THE HEIGHT OF INCREASING TREES

Michael Drmota

Inst. of Discrete Mathematics and Geometry

Vienna University of Technology, A 1040 Wien, Austria

michael.drmota@tuwien.ac.at

www.dmg.tuwien.ac.at/drmota/

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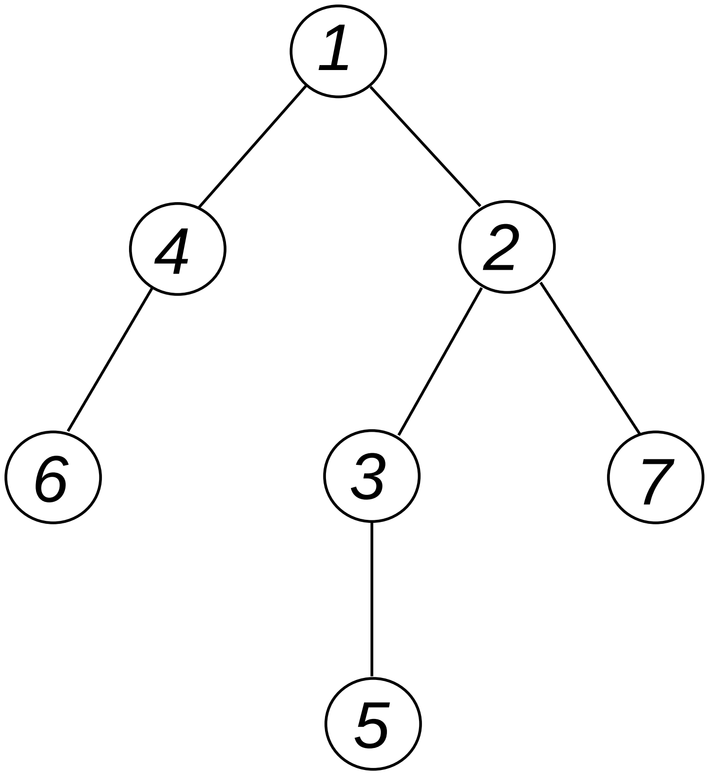
- Recursive Trees
- General Increasing Trees
- D-ary Recursive Trees
- "Travelling Wave"-Distribution of the Height
- Fixed Point Equation and Auxiliary Functions
- "Intersection Property"

Recursive Trees



Recursive Trees

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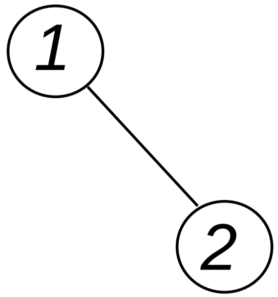

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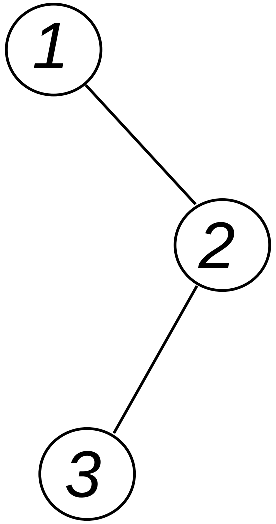

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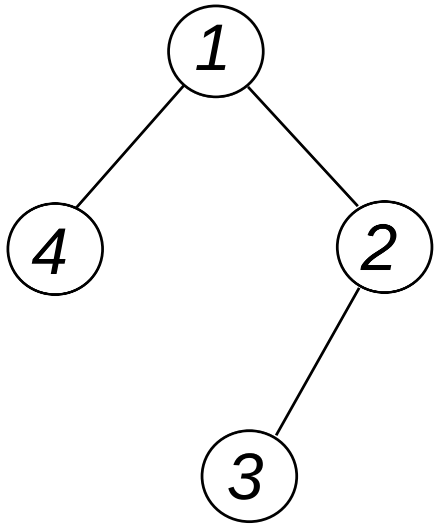

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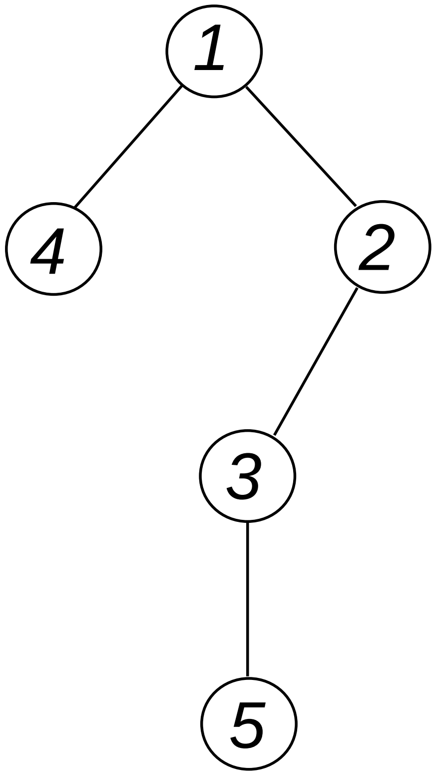

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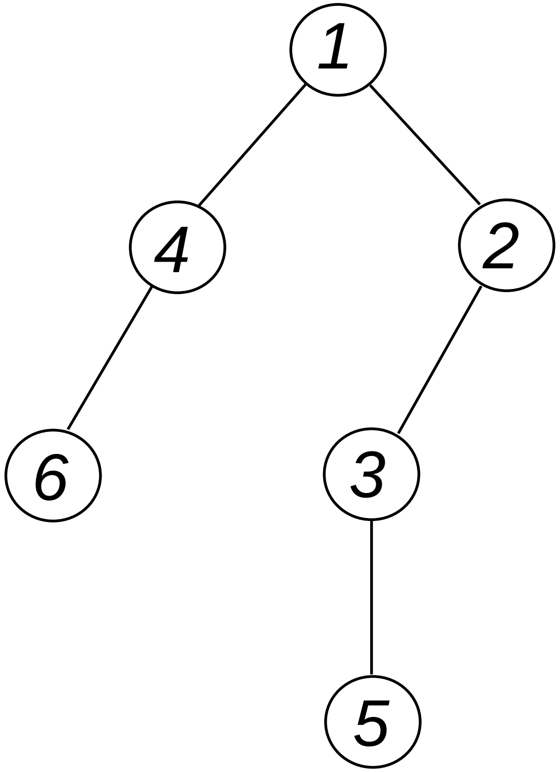

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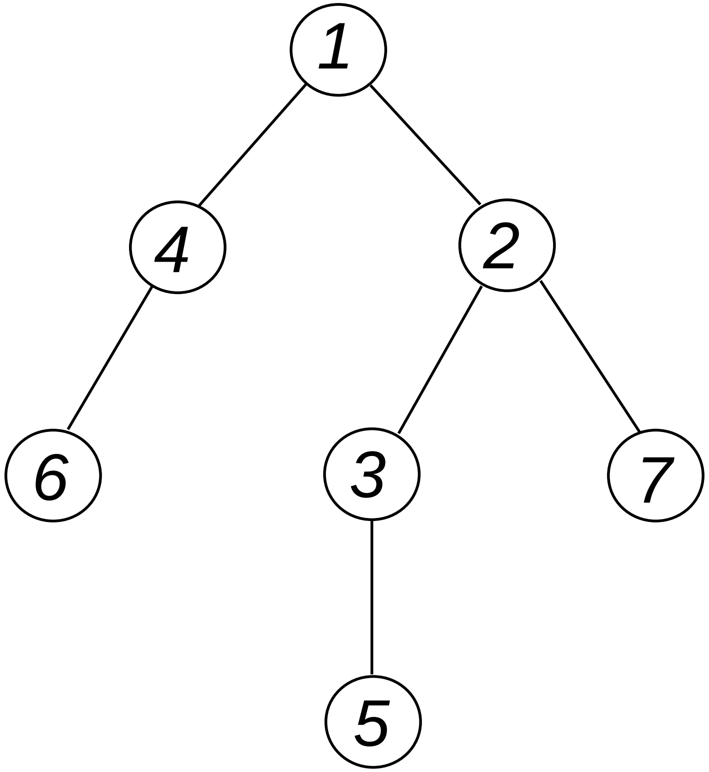


Recursive Trees

Combinatorial Description

- labeled rooted tree
- labels are strictly increasing
- no left-to-right order (non-planar)

Recursive Trees

Number of recursive trees

$$
\begin{aligned}
y_n &= \text{number of recursive trees of size } n \\
&= (n-1)!
\end{aligned}
$$

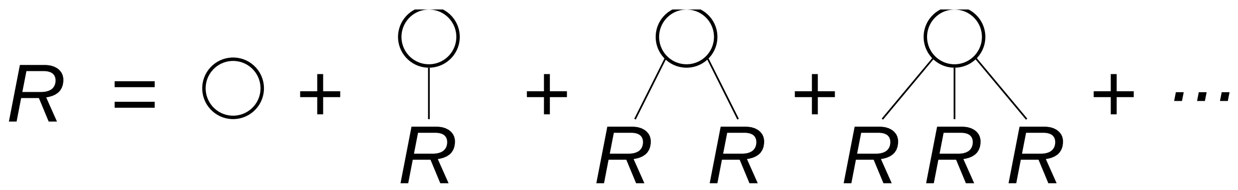
The node with label j has exactly $j-1$ possibilities to be inserted $\Longrightarrow y_n = 1 \cdot 2 \cdots (n-1)$.

