we have $|w_0(x,u)| = |u-1|\cdot|y(x)| \le C|u-1||x|$ since $y(x) = \mathcal{O}(x)$ as $x\to 0$. We will then use the trivial inequality

$$|e^x - 1| \le \frac{|x|}{1-\frac{|x|}{2}} \tag{13}$$

for the induction steps. However, in order to apply this tool we need some a-priori estimates.

Obviously we have for $|x|\le\rho$ and $|u|\le 1$

$$|w_k(x,u)| \le 2y(|x|)$$

and consequently

$$\left|\sum_{i\ge 1}\frac{w_k(x^i,u^i)}{i}\right| \le 2\sum_{i\ge 1}\frac{y(|x|^i)}{i} = 2\log\frac{y(|x|)}{|x|}.$$

Since the function $\frac{y(x)}{x}$ convex for $0\le x\le\rho$ and $y(\rho)=1$ we get $\frac{y(|x|)}{|x|} \le 1+\frac{|x|}{\rho}$ and consequently

$$\log\frac{y(|x|)}{|x|} \le \log\left(1+\frac{|x|}{\rho}\right) \le \frac{|x|}{\rho}.$$

Thus, if $|x|\le\rho^2+\varepsilon$ (for a sufficienly small $\varepsilon>0$ we have

$$\left|\sum_{i\ge 1}\frac{w_k(x^i,u^i)}{i}\right| \le 2\rho + 2\frac{\varepsilon}{\rho}.$$

By using (13) we thus obtain

$$\left|\exp\left(\sum_{i\ge 1}\frac{w_k(x^i,u^i)}{i}\right) - 1\right| \le \frac{1}{1-\rho-\frac{\varepsilon}{\rho}}\left|\sum_{i\ge 1}\frac{w_k(x^i,u^i)}{i}\right|.$$

Consequently, if we assume that we already know $|w_k(x,u)| \le C|u-1||x|L^k$ (for $|x|\le\rho$, $|u|\le 1$ and some L with $0<L<1$) then we also get

$$\begin{aligned} |w_{k+1}(x,u)| &\le |y(x)|\cdot\left|\exp\left(\sum_{i\ge 1}\frac{w_k(x^i,u^i)}{i}\right) - 1\right| \\ &\le \frac{|y(x)|}{1-\rho-\frac{\varepsilon}{\rho}}\left|\sum_{i\ge 1}\frac{w_k(x^i,u^i)}{i}\right| \\ &\le \frac{|y(x)|}{1-\rho-\frac{\varepsilon}{\rho}}CL^k\sum_{i\ge 1}\frac{|u^i-1|}{i}|x|^i \\ &\le \frac{|y(x)|}{1-\rho-\frac{\varepsilon}{\rho}}CL^k|u-1|\frac{|x|}{1-|x|}. \end{aligned}$$

By convexity we have $y(x) < x/\rho$ for $0<x<\rho$ and, thus, ther exists $\varepsilon>0$ with $y(\rho^2+\varepsilon)\le\rho$. Consequently we get for $|x|\le\rho^2+\varepsilon$ the estimate

$$|w_{k+1}(x,u)| \le CL'L^k|x||u-1|$$

with

$$L' = \frac{\rho}{\left(1-\rho-\frac{\varepsilon}{\rho}\right)(1-\rho^2-\varepsilon)}$$

that is smaller than 1 if $\varepsilon > 0$ is sufficiently small. Thus, an induction proof works for $L = L'$. □

For the following calculation let us use the abbreviation

$$\Sigma_k(x,u) := \sum_{i\ge 2} \frac{w_k(x^i,u^i)}{i}$$

Corollary 1. *For $|u| \le 1$ and $|x| \le \rho + \varepsilon$ ($\varepsilon > 0$ small enough) there is a positive constant $\tilde{C}$ such that (for all $k \ge 0$)*

$$|\Sigma_k(x,u)| \le \tilde{C}|u-1|L^k$$

with the constant L from the previous lemma.

Proof: We have

$$\begin{aligned}|\Sigma_k(x,u)| &\le \sum_{i\ge 2}\frac{1}{i}|w_k(x^i,u^i)| \le C\sum_{i\ge 2}\frac{1}{i}|u^i-1|\cdot|x|^iL^k \\ &\le C|u-1|L^k\frac{|x|^2}{1-|x|} \le C|u-1|L^k\frac{1}{1-(\rho+\varepsilon)} = \tilde{C}|u-1|L^k\end{aligned}$$

□

The asymptotic analysis of w_k resp. y_k enables us to apply Cauchy's integral formula and get the coefficients of $y_k(x,u)$ asymptotically (see the proof of Theorem 3 at the end of the previous section) which eventually leads to an integral of the form (8). Therefore estimates for $y(x)$, provided in the next lemma, and the other functions appearing appearing in our analysis are needed. Most of the estimates will refer to the domain

$$\Delta = \{x \in \mathbb{C} : |x| < \rho+\eta,\ |\arg(x-\rho)| > \theta\}, \tag{14}$$

with $\eta > 0$ and $0 < \theta < \frac{\pi}{2}$.

Lemma 2. *Provided that η in (14) is sufficiently small, the generating function $y(x)$ has the following properties:*

a) *For $x \in \Delta$ we have $|y(x)| \le 1$. Equality holds only for $x = \rho$.*
b) *Let $x = \rho\left(1-\frac{1+it}{n}\right)$ and $|t| \le C\log^2 n$ for some fixed $C > 0$. Then there is a $c > 0$ such that*

$$|y(x)| \le 1 - c\sqrt{\frac{\max(1,|t|)}{n}}.$$

Proof: The first statement, when restricted to $|x| \le \rho$, follows from the facts that $y(x)$ has only positive coefficients (except $y_0 = 0$), $y(\rho) = 1$ and there are no periodicities. Extension to Δ is easily established by using (1) and continuity arguments.

The second statement is an immediate consequence of the singular expansion (1) of $y(x)$. □

The next step is an extension of the bound of Lemma 1 to the region near the singularity. We will derive a lower bound as well.

Lemma 3. *Assume $|u| \le 1$, $|u-1| \le \varepsilon$ and $x \in \Delta$ with sufficiently small $\varepsilon > 0$. Then there are positive constants C_1, C_2, C_3, and C_4 such that*

$$|w_k(x,u)| \le C_1|w_0(x,u)|\cdot|y(x)|^k = C_1|u-1|\cdot|y(x)|^{k+1}$$

and

$$|w_k(x,u)| \ge C_2|w_0(x,u)|\cdot|y(x)|^k(1-C_3\varepsilon)^k = C_2|u-1|\cdot|y(x)|^{k+1}(1-C_3\varepsilon)^k$$

uniformly for $k \le \frac{C_4}{|u-1|}$.

Proof: We prove both statements at once by induction. The start is trivial since we can choose $C_1 = C_2 = 1$. Now introduce auxiliary bounds which depend on k and let $c_0 := 1$, $\tilde{c}_0 := 1$. In fact we work with the assumption

$$c_k|w_0| \cdot |y(x)|^k (1 - C_3\varepsilon)^k \le |w_k| \le \tilde{c}_k |w_0| \cdot |y(x)|^k \le C_1 |w_0| \cdot |y(x)|^k \tag{15}$$

and when referring to the induction hypothesis in the following we always mean (15). It will turn out that the sequence $(c_k)_{k\ge 0}$ can be chosen monotonically decreasing and convergent such that the limit serves as the constant C_2 in the lower bound of the assertion. Throughout the proof $C_5, C_6, \ldots$ denote suitable positive constants and L a suitable constant lying strictly between 0 and 1.

We start with the upper bound. From

$$w_{k+1}(x,u) = y(x)\left(e^{w_k(x,u)+\Sigma_k(x,u)} - 1\right)$$

we infer

$$\begin{aligned} |w_{k+1}| &= |y(x)| \cdot |w_k| \left|1 + \frac{\Sigma_k}{w_k}\right| \left(1 + \frac{w_k + \Sigma_k}{2} + \mathcal{O}\left((w_k + \Sigma_k)\right)^2\right) \\ &\le \tilde{c}_k |w_0| \cdot |y(x)|^{k+1} \left|1 + \frac{\Sigma_k}{w_k}\right| \left(1 + \frac{w_k + \Sigma_k}{2} + \mathcal{O}\left((w_k + \Sigma_k)\right)^2\right) \end{aligned} \tag{16}$$

where we used the induction hypothesis on the upper bound in the last step. By Corollary 1 and the induction hypothesis on the lower bound we deduce

$$\left|1 + \frac{\Sigma_k}{w_k}\right| \le 1 + C_5 L^k. \tag{17}$$

In order to estimate the factor containing the error term in (16), we remark that due to the induction hypothesis, Corollary 1, Lemma 2a, and the choice of ε we can guarantee that the error term is small enough. Furthermore, the induction hypothesis and Lemma 2a imply

$$\left|1 + \frac{w_k + \Sigma_k}{2} + \mathcal{O}\left((w_k + \Sigma_k)\right)^2\right| \le 1 + C_6|u-1|$$

and thus the upper bound holds with $C_1 = (1 + C_6|u-1|)^k \prod_{i\ge 1}\left(1 + C_4 L^i\right)$. Note that $(1+C_6|u-1|)^k$ is bounded for the allowed range of k.

