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WEYL SUMS OVER INTEGERS WITH LINEAR DIGIT RESTRICTIONS

by Michael DRMOTA and Christian MAUDUIT (*)

ABSTRACT. For any given integer $q \geqslant 2$, we consider sets $\mathcal{N}$ of non-negative integers that are defined by linear relations between their q-adic digits (for example, the set of non-negative integers such that the number of 1's equals twice the number of 0's in the binary representation). The main goal is to prove that the sequence $(\alpha n)_{n\in\mathcal{N}}$ is uniformly distributed modulo 1 for all irrational numbers α. The proof if based on a saddle point analysis of certain generating functions that allow us to bound the corresponding Weyl sums.

Sommes de Weyl et nombres entiers définis par des contraintes linéaires sur leurs chiffres

RÉSUMÉ. Le nombre entier $q \geqslant 2$ étant fixé, nous étudions les ensembles $\mathcal{N}$ de nombres entiers positifs définis par des relations linéaires entre les chiffres de leur représentation q-adique (par exemple l'ensemble des nombres entiers positifs dont la représentation binaire contient deux fois plus de 1 que de 0). Notre objectif principal est de démontrer que la suite $(\alpha n)_{n\in\mathcal{N}}$ est équirépartie modulo 1 pour tout nombre irrationnel α. La preuve s'appuie sur l'étude des points selles d'une certaine fonction génératrice qui permet de majorer les sommes de Weyl.

1. Introduction

Let $q \geqslant 2$ be a given integer and let

$$n = \sum_{j=0}^{L} \varepsilon_j(n) q^j$$

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be the q-ary digital expansions with digits $\varepsilon_j(n) \in \{0, 1, \ldots, q-1\}$ and $L = L(n) = [\log_q n]$ denotes the length of the expansion. Further, for $\ell \in \{0, 1, \ldots, q-1\}$ let

$$|n|_\ell := \mathrm{card}\{0 \leqslant j \leqslant L : \varepsilon_j(n) = \ell\}$$

denote the number of digits of n that equal ℓ.

For example, the q-ary sum-of-digits function is given by

$$s_q(n) = \sum_{j=0}^{L} \varepsilon_j(n) = \sum_{\ell=0}^{q-1} \ell\, |n|_\ell.$$

Several works concern the study of statistical properties of sequences of integers defined by digital properties: distribution in residue classes ([11, 14, 15, 20, 21, 22]) uniform distribution modulo 1 ([2, 5, 6, 7, 17, 12, 19]) and study of the associated exponential sums ([3, 1, 4, 13]); see also [23] for a description of the links to spectral analysis and properties of symbolic dynamical systems.

The purpose of this paper is to study, for any fixed irrational number α, the ditribution modulo 1 of the sequence $(n\alpha)_{n\in\mathcal{N}}$, where $\mathcal{N}$ is a set of integers defined by linear properties of their digits.

Definition 1.1. — *We say that the system $\mathcal{L} = (L_k)_{k=1,\ldots,K}$ of linear forms on $\mathbb{R}^q$ defined for every $(x_0, \ldots, x_{q-1}) \in \mathbb{R}^q$ by*

$$L_k(x_0, x_1, \ldots, x_{q-1}) = \sum_{\ell=0}^{q-1} a_{k,\ell} x_\ell, \qquad k = 1, \ldots, K$$

(with $a_{k,\ell} \in \mathbb{Z}$ for $(k, \ell) \in \{1, \ldots, K\} \times \{0, \ldots, q-1\}$) is complete if

(i) *the family of vectors formed by $(a_{1,0}, \ldots, a_{1,q-1}), \ldots, (a_{K,0}, \ldots, a_{K,q-1})$ and $(1, \ldots, 1)$ is linearly independent over $\mathbb{Q}$.*

(ii) $\mathrm{span}_{\mathbb{Z}} \{L_k(n_0, \ldots, n_{q-1}) : k = 1, \ldots, K,\ (n_0, \ldots, n_{q-1}) \in \mathbb{Z}^q\} = \mathbb{Z}^K$.

Let $\mathcal{L}$ be a complete system of linear forms over $\mathbb{R}^q$ and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_K)$ be a K-tuple of non negative real numbers.