Recursive Trees

Generating Functions:

$$y(x) = \sum_{n\geq 1} y_n \frac{x^n}{n!} = \sum_{n\geq 1} \frac{x^n}{n} = \log \frac{1}{1-x}$$

$$\boxed{y'(x) = 1 + y(x) + \frac{y(x)^2}{2!} + \frac{y(x)^3}{3!} + \cdots = e^{y(x)}}$$



A recursive tree can be interpreted as a root followed by an unordered sequence of recursive trees. $(y'(x) = \sum_{n\geq 0} y_{n+1} x^n/n!)$

Recursive Trees

Probability Model:

Wachstumsprozess:

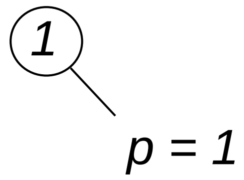
Process of growing trees

- The process starts with the root that is labeled with 1.
- At step j a new node (with label j) is attached to any previous node with probability $1/(j-1)$.

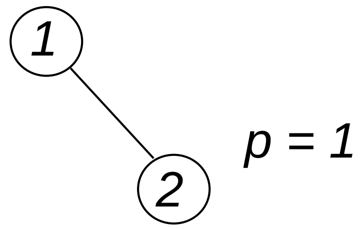
After n steps every tree (of size n) has equal probability $1/(n-1)!$.

Recursive Trees

Recursive Trees



Recursive Trees



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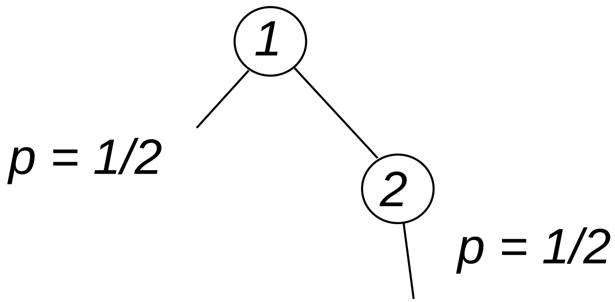
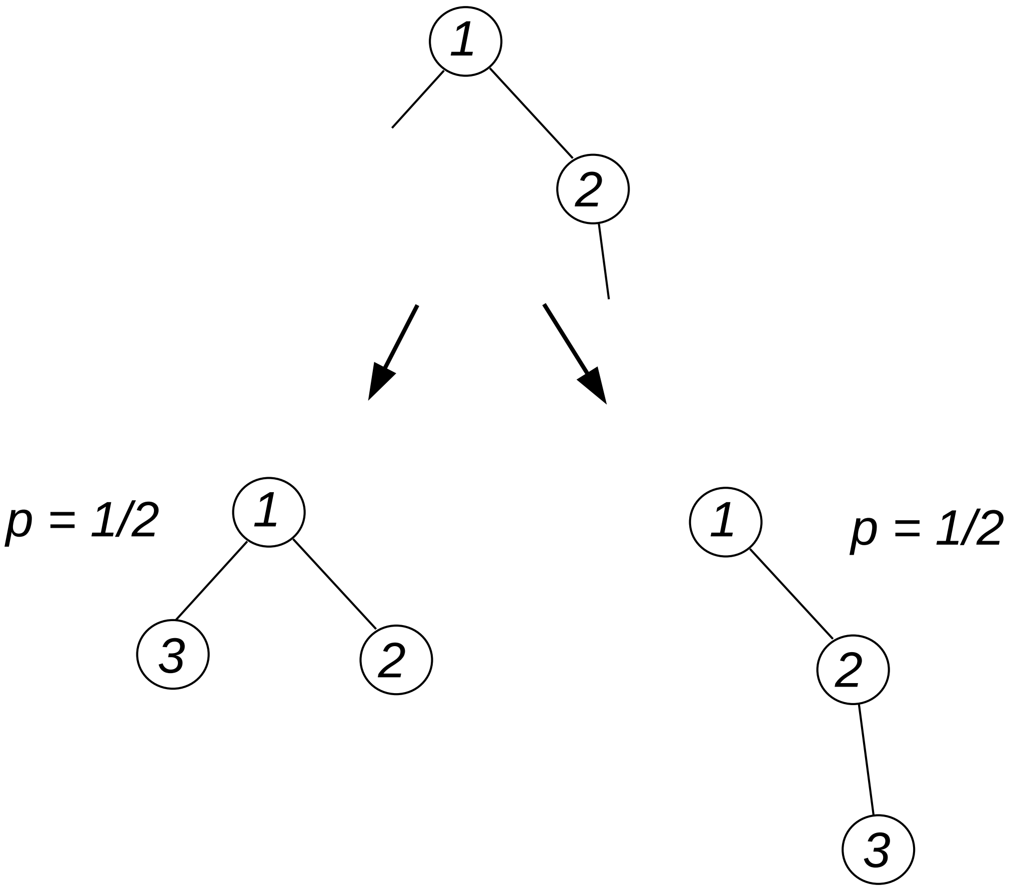
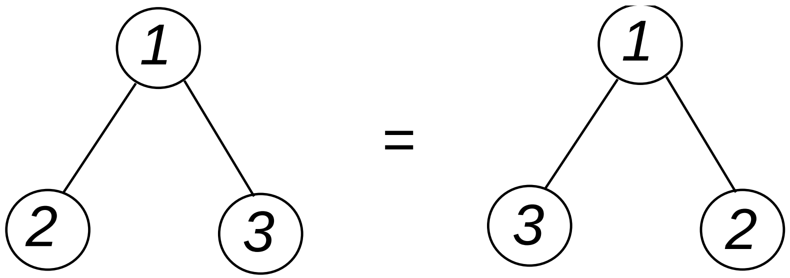


Recursive Trees



Recursive Trees

Remark: left-to-right order is irrelevant



Recursive Trees

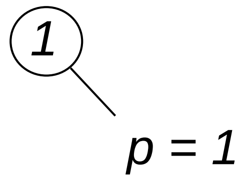
Height H_n

[Devroye 1987, Pittel 1994]

$$\boxed{\frac{H_n}{\log n} \to e \qquad (a.s.)}$$

Plane Oriented Trees

Plane Oriented Trees



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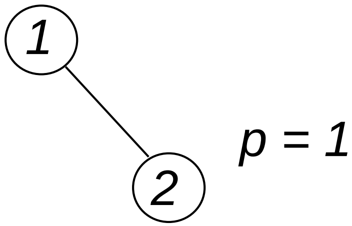

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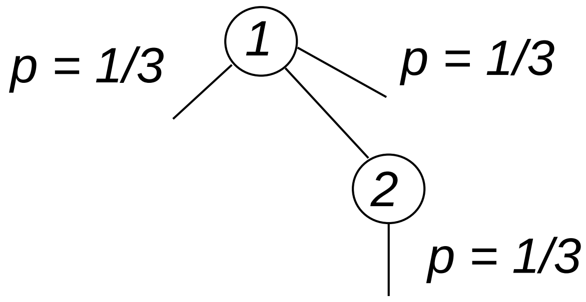