Let us turn to the lower bound now. Similarly as above, we use the recurrence relation for w_k to deduce

$$\begin{aligned} |w_{k+1}| &\ge |w_k| \cdot |y(x)| \left(1 - \left|\frac{\Sigma_k}{w_k}\right|\right) \left(1 + \frac{w_k + \Sigma_k}{2} + \mathcal{O}\left((w_k + \Sigma_k)\right)^2\right) \\ &\ge c_k |w_0| \cdot |y(x)|^{k+1} (1 - C_3\varepsilon)^k (1 - C_5 L^k) \left(1 + \frac{w_k + \Sigma_k}{2} + \mathcal{O}\left((w_k + \Sigma_k)\right)^2\right) \end{aligned}$$

where the last step follows from Corollary 1 and the induction hypothesis on the lower bound. As above, the error term can be made negligibly small. Corollary 1 again, this time in conjunction with the induction hypothesis on the upper bound yield

$$1 + \frac{w_k + \Sigma_k}{2} + \mathcal{O}\left((w_k + \Sigma_k)^2\right) \ge 1 - C_7 |w_0(x,u)||y(x)|^k \tag{18}$$

Since $|w_0(x,u)||y(x)|^k < \varepsilon$, the expression in (18) is bounded from below by $1 - C_3\varepsilon$ for $C_3 \ge C_7$. Setting $C_2 = \prod_{i\ge 1}\left(1 - C_5 L^k \rho^k\right)$ we obtain the lower bound after all. $\square$

Lemma 4. *Let $x \in \Delta$, $|u| \le 1$. Then there are a $C > 0$ and a constant $0 < \tilde{L} < 1$ such that*

$$\left|\frac{\Sigma_k(x,u)}{w_k(x,u)}\right| < C\tilde{L}^k.$$

Proof: First, let $|x-\rho|<\varepsilon$ and $|u-1|<\varepsilon$. Using Corollary 1 and the lower bound in Lemma 3 we get

$$\left|\frac{\Sigma_k(x,u)}{w_k(x,u)}\right| \le \frac{\tilde{C}}{C_2}\left(\frac{L}{1-C_4\varepsilon}\right)^k \frac{1}{|y(x)|^{k+1}}.$$

By the local expansion 1 $|y(x)|$ is bounded from below which implies the assertion.

Notice that $\Sigma_k(x,u)$ is an analytic function the power series of which has x^{2k+2} as its lowest power of x. The lowest power of x in the power series of $w_k(x,u)$ is x^{k+1}. At $u=1$ both functions have a first order zero. Thus the quotient is an analytic function as well with a power series starting with x^{k+1}. This implies the assertion for the whole domain. □

With these auxiliary results we are able get the first more precise result for $w_k(x,u)$

Proposition 1. *Let $|u-1|<\varepsilon$ and $|x-\rho|<\varepsilon$ (with $\varepsilon>0$ sufficiently small) such that $x\in\Delta$. Then we have*

(19)
$$w_k(x,u) = \frac{y(x)^k w_0}{1-w_0\left(\frac{1}{2}\frac{1-y(x)^k}{(1-y(x))}+\sum_{\ell=0}^{k-1}\frac{\exp\left(\sum_{i\ge 2} w_\ell(x^i,u^i)/i\right)-1}{w_\ell(x,u)^2}y^\ell\right)+\mathcal{O}\left(\left|w_0^2\frac{1-y^{2k}}{1-y^2}\right|\right)+\mathcal{O}\left(|w_0|\right)}$$

Proof: Observe that $w_k(x,u)$ satisfies the recurrence relation (we omit the arguments now)

$$w_{k+1} = yw_k\left(1+\frac{w_k}{2}+\mathcal{O}\left(w_k^2\right)+\frac{e^{\Sigma_k}-1}{w_k}\left(1+w_k+\mathcal{O}\left(w_k^2\right)\right)\right).$$

We know already from Lemma 4 that

$$\frac{e^{\Sigma_k}-1}{w_k} \sim \frac{\Sigma_k}{w_k} = \mathcal{O}\left(\tilde{L}^k\right).$$

Using this information and substituting $q_k = y^k/w_k$ (cf. [7, p. 156]) this recurrence is transformed into

$$\begin{aligned}
q_{k+1} &= q_k\left(1-\frac{w_k}{2}+\mathcal{O}\left(w_k^2\right)-\frac{e^{\Sigma_k}-1}{w_k}\left(1+w_k+\mathcal{O}\left(w_k^2\right)\right)\right)\\
&= q_k-\frac{y^k}{2}-\frac{e^{\Sigma_k}-1}{w_k^2}y^k+\mathcal{O}\left(w_0y^{2k}\right)+\mathcal{O}\left(\tilde{L}^k\right)
\end{aligned}$$

Thus we have

$$q_k = \frac{1}{w_0}-\frac{1}{2}\frac{1-y^k}{1-y}-\sum_{\ell=0}^{k-1}\frac{e^{\Sigma_\ell}-1}{w_\ell^2}y^\ell+\mathcal{O}\left(\frac{1-\tilde{L}^k}{1-\tilde{L}}\right)+\mathcal{O}\left(w_0\frac{1-y^{2k}}{1-y^2}\right)$$

which immediately implies the assertion. □

Looking at (19) more closely (i.e., near the singularity $(x,u)=(\rho,1)$ it will turn out that the term $w_0\frac{1-y(x)^k}{1-y(x)}$ appearing in the denominator behaves like a constant. The sum $w_0\sum_{\ell=0}^k(e^{\Sigma_\ell}-1)y^\ell w_\ell^{-2}$ is the term where the shape of $w_k(x,u)$ for Pólya trees differs from the analogue for simply generated trees. Since Pólya trees behave similar to simply generated trees, we might be tempted to expect that this term is asymptotically negligible. In that case, we would get an expression similar to (10) for w_k. A consequence of this would be that the limit of the profile process is of the form $Al(Bt)$ with different normalizing constants A and B. This implies that the sequence $l_n(t)$ cannot be tight, since the limiting process has points where mass is concentrated. Of course, such a strange behaviour would be a big surprise since we are not aware of any tree class exhibiting such a phenomenon. Therefore we expect the sum to be relevant. In fact, in order to get the nice weak limit theorem which we expect it should even behave like a constant. As we will show now, this is indeed the case.

Proposition 2. *Under the assumptions of Theorem 5, i.e.,* $x = \rho\left(1 + \frac{s}{n}\right)$ *with* $|\arg s| \ge \theta > 0$, $u = e^{it/\sqrt{n}}$, $k = \kappa\sqrt{n}$, $s = \mathcal{O}\left(\log^2 n\right)$, *and* t *bound, we have*

$$f_k(x,u) := w_0(x,u)\sum_{\ell=0}^{k} \frac{e^{\Sigma_\ell(x,u)} - 1}{w_\ell(x,u)^2} y(x)^\ell \to c_0$$

as $n \to \infty$.

Notice that continuity of f_k is not sufficient to prove Proposition 2, because the index k depends on n, too. Thus we need several auxiliary results first.

Lemma 5. *Let* $|u-1| < \varepsilon$ *and* $|x-\rho| < \varepsilon$ *(with* ε *sufficiently small) such that* $x \in \Delta$. *Furthermore set* $\Xi_k := w_k(x,u) - w_k(\rho,u)$. *Then, as* $n \to \infty$ *and* $k \to \infty$ *such that* $k = o(\sqrt{n})$, *we have* $\Xi_k = \mathcal{O}\left(k|u-1|\sqrt{\rho - x}\,\right)$.