Definition 1.2. — *We say that $\boldsymbol{\eta}$ is $\mathcal{L}$-admissible if the system of equations*

$$L_k(x_0, x_1, \ldots, x_{q-1}) = \eta_k, \qquad k = 1, \ldots, K,$$
$$x_0 + \cdots + x_{q-1} = 1$$

has a positive solution $x_0 > 0, x_1 > 0, \ldots, x_{q-1} > 0$.

Example 1.3. — If $K=1$, $\mathcal{L}=(L_1)$ with $L_1(x_0,\dots,x_{q-1})=\sum_{\ell=0}^{q-1}\ell x_\ell$, then $\boldsymbol{\eta}=(\frac{q-1}{2})$ is $\mathcal{L}$-admissible.

Example 1.4. — If $K\geqslant 1$, $\mathcal{L}=(L_1,\dots,L_k)$ with $L_k(x_0,\dots,x_{q-1})=x_0-x_k$ for $k=1,\dots,K$, then $\boldsymbol{\eta}=(0,\dots,0)$ is $\mathcal{L}$-admissible.

Example 1.5. — If $K=1$, $\mathcal{L}=(L_1)$ with $L_1(x_0,\dots,x_{q-1})=x_0-2x_1$, then $\boldsymbol{\eta}=(0,\dots,0)$ is $\mathcal{L}$-admissible.

For any complete system $\mathcal{L}$ of linear forms over $\mathbb{R}^q$, for any $\mathcal{L}$-admissible K-tuple $\boldsymbol{\eta}\in(\mathbb{R}^+)^K$ and for any K-tuple $\boldsymbol{\mu}=(\mu_1,\dots,\mu_K)\in\mathbb{Z}^K$ we define the set of integers

$$\begin{aligned}\mathcal{N}&=\mathcal{N}(\mathcal{L},\boldsymbol{\eta},\boldsymbol{\mu})\\ &=\left\{n\in\mathbb{N}: L_k(|n|_0,|n|_1,\dots,|n|_{q-1})=[\eta_k\log_q n]+\mu_k,\ k=1,\dots,K\right\}.\end{aligned}\tag{1.1}$$

In what follows we will always assume that $\mathcal{L}$ is complete and that $\boldsymbol{\eta}$ is $\mathcal{L}$-admissible.

In section 3 we will give the following estimate for $\operatorname{card}\{n\in\mathcal{N}: n<N\}$:

THEOREM 1.6. — *There exist positive constants C_1,C_2 and $\gamma<1$ depending only on $\mathcal{L}$, $\boldsymbol{\eta}$, and $\boldsymbol{\mu}$ such that for any integer $N\geqslant 2$ we have*

$$C_1\frac{N^\gamma}{(\log_q N)^{K/2}}\leqslant \operatorname{card}\{n\in\mathcal{N}: n<N\}\leqslant C_2\frac{N^\gamma}{(\log_q N)^{K/2}}.$$

In section 4 we prove our main result:

THEOREM 1.7. — *For any irrational number α the sequence $(n\alpha)_{n\in\mathcal{N}}$ is uniformly distributed modulo 1.*

Such a kind of theorem has been proved in [12] in the particular case of sequences of integers with an average sum of digits. More precisely, for any $b:\mathbb{N}\to\mathbb{R}$ such that $\frac{q-1}{2}\nu+b(\nu)\in\mathbb{N}$ for any $\nu\geqslant 1$ and such that the sequence $\left(\frac{b(\nu)}{\nu^{1/4}}\right)_{\nu\geqslant 1}$ is bounded, then Theorem 1.2 from [12] says that for any irrational number α the sequence $(n\alpha)_{n\in\mathcal{E}_b}$ is uniformly distributed modulo 1, where

$$\mathcal{E}_b=\left\{n\in\mathbb{N}: s_q(n)=\frac{q-1}{2}[\log_q n]+b\left([\log_q n]\right)\right\}.$$

It is easy to verify that in the particular case where $q=2$ or 3, $\mathcal{L}$ and $\boldsymbol{\eta}$ defined as in Example 1.3 and $\boldsymbol{\mu}=(0)$, our theorem is a consequence of Theorem 1.2 from [12] but that these results are formally disjoint when $q\geqslant 4$. Nevertheless the study of [12] concerns the case of integers whose