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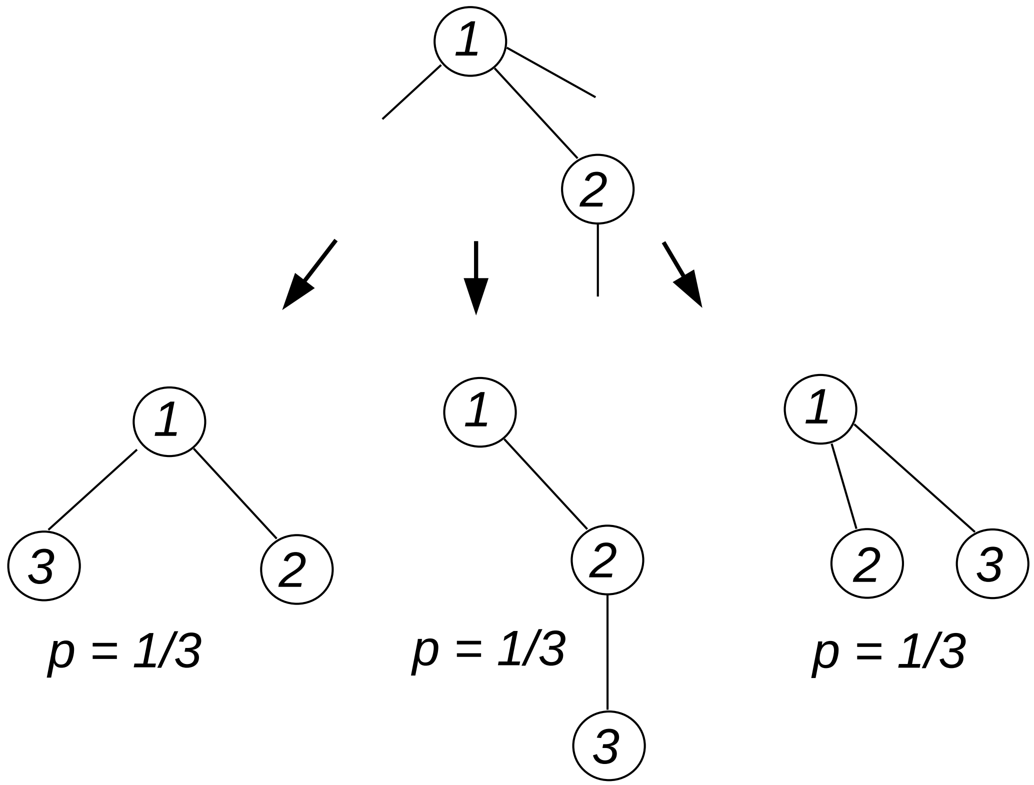
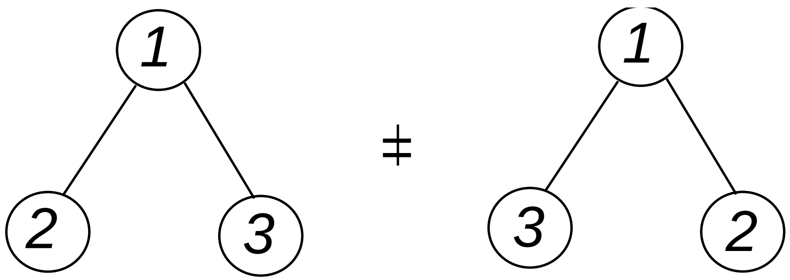


Plane Oriented Trees

Remark: left-to-right order is relevant



Plane Oriented Trees

Number of Plane Oriented Trees:

$$
\begin{aligned}
y_n &= \text{number of plane oriented trees of size } n \\
&= 1 \cdot 3 \cdot 5 \cdots (2n-3) = (2n-3)!! \\
&= \frac{(2n-2)!}{2^{n-1}(n-1)!}
\end{aligned}
$$

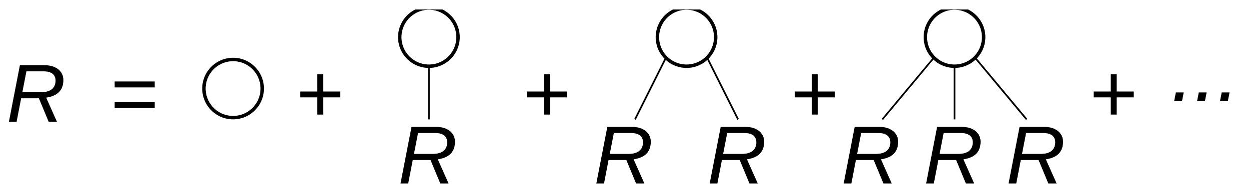
The node with label j has exactly $2j-3$ possibilities to be inserted $\Longrightarrow y_n = 1 \cdot 3 \cdots (2n-3)$.

Plane Oriented Trees

Generating Functions:

$$y(x) = \sum_{n\geq 1} y_n \frac{x^n}{n!} = \sum_{n\geq 1} \frac{1}{2^{n-1}}\binom{2(n-1)}{n-1}\frac{x^n}{n} = 1 - \sqrt{1-2x}$$

$$\boxed{y'(x) = 1 + y(x) + y(x)^2 + y(x)^3 + \cdots = \frac{1}{1-y(x)}}$$



A plane oriented tree can be interpreted as a root followed by an **ordered** sequence of plane oriented trees. $(y'(x) = \sum_{n\geq 0} y_{n+1} x^n / n!)$

Plane Oriented Trees

Probability Model:

Process of growing trees

- The process starts with the root that is labeled with 1.
- At step j a new node (with label j) is attached to any previous node of outdegree d with probability $(d+1)/(2j-3)$.

After n steps every tree (of size n) has equal probability $1/(2n-3)!!$.

Plane Oriented Trees

Height H_n

[Pittel 1994]

$$\boxed{\frac{H_n}{\log n} \to \frac{1}{2s} = 1.79556\ldots \qquad (a.s.)}$$

where $s = 0.27846\ldots$ is the positive solution of $se^{s+1} = 1$.

General Increasing Trees

[Bergeron & Flajolet & Salvy 1992]

$\mathcal{P}_n$: set of all *plane oriented trees* of size n

$\phi_0, \phi_1, \ldots$: weight sequence ($\phi_0 > 0$, $\phi_j > 0$ for some $j \geq 2$)

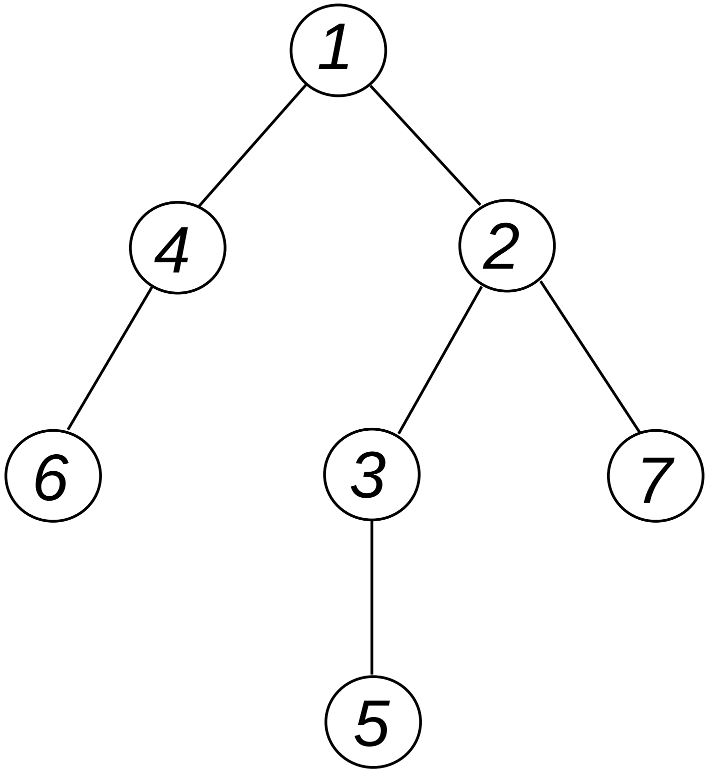
$$\phi(t) = \phi_0 + \phi_1 t + \phi_2 t^2 + \cdots$$

Weight of a tree $T \in \mathcal{P}_n$:

$$\boxed{\omega(T) = \prod_{j \geq 0} \phi_j^{N_j(T)}},$$

where $N_j(T) =$ the number of nodes in T with outdegree j.