Proof: For proving the assertion we will derive a recurrence relation for Ξ_k. First observe that

$$\Xi_0 = (u-1)(y(x) - y(\rho)) = \mathcal{O}\left(|u-1|\sqrt{\rho - x}\right). \tag{20}$$

Furthermore we have

$$\begin{aligned}
\Xi_{k+1} &= w_{k+1}(x,u) - w_{k+1}(\rho,u)\\
&= y(x)e^{w_k(x,u)+\Sigma_k(x,u)} - y(x) - e^{w_k(\rho,u)+\Sigma_k(\rho,u)} + 1\\
&= e^{w_k(x,u)+\Sigma_k(x,u)} - e^{w_k(\rho,u)+\Sigma_k(\rho,u)} - (1-y(x))\left(e^{w_k(x,u)+\Sigma_k(x,u)} - 1\right)
\end{aligned}$$

By Lemma 3 and Corollary 1 we know that $w_k(x,u) = \mathcal{O}\left(|u-1|\right)$ and $\Sigma_k(x,u) = \mathcal{O}\left(|u-1|L^k\right)$ with $0 < L < 1$. Hence expanding the exponentials gives

$$\begin{aligned}
\Xi_{k+1} &= w_k(x,u) - w_k(\rho,u) + \Sigma_k(x,u) - \Sigma_k(\rho,u)\\
&\qquad - (1-y(x))\left(e^{w_k(x,u)+\Sigma_k(x,u)} - 1\right) + \mathcal{O}\left(|u-1|^2\right)\\
&= \Xi_k + (1-y(x))\mathcal{O}\left(|u-1|\cdot|y|^k\right)
\end{aligned}$$

Solving this recurrence relation and using the expansion (1) for $y(x)$ gives

$$\Xi_k = \Xi_0 + \mathcal{O}\left(\sqrt{\rho - x}\cdot|u-1|\cdot\left|\frac{1-y^k(x)}{1-y(x)}\right|\right).$$

But $\frac{1-y^k(x)}{1-y(x)} = \mathcal{O}\left(k\right)$ for $k = o(\sqrt{n})$ and so using (20) completes the proof. $\square$

Lemma 6. *There exist functions* $C(x,u)$ *and* $D(x,u)$ *which are analytic in a neighbourhood of* $(x,u) = (\rho, 1)$ *such that* $f_\infty(x,u) := \lim_{k\to\infty} f_k(x,u)$ *admits the singular expansion*

$$f_\infty(x,u) = C(x,u) - D(x,u)\sqrt{1 - \frac{x}{\rho}}.$$

Proof: First observe that $w_k(x,u)$ admit similar expansions, i.e., there are analytic functions $g_k(x,u)$ and $h_k(x,u)$ such that

$$w_k(x,u) = g_k(x,u) - h_k(x,u)\sqrt{1 - \frac{x}{\rho}}. \tag{21}$$

This can be shown inductively by starting with $w_0(x,u) = (u-1)y(x)$ which obviously has the singular behaviour claimed above. Then use the recurrence relation (12) for $w_k(x,u)$ while keeping in mind that $\Sigma_k(x,u)$ is analytic. A consequence of (21) is that the terms in the sum defining $f_k(x,u)$, namely $\left(e^{\Sigma_\ell(x,u)} - 1\right)w_0(x,u)/w_\ell(x,u)^2$ have a singular expansion of the same type at $(x,u) = (\rho,1)$ and are analytic for $|u| \le 1$ and $|x| < \rho$.

By Corollary 1 and Lemmas 4 and 3, respectively, we infer that there are constants $C > 0$ and $0 < L < 1$ such that

$$e^{\Sigma_\ell(x,u)} - 1 \sim \Sigma_\ell(x,u) \quad\text{and}\quad \left|\frac{\Sigma_\ell(x,u)y(x)^\ell}{w_\ell(x,u)^2}\right| \le \frac{CL^\ell}{|u-1|}. \tag{22}$$

This implies that the series

$$w_0(x,u)\sum_{\ell=0}^{\infty}\frac{e^{\Sigma_\ell(x,u)-1}}{w_\ell(x,u)^2}$$

is uniformly convergent in the domain $|u|\le 1$, $|x|\le\rho$. Since the terms summed up are analytic functions in this domain except at $(\rho,1)$, by Weierstraß' double series theorem this series represents an analytic function there. However, the nature of the singularity could change by the infinite summation which requires further analysis.

Observe that by (22) again we have

$$f_k(x,u)\sim w_0(x,u)\sum_{\ell=0}^{k}\frac{\Sigma_\ell(x,u)}{w_\ell(x,u)^2}y(x)^\ell\sim w_0(x,u)\sum_{\ell<\log^2 n}\frac{\Sigma_\ell(x,u)}{w_\ell(x,u)^2}y(x)^\ell,$$

uniformly in x and u, i.e., we can restrict the sum to a range where Lemma 5 will be applicable. Now let us inspect what happens near the singularity. We have

$$\begin{aligned}|f_k(x,u)-f_k(\rho,u)|&\le C|w_0(x,u)|\sum_{\ell<\log^2 n}\left|\frac{1}{w_\ell(x,u)^2}-\frac{1}{w_\ell(\rho,u)^2}\right|\cdot L^\ell|u-1|\\ &=C|u-1|^2|y(x)|L^\ell\sum_{\ell<\log^2 n}\left|\frac{1}{w_\ell(x,u)^2}-\frac{1}{w_\ell(\rho,u)^2}\right|\end{aligned}\tag{23}$$

where $C>0$ and $0<L<1$. Moreover, by Lemma 3 and Lemma 5

$$\frac{1}{w_\ell(x,u)^2}-\frac{1}{w_\ell(\rho,u)^2}=\left(\frac{1}{w_\ell(x,u)w_\ell(\rho,u)^2}+\frac{1}{w_\ell(x,u)^2w_\ell(\rho,u)}\right)\Xi_\ell=\mathcal{O}\left(\ell\sqrt{\rho-x}\,\frac{1}{|u-1|^2}\right)$$

and combining this with (23) yields the assertion after all. □

Proof of Proposition 2: The previous lemma shows that $|f_k(x,u)-f_k(\rho,u)|=\mathcal{O}\left(\sqrt{\rho-x}\,\right)$ holds uniformly w.r.t. u and all nonnegative integers k. This immediately implies the statement of the proposition. □

What we know so far is the following: The characteristic function of the limiting distribution of the number of nodes in a certain level can be computed by taking the limit for $n\to\infty$ in (11) after inserting the local expansion given essentially by Proposition 1. In (11) only the contribution of the part γ of the integration is eventually relevant and there – setting $x=\rho\left(1+\frac{s}{n}\right)$ – we have the following expansions:

$$\begin{aligned}(u-1)y(x)&=\left(e^{it/\sqrt{n}}-1\right)y(x)\sim\frac{it}{\sqrt{n}}\\ 1-y(x)&\sim b\sqrt{-\frac{\rho s}{n}}\\ y(x)^k&\sim\exp\left(-\kappa b\sqrt{-\rho s}\right)\end{aligned}$$

By Lemma 5 this implies

$$w_k(x,u)\sim\frac{1}{\sqrt{n}}\cdot\frac{\frac{it}{1-c_0}\sqrt{-s}\exp\left(-\kappa b\sqrt{-\rho s}\,\right)}{\sqrt{-s}-\frac{it}{2(1-c_0)b\sqrt{\rho}}\left(1-\exp\left(-\kappa b\sqrt{-\rho s}\,\right)\right)}\tag{24}$$

and hence

$$l_n(t)\to\frac{1}{(1-c_0)b\sqrt{2\rho}}\,l\left(\frac{b\sqrt{\rho}}{2\sqrt{2}}\right)\tag{25}$$

for any fixed t. Of course, the generalization to several dimensions can be proved in a similar fashion as Theorem 3. So, our next task is to determine the constant c_0 in Proposition 2 which amounts to computing the normalization constant of the limiting process in (25).

4.2. **The normalization of the limiting process.** In order to get the correct normalization we will compute the limit of expected values $\mathbf{E} l_n(\kappa)$. Thus let

$$\gamma_k(x) = \left[\frac{\partial}{\partial u} y_k(x,u)\right]_{u=1}.$$

Then these expectations can be computed by evaluating the coefficient $[x^n]\gamma_k(x)$ with $k = \kappa\sqrt{n}$ which will be done by Cauchy's integral formula again. Using the shape (24) of $w_k(x,u)$ it is easy to show that $\mathbf{E} l_n(\kappa)^2$ is bounded. This amounts to estimating a Cauchy integral involving the second partial derivative $\frac{\partial^2}{\partial u^2} y_k(x,u)$. We leave this to the reader since Cauchy integrals of that kind will be analyzed in more detail in the next section. By [17, p.251, Example (e)] the boundedness of $\mathbf{E} l_n(\kappa)^2$ implies $\lim_{n\to\infty} \mathbf{E} l_n(\kappa) = \mathbf{E} \lim_{n\to\infty} l_n(\kappa)$ where the limit on the right-hand side means the weak limit. Therefore computing this will exhibit the correct normalizing constant.

Lemma 7. *For $|x| < \rho + \eta$ and $\arg(x-\rho) \neq 0$ (where $\eta > 0$ is sufficiently small) the functions $\gamma_k(x)$ can be represented as*

$$\gamma_k(x) = C_k(x) y(x)^k, \tag{26}$$

where $C_k(x)$ are analytic and converge uniformly to an analytic limit function $C(x)$ (for $|x| < \rho+\eta$ and $\arg(x-\rho) \neq 0$) with convergence rate

$$C_k(x) = C(x) + O(L^k),$$

for some L with $0 < L < 1$

Proof: A tree that has nodes at level k must have size larger than k. Thus $[x^r]y_k(x,u)$ does not depend on u for $r \le k$. Consequently, the lowest order non-vanishing term in the power series expansion of $\gamma_k(x)$ is of order $k+1$. The power series expansion of $y(x)$ starts with x. Hence $C_k(x) = \frac{\gamma_k(x)}{y(x)^k}$ is analytic for $|x| < \rho + \eta$ and $\arg(x-\rho) \neq 0$. We will show that the sequence $(C_k(x))_{k\ge 0}$ has a uniform limit $C(x)$ which has the desired properties.