General Increasing Trees



$$\omega(T) = \phi_0^3 \phi_1^2 \phi_2^2$$

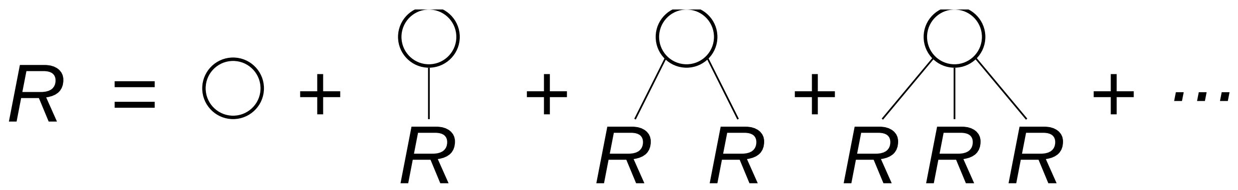
General Increasing Trees

Generating Functions:

$$y_n = \sum_{T \in \mathcal{P}_n} \omega(T)$$

$$y(x) = \sum_{n \geq 1} y_n \frac{x^n}{n!}$$

$$\boxed{y'(x) = \phi_0 + \phi_1\, y(x) + \phi_2\, y(x)^2 + \cdots = \phi(y(x))}$$



General Increasing Trees

Probability distribution on $\mathcal{P}_n$

For $T \in \mathcal{P}_n$ set:

$$\pi_n(T) := \frac{\omega(T)}{y_n}$$

Remark. In general it is not clear whether π_n is induced by a tree evolution process. It is just a sequence of probability measures.

General Increasing Trees

Examples

- **Recursive Trees:** $\phi(t) = \sum_{j \ge 0} \frac{t^j}{j!} = e^t$, $\phi_j = \frac{1}{j!}$

  The factor $1/j!$ "reduces" planar trees to non-planar ones.

- **Plane Oriented Trees:** $\phi(t) = 1 + t + t^2 + \cdots = \frac{1}{1-t}$, $\phi_j = 1$

- **Binary Search Trees:** $\phi(t) = (1+t)^2$, $\phi_0 = 1$, $\phi_1 = 2$, $\phi_2 = 1$.

For all these three examples, π_n is induced by a tree evolution process.

General Increasing Trees

Theorem [Panholzer & Prodinger]

The sequence π_n of probability measures on $\mathcal{P}_n$ is induced by a tree evolution process if and only if $\phi(t)$ has one of the three forms:

- $\phi(t) = \phi_0 e^{\frac{\phi_1}{\phi_0} t}$ with $\phi_0 > 0$, $\phi_1 > 0$.

  Recursive trees

- $\phi(t) = \dfrac{\phi_0}{\left(1 - \frac{\phi_1}{r\phi_0} t\right)^r}$ for some $r > 0$ and $\phi_0 > 0$, $\phi_1 > 0$.

  Scale free trees

- $\phi(t) = \phi_0 \left(1 + \dfrac{\phi_1}{D\phi_0} t\right)^D$ for some $D \in \{2, 3, \ldots\}$ and $\phi_0 > 0$, $\phi_1 > 0$.

  D**-ary recursive trees**

General Increasing Trees

Probabilistic tree evolution model

- The process starts with the root that is labeled with 1.
- At step j a new node (with label j) is attached to any previous node (with out-degree d) with probability proportional to

$$\frac{(d+1)\phi_{d+1}\phi_0}{\phi_d}$$

In order to obtain all possible probability distributions π_n it is sufficient to work with "normalized versions":

$$\boxed{\phi(t) = (1+t)^D, \quad \phi(t) = e^t, \quad \phi(t) = \frac{1}{(1-t)^r}}$$

General Increasing Trees

Recursive Trees: $\phi(t) = e^t$

$$\phi_d = \frac{1}{d!} \quad \Longrightarrow \quad \frac{(d+1)\phi_{d+1}\phi_0}{\phi_d} = 1$$

A new node is attached to previous nodes with equal probability.

General Increasing Trees

Scale Free Trees: $\phi(t) = 1/(1-t)^r$ for some $r > 0$

$$\phi_d = \binom{r+d-1}{d} \quad \Longrightarrow \quad \frac{(d+1)\phi_{d+1}\phi_0}{\phi_d} = \boxed{d+r}$$

A new node is attached to a previous nodes with probability proportional to $\boxed{d+r}$, where d is the out-degree (Barabasi-Albert model).

For $r = 1$ this these are (usual) plane oriented trees.

Scale Free Trees

$\phi(t) = 1/(1-t)^r \quad (r > 0)$

Height H_n

[Pittel 1994]

$$\boxed{\frac{H_n}{\log n} \to \frac{1}{(1+r)s} \quad (a.s.)}$$

where s is the positive solution of $rse^{s+1} = 1$.

The Degree Distribution

Theorem

Let $\phi(t) = 1/(1-t)^r$ for some $r > 0$ and set

$$\begin{aligned} \lambda_d &= \lim_{n\to\infty} \text{probability that a random node in } \mathcal{P}_n \text{ has out-degree } d \\ &= \lim_{n\to\infty} \frac{\text{expected number of nodes with out-degree } d}{n} \end{aligned}$$

Then

$$\boxed{\lambda_d = \frac{(r+1)\Gamma(2r+1)\Gamma(r+d)}{\Gamma(r)\Gamma(2r+d+2)}}$$

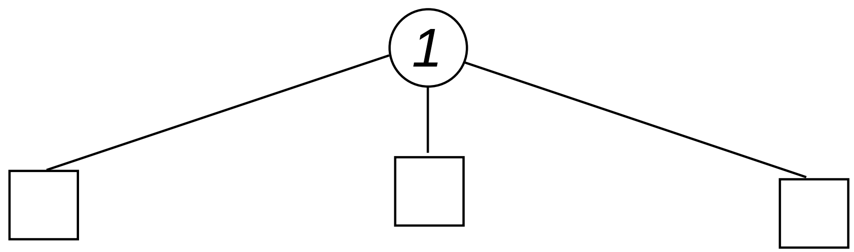
Note that

$$\lambda_d \sim \frac{(r+1)\Gamma(2r+1)}{\Gamma(r)} \cdot d^{-2-r}.$$

D-ary Recursive Trees

$\square$

D-ary Recursive Trees



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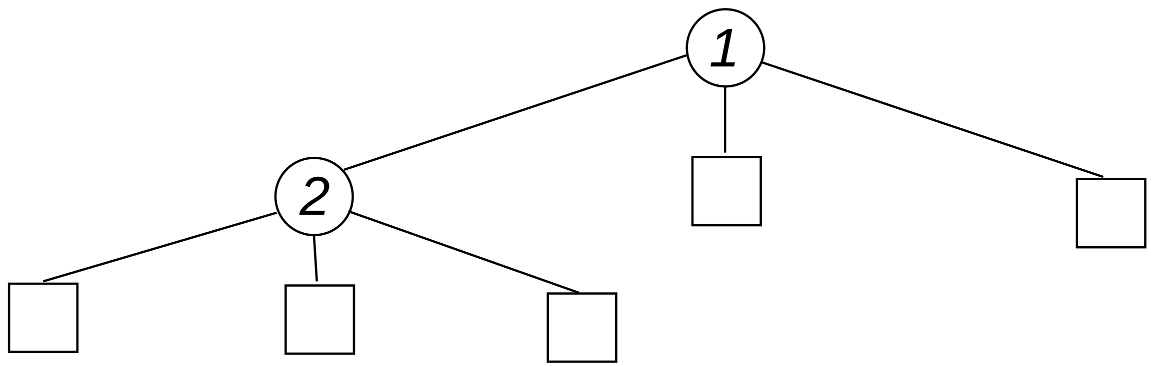