Using the recurrence relation of $y_k(x,u)$ we get

$$\begin{aligned}\gamma_{k+1}(x) &= \left[\frac{\partial}{\partial u} x e^{y_k(x,u)+\Sigma_k(x,u)}\right]_{u=1} \\ &= \left[x e^{y_k(x,u)+\Sigma_k(x,u)}\right]_{u=1} \sum_{i\ge 1} \left[\frac{\partial}{\partial u} y_k(x^i,u^i)u^{i-1}\right]_{u=1} = y(x)\sum_{i\ge 1}\gamma_k(x^i)\end{aligned}$$

which can be written as

$$C_{k+1}(x)y(x)^{k+1} = C_k(x)y(x)^{k+1} + y(x)\left(C_k(x^2)y(x^2)^k + C_k(x^3)y(x^3)^k + \dots\right). \tag{27}$$

resp. to

$$C_{k+1}(x) = \sum_{i\ge 1} C_k(x^i)\frac{y(x^i)^k}{y(x)^k}. \tag{28}$$

Set

$$L_k := \sup_{|x|<\rho+\eta,\ \arg(x-\rho)\neq 0} \sum_{i\ge 2} \frac{|y(x^i)|^k}{|y(x)|^k}.$$

If $\eta > 0$ is sufficiently small then

$$\sup_{|x|<\rho+\eta,\ \arg(x-\rho)\neq 0} \frac{|y(x^i)|}{|y(x)|} < 1 \quad \text{for all } i \ge 2 \text{ and } \quad \sup_{|x|<\rho+\eta,\ \arg(x-\rho)\neq 0} \frac{|y(x^i)|}{|y(x)|} = O(\overline{L}^i)$$

for some $\overline{L}$ with $0 < \overline{L} < 1$. Consequently we also get

$$L_k = O(L^k)$$

for some L with $0 < L < 1$. Thus, if we use the notation $\|f\| := \sup_{|x|<\rho+\eta,\ \arg(x-\rho)\neq 0} |f(x)|$ then (28) implies

$$\|C_{k+1}\| \le \|C_k\|(1 + L_k) \tag{29}$$

and also

$$\|C_{k+1} - C_k\| \le \|C_k\| L_k \tag{30}$$

Now (29) directly implies that the functions $C_k(x)$ are uniformly bounded in the given domain by

$$\|C_k\| \le c_0 := \prod_{\ell \ge 1} (1 + L_\ell)$$

Further (30) implies that there exists a limit $\lim_{k\to\infty} C_k(x) = C(x)$ that is analytic in the given domain and we have uniform exponential convergence rate

$$\|C_k - C\| \le c_0 \sum_{\ell \ge k} L_\ell = O(L^k)$$

Hence, we finally get (26) as desired. □

If we sum up all the level sizes of a tree, we obviously get the total number of nodes, i.e., the size of the tree. Hence the equations

$$\frac{1}{y_n}[x^n] \sum_{k\ge 0} \gamma_k(x) = \mathbf{E} \sum_{k \ge 0} L_n(k) = n = \frac{1}{y_n}[x^n] x y'(x)$$

hold. Equation (26) tells us that, as $x \to \rho$, we have

$$\sum_{k\ge 0} \gamma_k(x) = \frac{C(x)}{1 - y(x)} + \mathcal{O}(1)$$

and from this we infer

$$C(\rho) = \lim_{x\to\rho} x y'(x)(1 - y(x)) = \frac{b^2 \rho}{2}$$

where we used (1). Therefore $\gamma_k(x)$ is an analytic function with $\gamma_k(x) \sim C_k(\rho) y(x)^k$, as $x \to \rho$, allowing also an analytic continuation outside its disk of convergence. Standard transfer methods à la Flajolet and Odlyzko [18] apply now, where we use Cauchy's formula again with a Hankel-like contour complemented by a circular arc. This yields (for $k = \kappa\sqrt{n}$)

$$\frac{1}{y_n}[x^n]\gamma_k(x) \sim \frac{b^2\rho}{4\pi i y_n} \int_{\mathcal{H}} e^{-\kappa b\sqrt{-\rho s} - s}\, ds = \frac{b^2 \rho \kappa}{2} \exp\left(-\frac{b^2\rho\kappa^2}{4}\right)\sqrt{n}$$

where $\mathcal{H}$ is the Hankel-like contour around the singularity (see [18]). This integral can be transformed into the integral appearing in Hankel's representation of $\Gamma(x)$ (*cf.* [46, p.244]). The above given expression coincides with the first moment (see [44] for the computation of the moments and [25] for a study of the density) of the local time of the Brownian excursion, normalized as claimed in Theorem 3, and completes the proof of Theorem 5.

5. Tightness – Proof of Theorem 4

In this section we will show that the sequence of random variables $l_n(t) = n^{-1/2} L_n(t\sqrt{n})$, $t \ge 0$, is tight in $\mathrm{C}[0,\infty)$. By [30, p. 63] it suffices to prove tightness for $\mathrm{C}[0,T]$. Hence we consider $L_n(t)$ for $0 \le t \le A\sqrt{n}$, where $A > 0$ is an arbitrary real constant.

By [2, Theorem 12.3] tightness of $l_n(t) = n^{-1/2} L_n(t\sqrt{n})$, $0 \le t \le A$, follows from tightness of $L_n(0)$ (which is trivial) and from the existence of a constant $C > 0$ such that (5) holds for all non-negative integers n, r, h.

The fourth moment in the above equation can be expressed as the coefficient of a suitable generating function. We have

$$\mathbf{E}\left(L_n(r) - L_n(r+h)\right)^4 = \frac{1}{y_n}[x^n]\left[\left(\frac{\partial}{\partial u} + 7\frac{\partial^2}{\partial u^2} + 6\frac{\partial^3}{\partial u^3} + \frac{\partial^4}{\partial u^4}\right)\tilde{y}_{r,h}\left(x, u, \frac{1}{u}\right)\right]_{u=1}$$

where $\tilde{y}_{r,h}(x,u,v)$ is defined by (7). Thus, (5) is equivalent to

$$[x^n]\left[\left(\frac{\partial}{\partial u}+7\frac{\partial^2}{\partial u^2}+6\frac{\partial^3}{\partial u^3}+\frac{\partial^4}{\partial u^4}\right)\tilde{y}_{r,h}\left(x,u,\frac{1}{u}\right)\right]_{u=1}\le C\,\frac{h^2}{\sqrt{n}}\rho^{-n} \tag{31}$$

In order to prove (31) we use a result from [18] saying that

$$F(x)=\mathcal{O}\left((1-x/\rho)^{-\beta}\right)\qquad (x\in\Delta)$$

implies

$$[x^n]F(x)=\mathcal{O}\left(\rho^{-n}n^{\beta-1}\right),$$

where Δ is a region of i(14)

Hence, it is sufficient to show that

$$\left[\left(\frac{\partial}{\partial u}+7\frac{\partial^2}{\partial u^2}+6\frac{\partial^3}{\partial u^3}+\frac{\partial^4}{\partial u^4}\right)\tilde{y}_{r,h}\left(x,u,\frac{1}{u}\right)\right]_{u=1}=\mathcal{O}\left(\frac{h^2}{1-|y(x)|}\right)=\mathcal{O}\left(\frac{h^2}{\sqrt{|1-x/\rho|}}\right) \tag{32}$$

for $x\in\Delta$ and $h\ge 1$. (Note that $\theta<\frac{\pi}{2}$ implies that $1-|y(x)|\ge c\sqrt{|1-x/\rho|}$ for some constant $c>0$.)

We now define

$$\gamma_k^{[j]}(x)=\left[\frac{\partial^j y_k(x,u)}{\partial u^j}\right]_{u=1}\qquad\text{and}\qquad \gamma_{r,h}^{[j]}(x)=\left[\frac{\partial^j\tilde{y}_{r,h}\left(x,u,\frac{1}{u}\right)}{\partial u^j}\right]_{u=1}$$

and derive the following upper bounds.

Lemma 8. *We have*

$$\gamma_k^{[1]}(x)=\begin{cases}\mathcal{O}(1) & \text{uniformly for } x\in\Delta,\\ \mathcal{O}\left(|x/\rho|^k\right) & \text{uniformly for } |x|\le\rho\end{cases} \tag{33}$$

and

$$\gamma_{r,h}^{[1]}(x)=\begin{cases}\mathcal{O}\left(\frac{h}{r+h}\right) & \text{uniformly for } x\in\Delta,\\ \mathcal{O}\left(|x/\rho|^r\right) & \text{uniformly for } |x|\le\rho,\end{cases} \tag{34}$$

where L is constant with $0<L<1$.

Proof: We already know that $\gamma_k^{[1]}(x)=C_k(x)y(x)^k$, where $C_k(x)=\mathcal{O}(1)$ and $|y(x)|\le 1$ for $x\in\Delta$. Furthermore, by convexity we also have $|y(x)|\le|x/\rho|$ for $|x|\le\rho$. Hence, we obtain $\gamma_k^{[1]}(x)=\mathcal{O}\left(|x/\rho|^k\right)$ for $|x|\le\rho$.

The functions $\gamma_{r,h}^{[j]}(x)$ are given by the recurrence

$$\gamma_{r+1,h}^{[j]}(x)=y(x)\sum_{i\ge 1}\gamma_{r,h}^{[j]}(x^i)$$

with initial value $\gamma_{0,h}^{[j]}(x)=y(x)-\gamma_h(x)$. Hence, the representation $\gamma_{r,h}^{[1]}(x)=\gamma_r^{[1]}(x)-\gamma_{h+r}^{[1]}(x)$ follows by induction. Since, $\gamma_r^{[1]}(x)=(C(x)+O(L^r))y(x)^r$ we thus get that

$$\gamma_{r,h}^{[1]}(x)=\mathcal{O}\left(\sup_{x\in\Delta}|y(x)^r(1-y(x)^h)|+L^r\right)$$

However, it is an easy exercise to show that

$$\sup_{x\in\Delta}|y(x)^r(1-y(x)^h)|=\mathcal{O}\left(\frac{h}{r+h}\right). \tag{35}$$

For this purpose observe that if $x\in\Delta$ then we either have $|y(x)-1|\le 1$ and $|y(x)|\le 1$, or $|y(x)|\le 1-\eta$ for some $\eta>0$. In the second case we surely have

$$|y(x)^r(1-y(x)^h)|\le 2(1-\eta)^r=\mathcal{O}\left(L^r\right).$$

For the first case we set $y=1-\rho e^{i\varphi}$ and observe that

$$\left|1-(1-\rho e^{i\varphi})^h\right|\le(1+\rho)^h-1.$$

Hence, if $r \ge 3h$ we thus obtain that

$$|y(x)^r(1-y(x)^h)| \le \max_{0\le\rho\le 1}(1-\rho)^r\left((1+\rho)^h-1\right) \le \frac{h}{r-h} \le \frac{2h}{r+h}.$$

If $r < 3h$ we obviously have

$$|y(x)^r(1-y(x)^h)| \le 2 \le \frac{4h}{r+h}$$

which completes the proof of (35). Of course, we also have $L^r = \mathcal{O}\left(\frac{h}{h+r}\right)$. This completes the proof of the upper bound of $\gamma_{r,h}^{[1]}(x)$ for $x \in \Delta$.

Finally, the upper bound $\gamma_{r,h}^{[1]}(x) = \mathcal{O}\left(|x/\rho|^r\right)$ follows from (33). □

Lemma 9. *We have*

$$\gamma_k^{[2]}(x) = \begin{cases} \mathcal{O}\left(\min\left\{k, \frac{1}{1-|y(x)|}\right\}\right) & \textit{uniformly for } x\in\Delta, \\ \mathcal{O}\left(|x/\rho|^k\right) & \textit{uniformly for } |x|\le\rho-\eta \end{cases} \tag{36}$$

and

$$\gamma_{r,h}^{[2]}(x) = \begin{cases} \mathcal{O}\left(\min\left\{h, \frac{1}{1-|y(x)|}\right\}\right) & \textit{uniformly for } x\in\Delta, \\ \mathcal{O}\left(|x/\rho|^r\right) & \textit{uniformly for } |x|\le\rho-\eta \end{cases} \tag{37}$$

for every $\eta > 0$.

Remark. *By doing a more precise analysis similarly to Lemma 7 we can, for example show that $\gamma_k^{[2]}(x)$ can be represented as*

$$\gamma_k^{[2]}(x) = y(x)^k \sum_{\ell=1}^{k} D_{k,\ell}(x) y(x)^{\ell-1}, \tag{38}$$

where the functions $D_{k,\ell}(x)$ are analytic in Δ. For every ℓ there is a limit $D_\ell(x) = \lim_{k\to\infty} D_{k,\ell}(x)$ with

$$D_{k,\ell}(x) = D_\ell(x) + O(\tilde{L}^{k+\ell}),$$

where $0 < \tilde{L} < 1$. Furthermore these limit functions $D_\ell(x)$ satisfy

$$D_\ell(x) = C(x)^2 + O(\tilde{L}^\ell).$$

Since we will not make use of this precise representation we leave the details to the reader.

Proof: We start with the analysis of $\gamma_k^{[2]}(x)$. We proceed by induction with help of the recurrenc

$$\gamma_{k+1}^{[2]}(x) = y(x)\sum_{i\ge 1} i\gamma_k^{[2]}(x^i) + y(x)\left(\sum_{i\ge1}\gamma_k^{[1]}(x^i)\right)^2 + y(x)\sum_{i\ge 2}(i-1)\gamma_k^{[1]}(x^i) \tag{39}$$

Since $\gamma_0^{[2]}(x) = 0$ it follows that (36) is satisfied for $k = 0$.

Now assume that $|\gamma_k^{[2]}(x)| \le D_k|x/\rho|^k$ for $|x| \le \rho - \eta$. Since $\gamma_0^{[2]}(x) = 0$ we can set $D_0 = 0$. Furthermore, we use the bound from Lemma 8: $|\gamma_k^{[1]}(x)| \le C|x/\rho|^k$ for $|x| \le \rho - \eta$. By using the recurrence (39) we thus obtain

$$\begin{aligned}
|\gamma_{k+1}^{[2]}(x)| &\le D_k|x/\rho|^{k+1} + D_k|x/\rho| \sum_{i\ge 2} i|x^i/\rho|^k \\
&\quad + C^2|x/\rho| \left(\sum_{i\ge 1} |x^i/\rho|^k\right)^2 + C\sum_{i\ge 2}(i-1)|x^i/\rho|^k \\
&\le D_k|x/\rho|^{k+1} + D_k \frac{2|x|^{2k+1}/\rho^{k+1}}{(1-|x|^k)^2} \\
&\quad + C^2|x/\rho| \left(\frac{|x/\rho|^k}{1-|x|^k}\right)^2 + C|x/\rho|\frac{2|x|^{2k}/\rho^k}{(1-|x|^k)^2}.
\end{aligned}$$

Consequently we can set

$$D_{k+1} = D_k\left(1+\frac{2(\rho-\eta)^k}{(1-\rho^k)^2}\right) + C^2\frac{(\rho-\eta)^k/\rho^k}{(1-|\rho|^k)^2} + C\frac{2(\rho-\eta)^k}{(1-|\rho|^k)^2}$$

and obtain that $D_k = \mathcal{O}(1)$ as $k\to\infty$.

Now assume that $|\gamma_k^{[2]}(x)| \le \bar{D}_k$ for $x\in\Delta$. We also use the bound $|\gamma_k^{[1]}(x)| \le C$ for $x\in\Delta$. Without loss of generality we can also assume that $\rho+\eta<\frac{1}{2}$ and that $|x|^i \le \rho-\eta$ for $x\in\Delta$ and $i\ge 2$. Then we get inductively for $x\in\Delta$

$$\begin{aligned}
|\gamma_{k+1}^{[2]}(x)| &\le \bar{D}_k + D_k\sum_{i\ge 2} i((\rho+\eta)^i/\rho)^k \\
&\quad + C^2\left(1+\sum_{i\ge 2}((\rho+\eta)^i/\rho)^k\right)^2 + C\sum_{i\ge 2}(i-1)((\rho+\eta)^i/\rho)^k \\
&\le \bar{D}_k + 8D_k(\rho+\eta)^{2k}/\rho^k \\
&\quad + C^2\left(1+2(\rho+\eta)^{2k}/\rho^k\right)^2 + 4C(\rho+\eta)^{2k}/\rho^k.
\end{aligned}$$

Hence, we can set

$$\bar{D}_{k+1} = \bar{D}_k + 8D_k(\rho+\eta)^{2k}/\rho^k + C^2\left(1+2(\rho+\eta)^{2k}/\rho^k\right)^2 + 4C(\rho+\eta)^{2k}/\rho^k$$

and obtain easily that

$$\bar{D}_k = \mathcal{O}(k).$$

Thus, in order to complete the proof of (36) we just have to show that $\gamma_k^{[2]}(x) = \mathcal{O}(1/(1-|y(x)))$, too, for $x\in\Delta$. We rewrite the recurrence (39) as

$$\gamma_{k+1}^{[2]}(x) = y(x)\gamma_k^{[2]}(x) + b_k(x),$$

where

$$b_k(x) = y(x)\sum_{i\ge 2} i\gamma_k^{[2]}(x^i) + y(x)\left(\sum_{i\ge 1}\gamma_k^{[1]}(x^i)\right)^2 + y(x)\sum_{i\ge 2}(i-1)\gamma_k^{[1]}(x^i).$$

Since $\gamma_0^{[2]}(x)=0$ the solution of this recurrence can be written as

$$\gamma_k^{[2]}(x) = b_{k-1}(x) + y(x)b_{k-2}(x) + \cdots + y(x)^{k-1}b_0(x).$$

From what we know we directly get $b_k(x) = \mathcal{O}(1)$ uniformly for $x\in\Delta$. Hence,

$$\gamma_k^{[2]}(x) = \mathcal{O}\left(\sum_{j=0}^{k-1}|y(x)|^j\right) = \mathcal{O}\left(\frac{1}{1-|y(x)|}\right).$$

This completes the proof of (36).