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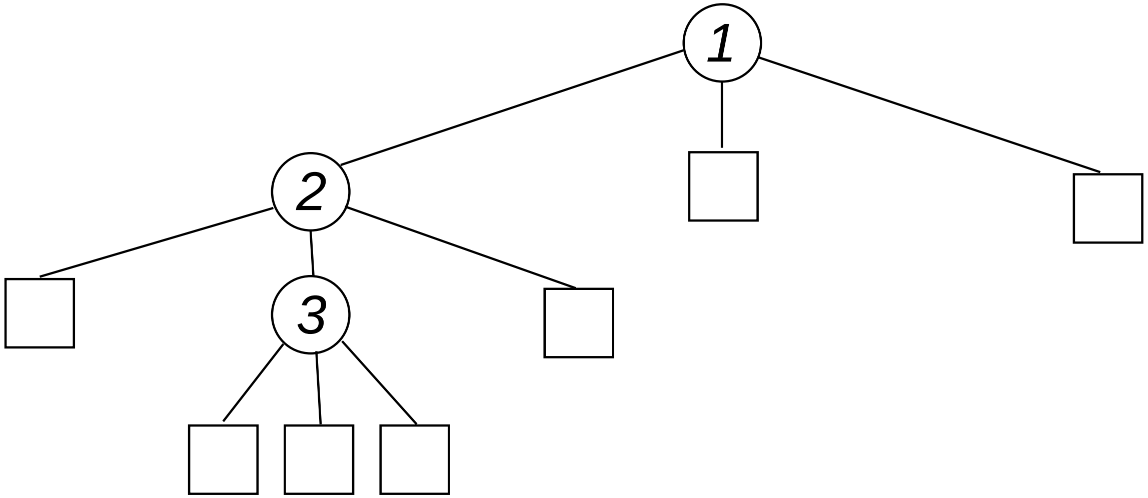

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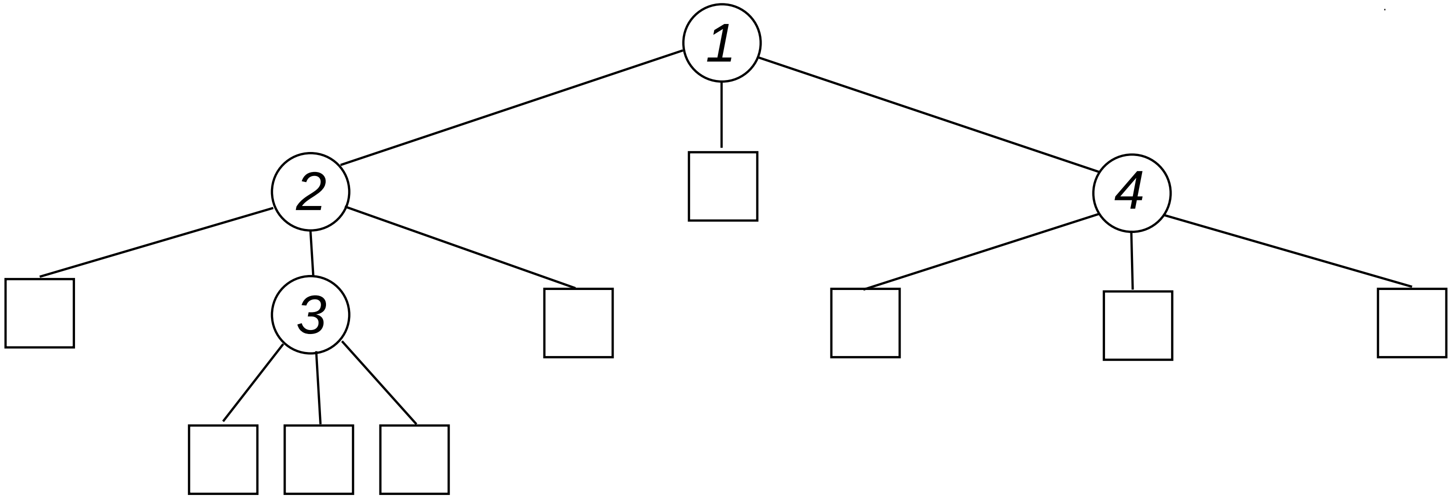

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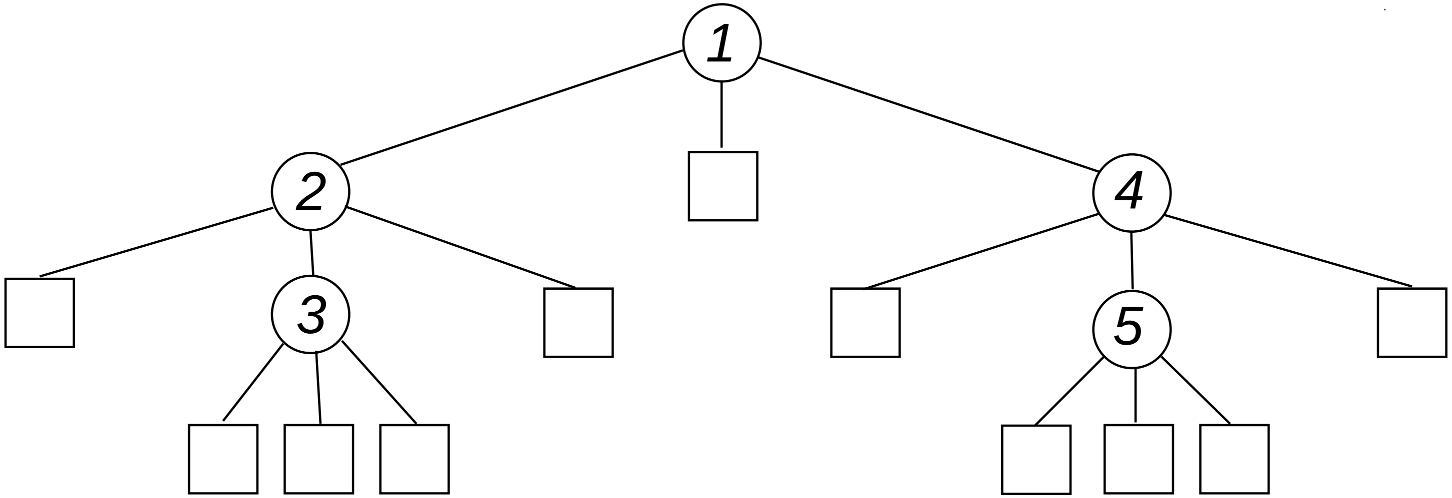

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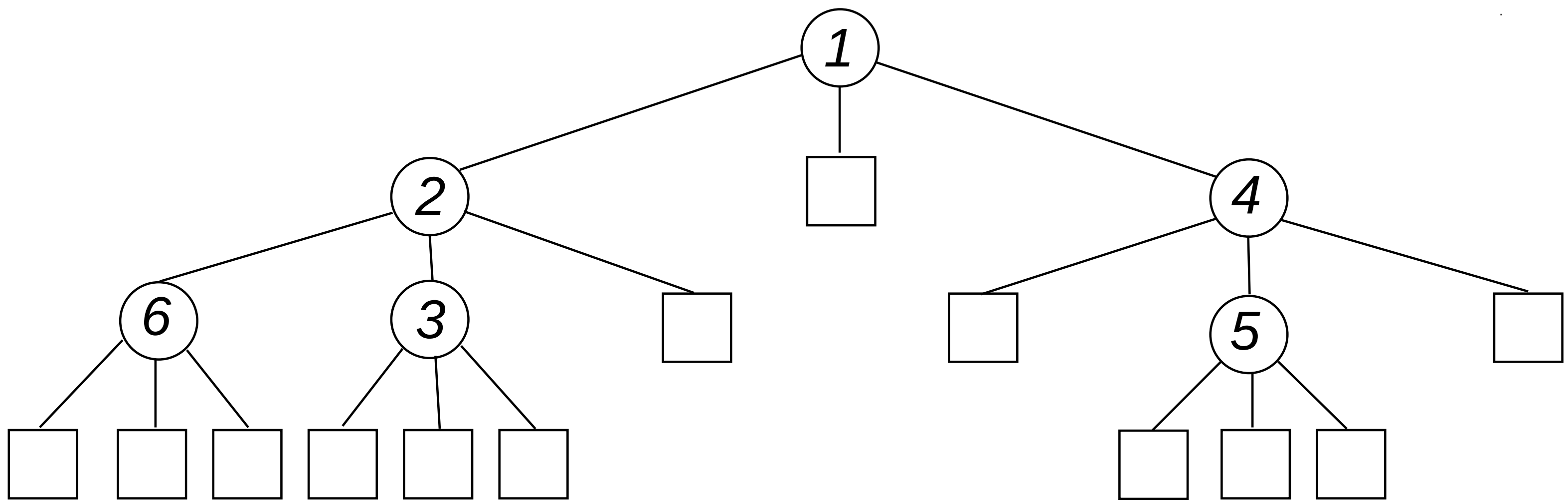