The recurrence for $\gamma_{h,r}^{[2]}(x)$ is similar to that of $\gamma_k^{[2]}(x)$.

$$\gamma_{r+1,h}^{[2]}(x) = y(x)\sum_{i\ge 1} i\gamma_{r,h}^{[2]}(x^i) + y(x)\left(\sum_{i\ge 1}\gamma_{r,h}^{[1]}(x^i)\right)^2 + y(x)\sum_{i\ge 2}(i-1)\gamma_{r,h}^{[1]}(x^i) \tag{40}$$

with initial value $\gamma_{0,h}^{[2]}(x) = \gamma_h^{[2]}(x)$. Now assume that we already know that $|\gamma_{r,h}^{[2]}(x)| \le D_{r,h}|x/\rho|^r$ for $|x| \le \rho-\eta$. By (37) we can set $D_{0,h} = D_h$ which is bounded as $h\to\infty$. We also assume that $|\gamma_{r,h}^{[1]}(x)| \le C|x/\rho|^k$ for $|x|\le \rho-\eta$. Then by (40) we get

$$\begin{aligned}|\gamma_{r+1,h}^{[2]}(x)| &\le D_{r,h}|x/\rho|^{k+1} + D_{r,h}|x/\rho|\frac{2|x|^{2k}/\rho^k}{(1-|x|^k)^2}\\ &\quad + C^2|x/\rho|\left(\frac{|x/\rho|^k}{1-|x|^k}\right)^2 + C|x/\rho|\frac{2|x|^{2k}/\rho^k}{(1-|x|^k)^2}\end{aligned}$$

Thus, we can set

$$D_{r+1,h} = D_{r,h}\left(1+\frac{2(\rho-\eta)^k}{(1-\rho^k)^2}\right) + C^2\frac{(\rho-\eta)^k}{(1-\rho^k)^2} + C\frac{2(\rho-\eta)^k}{(1-\rho^k)^2}$$

and obtain $D_{r,h}$ uniformly bounded. Consequently $\gamma_{r,h}^{[1]}(x) = \mathcal{O}\left(|x/\rho|^r\right)$ for $|x|\le\rho-\eta$.

Next we assume that $|\gamma_{r,h}^{[2]}(x)| \le \bar{D}_{r,h}$ for $x\in\Delta$, and of course we already know that $|\gamma_{r,h}^{[1]}(x)| \le C\frac{h}{h+r}$ for $x\in\Delta$. Hence

$$\begin{aligned}|\gamma_{r+1,h}^{[2]}(x)| &\le \bar{D}_{r,h} + D_{r,h}\sum_{i\ge 2} i|x^i/\rho|^r\\ &\quad + C^2\left(\frac{h}{r+h} + \sum_{i\ge 2}|x^i/\rho|^r\right)^2 + C\sum_{i\ge 2}(i-1)|x^i/\rho|^r\\ &\le \bar{D}_{r,h} + 8D_{r,h}(\rho+\eta)^{2r}/\rho^r\\ &\quad + C^2\left(\frac{h}{r+h} + 2(\rho+\eta)^{2r}/\rho^r\right)^2 + 4C(\rho+\eta)^{2r}/\rho^r.\end{aligned}$$

Thus, we can set

$$\bar{D}_{r+1,h} = \bar{D}_{r,h} + 8D_{r,h}(\rho+\eta)^{2r}/\rho^r + C^2\left(\frac{h}{r+h}+2(\rho+\eta)^{2r}/\rho^r\right)^2 + 4C(\rho+\eta)^{2r}/\rho^r$$

with initial value $\bar{D}_{0,h} = \bar{D}_h = \mathcal{O}(h)$ and obtain a uniform upper bound of the form

$$\bar{D}_{r,h} = \mathcal{O}(h).$$

Consequently $\gamma_{r,h}^{[2]}(x) = \mathcal{O}(h)$ for $x\in\Delta$.

Thus, in order to complete the proof of (37) it remains to prove $\gamma_{r,h}^{[2]}(x) = \mathcal{O}(1/(1-|y(x)|))$, too, for $x\in\Delta$. Similarly to the above we represent $\gamma_{r,h}^{[2]}(x)$ as

$$\gamma_{r,h}^{[2]}(x) = \gamma_{0,h}^{[2]}(x) + c_{r-1,h}(x) + y(x)c_{r-2,h}(x) + \cdots + y(x)^{r-1}c_{0,h}(x), \tag{41}$$

where

$$c_{j,h}(x) = y(x)\sum_{i\ge 2} i\gamma_{j,h}^{[2]}(x^i) + y(x)\left(\sum_{i\ge 1}\gamma_{j,h}^{[1]}(x^i)\right)^2 + y(x)\sum_{i\ge 2}(i-1)\gamma_{j,h}^{[1]}(x^i).$$

Observe that there exists $\eta>0$ such that $|x^i|\le\rho-\eta$ for $i\ge 2$ and $x\in\Delta$. Hence it directly follows that $c_{j,h}(x) = \mathcal{O}(1)$ for $x\in\Delta$. Since $\gamma_{0,h}^{[2]}(x) = \gamma_h^{[2]}(x) = \mathcal{O}(1/(1-|y(x)|))$ we consequently get

$$\gamma_{r,h}^{[2]}(x) = \gamma_h^{[2]}(x) + \mathcal{O}\left(\frac{1}{1-|y(x)|}\right) = \mathcal{O}\left(\frac{1}{1-|y(x)|}\right).$$

□

Remark. *Note that the estimates of Lemma 9 already prove that*

$$\mathbf{E}\left(L_n(r) - L_n(r+h)\right)^2 = \mathcal{O}\left(h\sqrt{n}\right).$$

Unfortunately this estimate is not sufficient to prove tightness. In fact, we have to deal with the 4-th moments.

Before we start with bounds for $\gamma_k^{[3]}(x)$ and $\gamma_k^{[4]}(x)$ we need an auxiliary bound.

Lemma 10. *We have uniformly for $x \in \Delta$*

$$\sum_{r\ge 0} |\gamma_{r,h}^{[1]}(x)\gamma_{r,h}^{[2]}(x)| = \mathcal{O}\left(h^2\right). \tag{42}$$

Proof: We use the representation (41), where we can approximate $c_{j,h}(x)$ by

$$c_{j,h}(x) = y(x)\gamma_{j,h}^{[1]}(x)^2 + \mathcal{O}\left(L^j\right) = \mathcal{O}\left(\frac{h^2}{(r+h)^2}\right)$$

uniformly for $x \in \Delta$ with some constant L that satisfies $0 < L < 1$. Furthermore, we use the approximation

$$\gamma_{r,h}^{[1]}(x) = C(x)y(x)^r(1-y(x)^h) + \mathcal{O}\left(L^r\right)$$

that is uniform for $x \in \Delta$. For example, this shows

$$\sum_{r\ge 0} |\gamma_{r,h}^{[1]}(x)| = |C(x)|\frac{|1-y(x)^h|}{1-|y(x)|} + \mathcal{O}\left(1\right).$$

Now observe that for $x \in \Delta$ there exists a constant $c > 0$ with $|1-y(x)| \ge c(1-|y(x)|)$. Hence it follows that

$$\frac{|1-y(x)^h|}{1-|y(x)|} = \mathcal{O}\left(\frac{1-y(x)^h}{1-y(x)}\right) = \mathcal{O}\left(h\right)$$

and consequently

$$\sum_{r\ge 0} |\gamma_{r,h}^{[1]}(x)| = \mathcal{O}\left(h\right).$$

Similarly we get

$$\begin{aligned}
\sum_{r\ge 1}|\gamma_{r,h}^{[1]}(x)| \left|\sum_{j<r} y(x)^{r-j-1}c_{j,h}(x)\right| &\le \sum_{j\ge 0}|c_{j,h}(x)|\,|y(x)|^{-j-1}\sum_{r>j}|y(x)|^r|\gamma_{r,h}^{[1]}(x)| \\
&= \sum_{j\ge 0}|c_{j,h}(x)|\,|y(x)|^{-j-1}\left(|C(x)|\,|y(x)|^{2j+2}\frac{|1-y(x)^h|}{1-|y(x)|^2} + \mathcal{O}\left(|y(x)|^{j+1}L^j\right)\right) \\
&= \mathcal{O}\left(\sum_{j\ge 0}\frac{h^3}{(j+h)^2}\right) \\
&= \mathcal{O}\left(h^2\right).
\end{aligned}$$

Hence, we finally obtain

$$\begin{aligned}
\sum_{r\ge 0}|\gamma_{r,h}^{[1]}(x)\gamma_{r,h}^{[2]}(x)| &\le \sum_{r\ge 0}|\gamma_{r,h}^{[1]}(x)|\,|\gamma_{r,h}^{[2]}(x)| + \sum_{r\ge 1}|\gamma_{r,h}^{[1]}(x)|\left|\sum_{j<r} y(x)^{r-j-1}c_{j,h}(x)\right| \\
&= \mathcal{O}\left(h^2\right).
\end{aligned}$$

□

Lemma 11. *We have*

$$\gamma_k^{[3]}(x) = \begin{cases} \mathcal{O}\left(\min\{k^2, \frac{k}{1-|y(x)|}\}\right) & \textit{uniformly for } x \in \Delta, \\ \mathcal{O}\left(|x/\rho|^k\right) & \textit{uniformly for } |x| \le \rho - \eta \end{cases} \tag{43}$$

and

$$\gamma_{r,h}^{[3]}(x) = \begin{cases} \mathcal{O}\left(\min\{h^2, \frac{h}{1-|y(x)|}\}\right) & \textit{uniformly for } x \in \Delta, \\ \mathcal{O}\left(|x/\rho|^r\right) & \textit{uniformly for } |x| \le \rho - \eta \end{cases} \tag{44}$$

for every $\eta > 0$.

Proof: The recurrence for $\gamma_k^{[3]}(x)$ is given by

$$\begin{aligned} \gamma_{k+1}^{[3]}(x) &= y(x)\sum_{i\ge 1} i^3\gamma_k^{[3]}(x^i) + y(x)\left(\sum_{i\ge i}\gamma_k^{[1]}(x^i)\right)^3 + 3y(x)\left(\sum_{i\ge 1}\gamma_k^{[1]}(x^i)\right)\left(\sum_{i\ge 1} i\gamma_k^{[2]}(x^i)\right) \\ &\quad + 3y(x)\left(\sum_{i\ge 1}\gamma_k^{[1]}(x^i)\right)\left(\sum_{i\ge 1}(i-1)\gamma_k^{[i]}(x^i)\right) + 3y(x)\sum_{i\ge 1} i(i-1)\gamma_k^{[2]}(x^i) \\ &\quad + y(x)\sum_{i\ge 1}(i-1)(i-2)\gamma_k^{[1]}(x^i) \end{aligned} \tag{45}$$

By inspecting the proof of Lemmas 8 and 9 one expects that the only *important* part of this recurrence if given by

$$\gamma_{k+1}^{[3]}(x) = y(x)\gamma_k^{[3]}(x) + y(x)\gamma_k^{[1]}(x)^3 + 3y(x)\gamma_k^{[1]}(x)\gamma_k^{[2]}(x) + R_k \tag{46}$$

and R_k collects the *less important* remainder terms that only contributes exponentially small terms. Thus, in order to shorten our presentation we will only focus on these terms. In particular it is easy to show the bound $\gamma_k^{[3]}(x) = \mathcal{O}\left(|x/\rho|^k\right)$ for $|x| \le \rho - \eta$. (We omit the details.)

Next, since $y(x)\gamma_k^{[1]}(x)^3 + 3y(x)\gamma_k^{[1]}(x)\gamma_k^{[2]}(x) + R_k = \mathcal{O}(k)$ it directly follows that $\gamma_k^{[3]}(x) = \mathcal{O}\left(k^2\right)$.

Now we proceed by induction and observe that a bound of the form $|\gamma_k^{[3]}(x)| \le E_k/(1-|y(x)|)$ leads to

$$|\gamma_{k+1}^{[3]}(x)| \le \frac{E_k}{1-|y(x)|} + \mathcal{O}\left(\frac{1}{1-|y(x)|}\right) + |R_k|$$

and consequently to $E_{k+1} \le E_k + \mathcal{O}(1)$. Hence, $E_k = \mathcal{O}(k)$ and $\gamma_k^{[3]}(x) = \mathcal{O}(k/(1-|y(x)|))$.

Similarly, the leading part of the recurrence for $\gamma_{r,h}^{[3]}(x)$ is given by

$$\begin{aligned} \gamma_{r+1,h}^{[3]}(x) &= y(x)\gamma_{r,h}^{[3]}(x) + y(x)\gamma_{r,h}^{[1]}(x)^3 + 3y(x)\gamma_{r,h}^{[1]}(x)\gamma_{r,h}^{[2]}(x) + \bar{R}_{r,h} \\ &= y(x)\gamma_{r,h}^{[3]}(x) + d_{r,h}(x), \end{aligned} \tag{47}$$

where

$$d_{r,h}(x) = y(x)\gamma_{r,h}^{[1]}(x)^3 + 3y(x)\gamma_{r,h}^{[1]}(x)\gamma_{r,h}^{[2]}(x) + \bar{R}_{r,h} = \mathcal{O}(h)$$

and the initial value is given by

$$\gamma_{0,h}^{[3]}(x) = -\gamma_h^{[3]}(x) - 3\gamma_h^{[2]}(x) = \mathcal{O}\left(\min\left\{h^2, \frac{h}{1-|y(x)|}\right\}\right).$$

Note that we also assume that $\gamma_{r,h}^{[3]}(x) = \mathcal{O}(|x/\rho|^r)$ for $|x| \le \rho - \eta$ (which can be easily proved). Consequently it directly follows that

$$\begin{aligned} \gamma_{r,h}^{[3]}(x) &= \gamma_{0,h}^{[3]}(x) + d_{r-1,h}(x) + y(x)d_{r-1,h}(x) + \cdots + y(x)^{r-1}d_{0,h}(x) \\ &= \mathcal{O}\left(\frac{h}{1-|y(x)|}\right). \end{aligned}$$

Next observe that Lemmas 8–10 ensure that

$$\sum_{j\ge 0} |d_{j,h}(x)| = \mathcal{O}\left(h^2\right)$$

uniformly for $x \in \Delta$. Hence, we finally get

$$\gamma_{r,h}^{[3]}(x) = \mathcal{O}\left(h^2\right)$$

which completes the proof of Lemma 11. □

Lemma 12. *We have*

$$\gamma_k^{[4]}(x) = \begin{cases} \mathcal{O}\left(\frac{k^2}{1-|y(x)|}\right) & \text{uniformly for } x \in \Delta, \\ \mathcal{O}\left(|x/\rho|^k\right) & \text{uniformly for } |x| \le \rho - \eta \end{cases} \tag{48}$$

and

$$\gamma_{r,h}^{[4]}(x) = \begin{cases} \mathcal{O}\left(\frac{h^2}{1-|y(x)|}\right) & \text{uniformly for } x \in \Delta, \\ \mathcal{O}\left(|x/\rho|^r\right) & \text{uniformly for } |x| \le \rho - \eta \end{cases} \tag{49}$$

for every $\eta > 0$.

Proof: The proof is very similar to that of Lemma 11. First, the recurrence for $\gamma_k^{[4]}(x)$ is essentially of the form

$$\begin{aligned} \gamma_{k+1}^{[4]}(x) &= y(x)\gamma_k^{[4]}(x) + y(x)\gamma_k^{[1]}(x)^4 + 4y(x)\gamma_k^{[1]}(x)\gamma_k^{[3]}(x) \\ &\quad + 6y(x)\gamma_k^{[1]}(x)^2\gamma_k^{[2]}(x) + 3y(x)\gamma_k^{[2]}(x)^2 + R_k, \end{aligned} \tag{50}$$

where R_k collects all exponentially small summands. We assume that we have already proved the upper bound $\gamma_k^{[4]}(x) = \mathcal{O}\left(|x/\rho|^k\right)$ for $|x| \le \rho - \eta$. Now, by induction it the assumption $|\gamma_k^{[4]}(x)| \le F_k/(1-|y(x)|)$ and the known estimates $\gamma_k^{[1]}(x) = \mathcal{O}(1)$, $\gamma_k^{[2]}(x) = \mathcal{O}(\min\{k, 1/(1-|y(x)|)\})$, and $\gamma_k^{[3]}(x) = \mathcal{O}(k/(1-|y(x)|))$ we get

$$|\gamma_{k+1}^{[4]}(x)| \le \frac{F_k}{1-|y(x)|} + \mathcal{O}\left(|y(x)|^k\right) + \mathcal{O}\left(\frac{k}{1-|y(x)|}\right) + \mathcal{O}\left(\frac{1}{1-|y(x)|}\right) + |R_k|$$

and consequently $F_k = \mathcal{O}\left(k^2\right)$.