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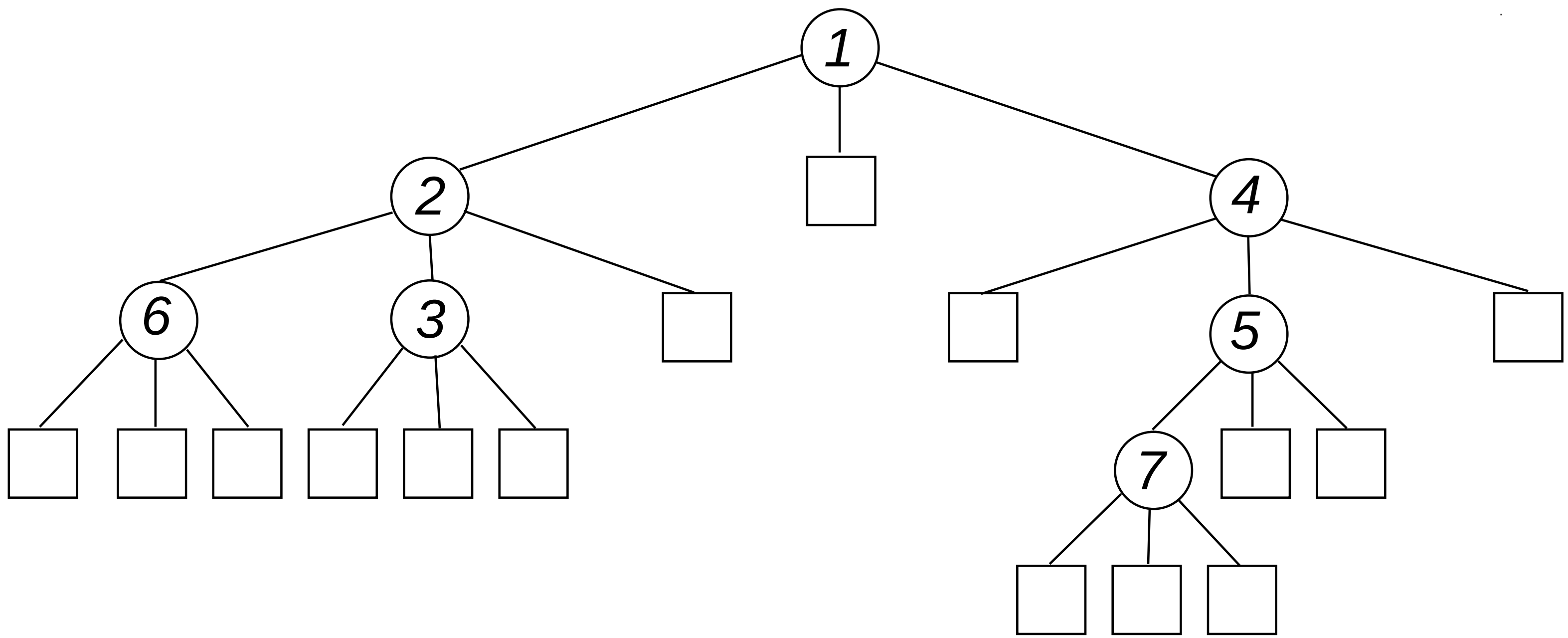

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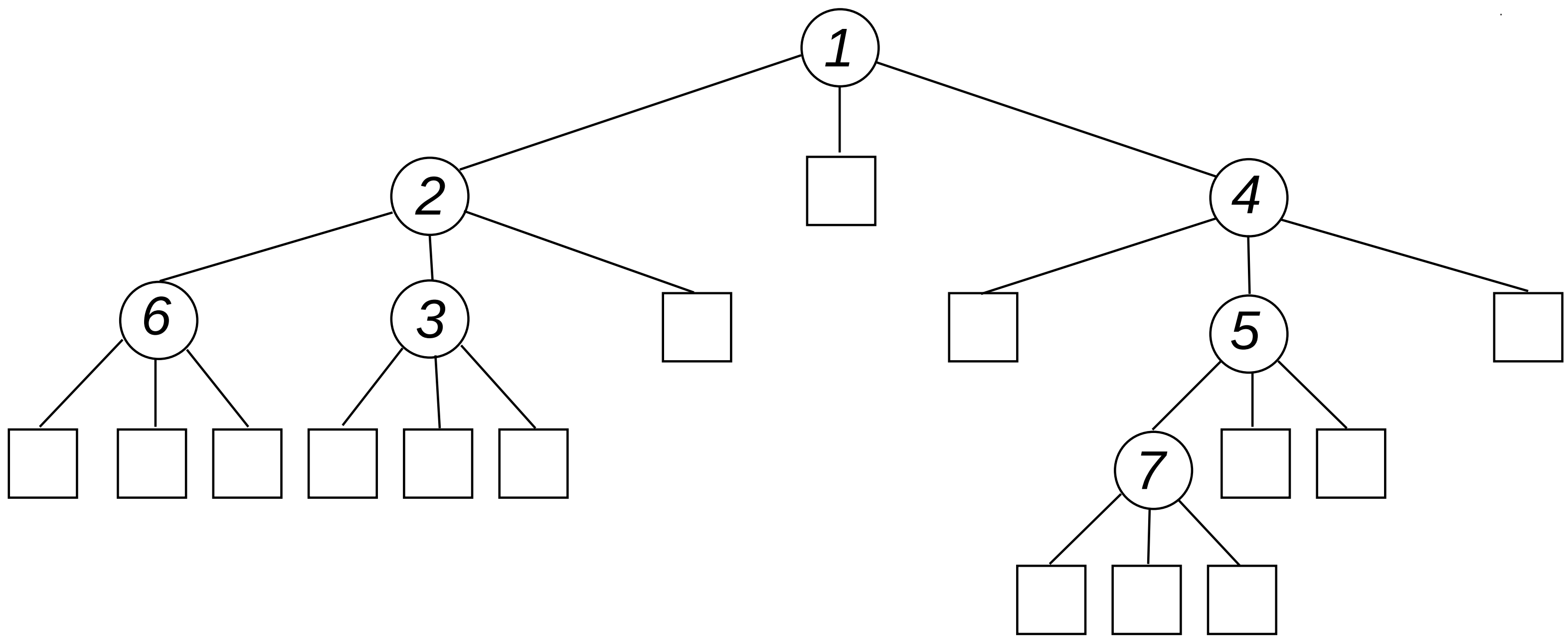

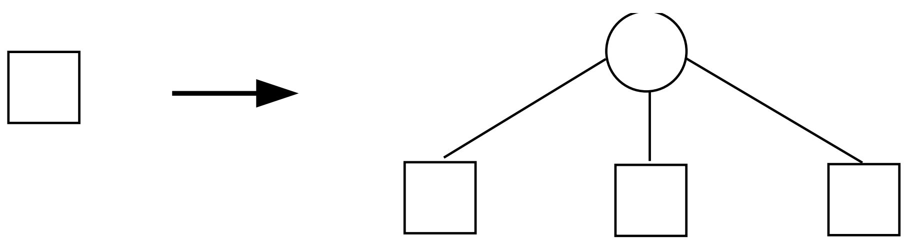

D-ary Recursive Trees

$\phi(t) = (1+t)^D$

Height H_n

[Devroye et al. 2005+?]

$$\boxed{\frac{H_n}{\log n} \to c_D \qquad (a.s.)}$$

where $c = c_D > 1$ satisfies the equation $c \log \dfrac{De}{c(D-1)} = \dfrac{1}{D-1}$.

Special Case: Binary Search Trees $(D = 2)$
[Pittel, Devroye, Robson, Reed, ...]

Polynomial Increasing Trees

$$\phi(t) = \varphi_0 + \varphi_1 t + \cdots + \varphi_D t^D \qquad (\varphi_0 \neq 0,\ \varphi_D \neq 0)$$

Height H_n

[Devroye et al. 2005+?]

$$\boxed{\frac{H_n}{\log n} \to c_D \quad (a.s.)}$$

where $c = c_D > 1$ satisfies the equation $c \log \dfrac{De}{c(D-1)} = \dfrac{1}{D-1}$.