Finally, the essential part of the recurrence for $\gamma_{r,h}^{[4]}(x)$ is given by

$$\begin{aligned} \gamma_{r+1,h}^{[4]}(x) &= y(x)\gamma_{r,h}^{[4]}(x) + y(x)\gamma_{r,h}^{[1]}(x)^4 + 4y(x)\gamma_{r,h}^{[1]}(x)\gamma_{r,h}^{[3]}(x) \\ &\quad + 6y(x)\gamma_{r,h}^{[1]}(x)^2\gamma_{r,h}^{[2]}(x) + 3y(x)\gamma_{r,h}^{[2]}(x)^2 + \bar{R}_{r,h} \\ &= y(x)\gamma_{r,h}^{[4]}(x) + e_{r,h}(x), \end{aligned} \tag{51}$$

where

$$\begin{aligned} e_{r,h}(x) &= y(x)\gamma_{r,h}^{[1]}(x)^4 + 4y(x)\gamma_{r,h}^{[1]}(x)\gamma_{r,h}^{[3]}(x) \\ &\quad + 6y(x)\gamma_{r,h}^{[1]}(x)^2\gamma_{r,h}^{[2]}(x) + 3y(x)\gamma_{r,h}^{[2]}(x)^2 + \bar{R}_{r,h}. \end{aligned}$$

As above, $\bar{R}_{r,h}$ collects all exponentially small terms. Thus,

$$\gamma_{r,h}^{[4]}(x) = \gamma_{0,h}^{[4]}(x) + e_{r-1,h}(x) + y(x)e_{r-1,h}(x) + \cdots + y(x)^{r-1}e_{0,h}(x).$$

If we use the known estimates $\gamma_{r,h}^{[1]}(x) = \mathcal{O}(1)$, $\gamma_{r,h}^{[2]}(x) = \mathcal{O}(h)$, and $\gamma_{r,h}^{[3]}(x) = \mathcal{O}\left(h^2\right)$ which gives $d_{r,h} = \mathcal{O}\left(h^2\right)$ and the initial condition

$$\gamma_{0,h}^{[4]}(x) = 12\gamma_h^{[2]}(x) + 8\gamma_h^{[3]}(x) + \gamma_h^{[4]}(x) = \mathcal{O}\left(\frac{h^2}{1-|y(x)|}\right)$$

we obtain

$$\gamma_{r,h}^{[4]}(x) = \mathcal{O}\left(\frac{h^2}{1-|y(x)|}\right).$$

This completes the proof of Lemma 12. □

The proof of (32) is now immediate. As already noted this implies (31) and proves Theorem 4.

6. The Height

Let y_{kn} denote the number of trees with n nodes and height at most k. Then the generating function $y_k(x) = \sum_{n\geq 1} y_{kn}x^n$ satisfies the recurrence relation

$$y_0(x) = 0$$

$$y_{k+1}(x) = x\exp\left(\sum_{i\geq 1}\frac{y_k(x^i)}{i}\right), \quad k \geq 0.$$

Obviously $y_k(x) = y_k(x,0)$ where the function on the right-hand side is the generating function of (6) which we used to analyze the profile in the previous sections. So w_k and Σ_k can be defined accordingly, i.e., we write

$$w_k(x) := w_k(x,0) = y(x) - y_k(x) = \sum_{n\geq 0}\mathbf{P}\left\{H_n > k\right\} y_n x^n$$

and

$$\Sigma_k(x) := \Sigma_k(x,0) = \sum_{i\geq 2}\frac{w_k(x^i)}{i}.$$

So, $w_k(x)$ encodes the distribution of the height and thus it suffices to analyze this function.

Lemma 13. *For $|x| \leq \rho^2 + \varepsilon$ ($\varepsilon > 0$ sufficiently small) we have $|w_k(x)| \leq C|x|L^k$ for all k. The constants L and C are those of Lemma 1. Consequently, for $|x| < \rho + \varepsilon$ we have $|\Sigma_k(x)| \leq CL^k$.*

Proof: This is a special case of Lemma 1 and Corollary 1. □

Note that Lemma 4 holds for $u = 0$. So the next task is to derive upper and lower bounds for $w_k(x)$.

Lemma 14. *Let $x \in \Delta$. Then there is a $C > 0$ such that*

$$|w_k(x)| \leq \frac{C}{\sqrt{k}}\left|\frac{x}{\rho}\right|^k$$

Proof: Obviously, for $|x| \leq \rho$ we have

$$|w_k(x)| = \sum_{n>k}(y_n - y_{kn})|x|^n \leq \sum_{n>k} y_n|x|^n.$$

The assertion for this domain follows now from $y_n \asymp \rho^{-n}n^{-3/2}$.

For $x \in \Delta$ but outside the circle $|x| \leq \rho$ observe that in view of Lemma 13 it is easy to show $|y_k(x)| \leq |y(x)|$ inductively. Thus by Lemma 2a $|w_k(x)|$ is bounded whereas the bound in the assertion is not ($|x| > \rho$ in the considered case). □

Lemma 15. *Let $x \in \Delta$ and $|x - \rho| < \varepsilon$. Then*

$$|w_k(x)| \geq C|y(x)|^{k+1}e^{-\sqrt{k}}$$

for some $C > 0$.

Proof: Throughout the proof C_i and $0 < L < 1$ denote again suitable positive constants. We have

$$|w_{k+1}(x)| \geq |w_k(x)||y(x)|\left(1 - \left|\frac{\Sigma_k(x)}{w_k(x)}\right|\right)\left(1 + \mathcal{O}\left(w_k\right)\right)$$

By Lemma 4 we have $1-\left|\frac{\Sigma_k(x)}{w_k(x)}\right| \ge 1 - C_1 L^k$. With the upper bound of Lemma 14 we obtain

$$1+\mathcal{O}\left(w_k\right) \ge 1 - \frac{C_2}{\sqrt{k}}.$$

Hence

$$|w_{k+1}(x)| \ge |y(x)|^{k+1} \prod_{i=1}^{k}(1-C_1L^i)\prod_{i=1}^{k}\left(1-\frac{C_2}{\sqrt{i}}\right)$$

and since the latter product is bounded from below by $C_3 e^{-\sqrt{k}}$ we are done. □

The essential property of $w_k(x)$ that is needed is the following one.

Proposition 3. *Let $k=\kappa\sqrt{n}$. Then, as $n\to\infty$, we have*

$$w_k(x) = \frac{-y(x)^k}{\frac{1}{2}\frac{1-y(x)^k}{1-y(x)} + \mathcal{O}\left(\sqrt{k}\right)} \tag{52}$$

uniformly for $|x-\rho|<\varepsilon$ such that $x\in\Delta$.

Proof: We use essentially the same arguments as in the proof of Proposition 1. As there we set $q_k = y^k/w_k$ and get

$$q_{k+1} = q_k - \frac{y^k}{2} - \frac{e^{\Sigma_k}-1}{w_k^2}y^k + \mathcal{O}\left(w_k\right) + \mathcal{O}\left(\tilde{L}^k\right)$$

Using Lemma 14 we obtain

$$\begin{aligned} q_k &= \frac{1}{w_0} - \frac{1}{2}\frac{1-y^k}{1-y} - \sum_{\ell=0}^{k-1}\frac{e^{\Sigma_\ell}-1}{w_\ell^2}y^\ell + \mathcal{O}\left(\frac{1-\tilde{L}^k}{1-\tilde{L}}\right) + \mathcal{O}\left(\sqrt{k}\right) \\ &= -\frac{1}{2}\frac{1-y^k}{1-y} - \sum_{\ell=0}^{k-1}\frac{e^{\Sigma_\ell}-1}{w_\ell^2}y^\ell + \mathcal{O}\left(\sqrt{k}\right) \end{aligned}$$

Applying Lemma 15 yields

$$\sum_{\ell=0}^{k-1}\frac{e^{\Sigma_\ell(x)}-1}{w_\ell(x)^2}y(x)^\ell = \mathcal{O}\left(1\right)$$

and this completes the proof. □

Now we are able to complete the proof of Theorem 1. In fact, it is an immediate consequence of the following proposition.

Proposition 4. *The asymptotic behaviour* (52) *in* Δ *implies* (3) *for the average height and* (4) *for the distribution of the height of Pólya trees.*

Proof: Note that, when computing $[x^n]w_k(x)$ by a Cauchy integral, we can choose an integration contour following more or less the boundary of Δ (the singularity has to be avoided of course). Since $w_k(x)$ is bounded on Δ the contribution of the circular part is – after normalization by $1/y_n$ – exponentially small. This fact in conjunction with Proposition 3 means, that the shape of $w_k(x)$ precisely matches that of the corresponding quantity for simply generated trees: Flajolet and Odlyzko showed that (52) implies (3), see [20, p. 204] where this argument was used to derive the average height of simply generated trees. When computing the distribution of the height of simply generated tree, Flajolet et al. showed that (52) implies (4), see [19, end of Section 2]. □

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