Generating Functions

Let $y(z) = \sum_{n\ge 0} y_n z^n/n!$ be the generating function of $y_n = \sum_{T\in\mathcal{P}_n} \omega(T)$:

$$y'(z) = \phi(y(z)), \quad y(0) = 0.$$

$$\mathbf{P}\{H_n \le k\} = \frac{1}{y_n} \sum_{T\in\mathcal{P}_n,\, H(T)\le k} \omega(T)$$

$$y_k(z) = \sum_{n\ge 0} y_n \mathbf{P}\{H_n \le k\} \frac{z^n}{n!}.$$

$\Longrightarrow$

$$\boxed{y'_{k+1}(z) = \phi(y_k(z))}$$

with initial conditions $y_0(x) = 0$ and $y_{k+1}(0) = 0$.

Height Distribution

D-ary recursive trees

$\phi(t) = (1+t)^D$, ($D \ge 2$ positive integer), $y'_{k+1}(z) = (1+y_k(z))^D$

$\rho = 1/(D-1)$ radius of convergence of $y(z) = (1-(D-1)z)^{1/(D-1)} - 1$

$$c_D \log \frac{De}{c_D(D-1)} = \frac{1}{D-1}$$

$F(y)$ solution of integral equation

$$\boxed{y^{\frac{1}{D-1}} F(ye^{-1/c_D}) = \frac{\Gamma\left(\frac{D}{D-1}\right)}{\Gamma\left(\frac{1}{D-1}\right)^d} \int_{y_1+\cdots+y_D=y, y_j\ge 0} \prod_{j=1}^{D} \left(F(y_j) y_j^{\frac{1}{D-1}-1} \right) d\mathbf{y}}$$

Height Distribution

D**-ary recursive trees**

THEOREM 1 $\phi(t) = (1+t)^D$

$$\mathbf{E}\, H_n = c_D \log n + O\Big(\sqrt{\log n}\,(\log\log n)\Big)$$

$$\boxed{\mathbf{P}\{H_n \le k\} = F\Big((D-1)n/y_k(\rho)^{D-1}\Big) + o(1)}$$

$$\mathbf{P}\{|H_n - \mathbf{E}\, H_n| \ge \eta\} \ll e^{-c\eta} \qquad (c > 0)$$

Height Distribution

Remark 1:

$$\mathbf{Var}\, H_n = O(1)$$

Remark 2:

$h_n = \max\{k : y_k(\rho)^{d-1} \le n\}$

$W(x) = F(e^{-x})$ *"travelling wave"*

$$\boxed{\mathbf{P}\{H_n \le h_n + r\} = W\left(\log \frac{y_{h_n}(\rho)^{D-1}}{(D-1)n} + \frac{r}{c_D}\right) + o(1)}$$

$(-1/c_D \le \log \frac{y_{h_n}(\rho)^{D-1}}{(D-1)n} \le 1/c_D$ is bounded)

Height Distribution

Recursive Trees

$\phi(t) = e^t$, $\quad y'_{k+1}(z) = e^{y_k(z)}$

$y(z) = \log \frac{1}{1-z}$

$F(z)$ solution of

$$\boxed{y\,F(y/e^{1/e}) = \int_0^y F(z/e^{1/e})F(y-z)\,dz}$$

Height Distribution

Recursive Trees

THEOREM 2 $\phi(t) = e^t$

$$\mathbf{E}\, H_n = e \log n + O\Big(\sqrt{\log n}\, (\log\log n)\Big).$$

$$\boxed{\mathbf{P}\{H_n \le k\} = F(n/y_k'(\rho)) + o(1)}$$

$$\mathbf{P}\{|H_n - \mathbf{E}\, H_n| \ge \eta\} \ll e^{-c\eta} \quad (c > 0)$$

Height Distribution

Scale Free Trees

$\phi(t) = 1/(1-t)^r$, $r = \frac{A}{B} > 0$ rational number

$\rho = 1/(r+1)$ radius of convergence of $y(z) = 1 - (1-(r+1)z)^{1/(r+1)}$

$c_r' = 1/((r+1)s)$ with $r\, s\, e^{s+1} = 1$

$$y^{\frac{1}{A+B}} F(ye^{-1/c_r'}) = \frac{\Gamma\left(1+\frac{1}{A+B}\right)}{\Gamma\left(\frac{1}{A+B}\right)^{A+B+1}} \times$$
$$\times \int_{y_1+\cdots+y_{A+B+1}=y, y_j \ge 0} \prod_{j=1}^{B+1} \left(F(y_j e^{-1/c_r'}) y_j^{\frac{1}{A+B}-1} \right)$$
$$\times \prod_{\ell=B+2}^{A+B+1} \left(F(y_\ell) y_\ell^{\frac{1}{A+B}-1} \right) d\mathbf{y}$$

Height Distribution

Scale Free Trees

$$G(y) = \frac{\Gamma\left(\frac{A}{A+B}\right)}{\Gamma\left(\frac{1}{A+B}\right)^A} \int_{z_1+\cdots+z_A=1, z_j \geq 0} \prod_{j=1}^{A} \left(F(yz_j) z_j^{\frac{1}{A+B}-1} \right) d\mathbf{z}$$

Height Distribution

Scale Free Trees

THEOREM 3 $r = \frac{A}{B} > 0$ rational number, $\phi(t) = 1/(1-t)^r$

$$\mathbf{E}\, H_n \sim c_r' \log n.$$

$$\boxed{\mathbf{P}\{H_n \le k\} = G\Big((r+1)n/(y_k'(\rho))^{1+\frac{1}{r}}\Big) + o(1)}$$

$$\mathbf{P}\{|H_n - \mathbf{E}\, H_n| \ge \eta\} \ll e^{-c\eta} \quad (c > 0)$$

Auxiliary Functions

D-ary recursive trees $\phi(t) = (1+t)^D$

$$\tilde{y}_k(z) = y_k(z) + 1 = 1 + \sum_{n \geq 0} \mathbf{P}\{H_n \leq k\} y_n \frac{z^n}{n!}$$

$$\boxed{\tilde{y}'_{k+1}(z) = \tilde{y}_k(z)^D}$$

with initial conditions $\tilde{y}_0(z) = 1$, $\tilde{y}_k(0) = 1$.

Auxiliary Functions

D-ary recursive trees

$$y^{\frac{1}{D-1}}F(ye^{-1/c_D}) = \frac{\Gamma\left(\frac{D}{D-1}\right)}{\Gamma\left(\frac{1}{d-1}\right)^d} \int_{y_1+\cdots+y_D=y, y_j\ge 0} \prod_{j=1}^{D} \left(F(y_j) y_j^{\frac{1}{D-1}-1}\right) d\mathbf{y}$$

$$\Psi(u) = \frac{1}{(D-1)^{\frac{1}{D-1}}\Gamma\left(\frac{1}{D-1}\right)} \int_0^\infty F(y)\, y^{\frac{1}{D-1}-1} e^{-uy}\, dy$$

$$\boxed{\overline{y}_k(z) := e^{k/(c_D(D-1))} \cdot \Psi\left(e^{k/c_D}(\rho - z)\right)}$$

$(\rho = 1/(D-1))$

Auxiliary Functions

D**-ary recursive trees**

- $1-\overline{y}_k(0) \sim Ck\left(\frac{D}{c_D}\right)^k, \qquad \overline{y}_k(\rho) = e^{k/(c_D(D-1))}.$

-
$$\boxed{\overline{y}_{k+1}'(z) = \overline{y}_k(z)^D}$$

- For every positive integer ℓ and for every real number $k > 0$ the difference
$$\tilde{y}_\ell(z) - \overline{y}_k(z)$$
has exactly one zero (**"Intersection Property"**)

Auxiliary Functions

D-ary recursive trees

- $\overline{y}_k(z) = \sum_{n\ge 0} \overline{y}_{k,n} \frac{z^n}{n!}$ is an entire function with coefficients

$$\overline{y}_{k,n} = \frac{(D-1)^n}{\Gamma\left(\frac{1}{D-1}\right)} \int_0^\infty F\left((D-1)ve^{-k/c_D}\right) v^{\frac{1}{D-1}-1+n} e^{-v}\, dv$$

and asymptotically we have

$$\boxed{\frac{\overline{y}_{k,n}}{y_n} = F\left((D-1)ne^{-k/c_D}\right) + o(1)}$$

Auxiliary Functions

D-**ary recursive trees**

Proof idea

- $\tilde{y}_k(z) = y_k(z) + 1$ is approximated by the *auxiliary function* $\overline{y}_{e_k}(z)$:

$$\tilde{y}_k(\rho) = \overline{y}_{e_k}(\rho) \quad \Longleftrightarrow \quad e_k = c_D(D-1)(\log \tilde{y}_k(\rho)) \sim k.$$

- $\tilde{y}_k(z) \approx \overline{y}_{e_k}(z)$ in a neighbourhood of $z = \rho$

$$\Longrightarrow \quad \boxed{\mathbf{P}\{H_n \le k\} \approx \overline{y}_{n,e_k} = F\Big((D-1)n/y_k(\rho)^{d-1}\Big) + o(1)}$$

Auxiliary Functions

Recursive Trees $\phi(t) = e^t$

$$y_k(z) = \sum_{n \ge 0} \mathbf{P}\{H_n \le k\} \frac{z^n}{n}$$

$$\boxed{y'_{k+1}(z) = e^{y_k(z)}}$$

$$Y_k(z) = y'_k(z) = \sum_{n \ge 0} \mathbf{P}\{H_{n+1} \le k\} z^n$$

$$\boxed{Y'_{k+1}(z) = Y_{k+1}(z) Y_k(z)}$$

$(Y_{k+1}(0) = 1)$

Auxiliary Functions

Recursive Trees

$$y\,F(y/e^{1/e}) = \int_0^y F(z/e^{1/e})F(y-z)\,dz$$

$$\Psi(u) = \int_0^\infty F(y)e^{-yu}\,dy$$

$$\boxed{\overline{Y}_k(z) = e^{k/e} \cdot \Psi\left(e^{k/e}(1-z)\right)}$$

Auxiliary Functions

Recursive Trees

- $1 - \overline{Y}_k(0) \sim Ck\left(\frac{2}{e}\right)^k, \qquad \overline{Y}_k(1) = e^{k/e}.$

- $$\boxed{\overline{Y}'_{k+1}(z) = \overline{Y}_{k+1}(z)\overline{Y}_k(z)}$$

- For every positive integer ℓ and for every real number $k > 0$ the difference

  $$Y_\ell(z) - \overline{Y}_k(z)$$

  has exactly one zero (**"Intersection Property"**).

Auxiliary Functions

Recursive Trees

- $\overline{Y}_k(z) = \sum_{n\geq 0} \overline{Y}_{k,n} z^n$ is an entire function with coefficients

$$\overline{y}_{k,n} = \int_0^\infty F\left(ve^{-k/e}\right) v^n e^{-v}\, dv$$

and asymptotically we have

$$\boxed{\overline{Y}_{k,n} = F\left(ne^{-k/e}\right) + o(1)}$$

Auxiliary Functions

Recursive Trees

Remark:

The functions

$$\overline{y}_k(z) = \int_0^z \overline{Y}_k(t)\,dt = \log \overline{Y}_{k+1}(z)$$

satisfy the recurrence

$$\boxed{\overline{y}_{k+1}(z) = e^{\overline{y}_k(z)}}$$

Auxiliary Functions

Scale Free Trees $\phi(t) = (1-t)^{-r}$, $r = \frac{A}{B}$

$$y_k(z) = \sum_{n\ge 0} y_n \mathbf{P}\{H_n \le k\} z^n/n!$$

$$\boxed{y'_{k+1}(z) = \frac{1}{(1-y_k(z))^r}}$$

$$Y_k(z) = \left(y'_k(z)\right)^{\frac{1}{A}} \qquad \left(\text{d.h. } y'_k(z) = Y_k(z)^A = \sum_{n\ge 0} y_{n+1} \mathbf{P}\{H_{n+1} \le k\} \frac{z^n}{n!}\right)$$

$$\boxed{Y'_{k+1}(z) = \frac{1}{B} Y_{k+1}(z)^{B+1} Y_k(z)^A}$$

$(Y_{k+1}(0) = 1)$

Auxiliary Functions

Scale Free Trees

$$y^{\frac{1}{d-1}} F(ye^{-1/c'_r}) = \frac{\Gamma\left(1+\frac{1}{A+B}\right)}{\Gamma\left(\frac{1}{A+B}\right)^{A+B+1}} \times$$

$$\times \int_{y_1+\cdots+y_{A+B+1}=y, y_j \geq 0} \prod_{j=1}^{B+1} \left(F(y_j e^{-1/c'_r}) y_j^{\frac{1}{A+B}-1} \right)$$

$$\times \prod_{\ell=B+2}^{A+B+1} \left(F(y_\ell) y_\ell^{\frac{1}{A+B}-1} \right) d\mathbf{y}$$

Auxiliary Functions

Scale Free Trees

$$\overline{\Psi}(u) = \frac{1}{(r+1)^{\frac{1}{A+B}}\Gamma\left(\frac{1}{A+B}\right)} \int_0^\infty F(y)\, y^{\frac{1}{A+B}-1} e^{-uy}\, dy$$

$$\boxed{\overline{Y}_k(z) = e^{k/(c'_r(A+B))} \cdot \overline{\Psi}\left(e^{k/c'_r}\left(\frac{1}{r+1} - z\right)\right)}$$

Auxiliary Functions

Scale Free Trees

$$G(y) = \frac{\Gamma\left(\frac{A}{A+B}\right)}{\Gamma\left(\frac{1}{A+B}\right)^A} \int_{z_1+\cdots+z_A=1, z_j \geq 0} \prod_{j=1}^{A} \left(F(yz_j) z_j^{\frac{1}{A+B}-1} \right) d\mathbf{z}$$

$$\Psi(u) = \overline{\Psi}(u)^A = \frac{1}{(r+1)^{\frac{r}{1+r}} \Gamma\left(\frac{r}{1+r}\right)} \int_0^\infty G(y) y^{-\frac{1}{1+r}} e^{-yu} \, dy$$

$$\boxed{\overline{y}_k(z) = \int_0^z e^{\frac{rk}{c'_r(1+r)}} \cdot \Psi\left(e^{k/c'_r}\left(\frac{1}{r+1} - t\right)\right)} dt$$

Auxiliary Functions

Scale Free Trees

- $$\overline{Y}'_{k+1}(z) = \frac{1}{B}\overline{Y}_{k+1}(z)^{B+1}\overline{Y}_k(z)^A$$

- $$\overline{y}'_{k+1}(z) = \frac{1}{(1-\overline{y}_k(z))^r}$$

etc.

“Intersection Property”

Lemma

$\tilde{y}_0(x) = 1$, $\boxed{\tilde{y}'_{k+1}(x) = \tilde{y}_k(x)^D}$ with $\tilde{y}_{k+1}(0) = 1$.

$\overline{y}_k(z) := e^{k/(c_D(D-1))} \cdot \Psi\left(e^{k/c_D}(\rho - z)\right) \qquad (k \in \mathbb{R})$

$\boxed{\overline{y}'_{k+1}(x) = \tilde{y}_k(x)^D}$ with $0 < \overline{y}_{k+1}(0) < 1$

$\Longrightarrow$ For every integer $\ell \geq 0$ and for every real number $k > 0$ the difference

$$\tilde{y}_\ell(z) - \overline{y}_k(z)$$

has exactly one zero.

“Intersection Property”

Proof

The case $\ell = 0$ is (trivially) true for all $k > 0$.

$\ell \to \ell + 1$:

$$\tilde{y}'_{\ell+1}(x) - \overline{y}'_{k+1}(x) = (\tilde{y}_\ell(x) - \overline{y}_k(x)) \underbrace{\sum_{j=0}^{D-1} \tilde{y}_\ell(x)^j \overline{y}_k(x)^{D-1-j}}_{>0}$$

$\Longrightarrow \tilde{y}'_{\ell+1}(x) - \overline{y}'_{k+1}(x)$ has exactly one zero.

$\Longrightarrow \tilde{y}_{\ell+1}(x) - \overline{y}_{k+1}(x)$ has exactly one zero.

Thank You