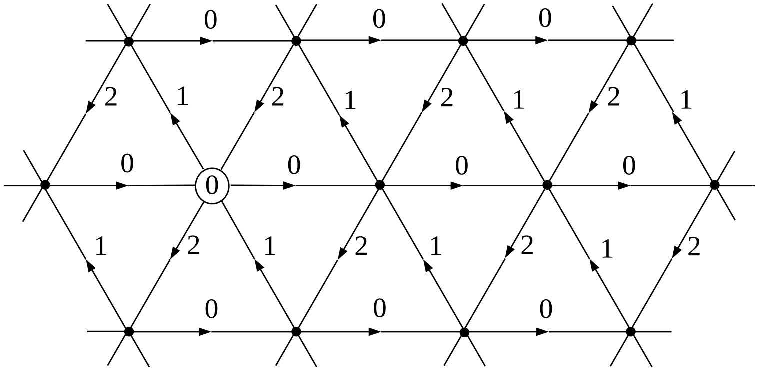

Figure 1.1. Infinite 3-automaton generating $n \in \mathbb{N}$ with $|n|_0 = |n|_1 = |n|_2$

sum of digits is "close" to the expected value and our work generalizes this study to the case of integers whose sum of digits (or any other linear combination of digits) is "far" from the expected value.

In the particular case where $\mathcal{L}$ and $\boldsymbol{\eta}$ are defined as in Example 1.4 and for $\boldsymbol{\mu} = (0, \ldots, 0)$, our theorem corresponds to Theorem 4.2 (for the set $\mathcal{E}_{q-1}$) from [19]. The main theorem from [19] can be understood as an uniform distribution result in the case where the set $\mathcal{N}$ defined by (1.1) is generated by a deterministic q-infinite automaton corresponding to a random walk of zero average on a d-dimensional lattice (see [19] for definitions of these notions).

For example when $q = 3$, $\mathcal{L}$ and $\boldsymbol{\eta}$ as in Example 1.4 and $\boldsymbol{\mu} = (0, \ldots, 0)$, the set $\mathcal{N} = \{n \in \mathbb{N} : |n|_0 = |n|_1 = |n|_2\}$ is generated by the deterministic 3-infinite automaton (that is depicted in Figure 1.1) with 0 as initial state and 0 as unique final state.

The theorem we prove here is a generalization of this result to the case of any random walk on a q-dimensional lattice (the more general case of d-dimensional lattices, with $d \leqslant q$, corresponds to the generalization suggested in section 5 of our paper).

Indeed for $\mathcal{L}$ and $\boldsymbol{\eta}$ as in Example 1.5 and $\boldsymbol{\mu} = (0)$, the set $\mathcal{N} = \{n \in \mathbb{N} : |n|_0 = 2|n|_1\}$ is generated by the deterministic q-infinite automaton that is depicted in Figure 1.2 with 0 as initial state and 0 is unique final state. It is also linked to the random walk on the lattice $\mathbb{Z}$ with probability transition $(\frac{1}{3}, \frac{2}{3})$. It follows in particular from our main theorem that for any irrational number α the sequence $(n\alpha)_{|n|_0 = 2|n|_1}$ is uniformly distributed modulo 1

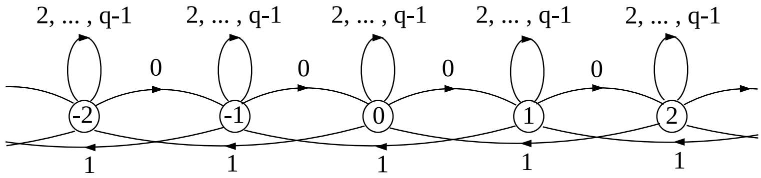

Figure 1.2. Infinite q-automaton generating $n \in \mathbb{N}$ with $|n|_0 = 2|n|_1$

(such a kind of result was out of reach from the methods developped in [12] and [19]).

It follows from Weyl's criterion that in order to prove Theorem 1.7, it is enough to show that for every irrational number α we have

$$\sum_{n\in\mathcal{N}, n\leqslant N} e(\alpha n) = o(\mathrm{card}\{n \in \mathcal{N} : n \leqslant N\})$$

as $N \to \infty$, where we denote $e(x) = e^{2\pi i x}$ (for general references to the theory of uniformly distributed sequences we refer to [16] and [10]). For this purpose we use a sophisticated saddle point method applied to properly chosen generating functions. In Section 2 we set up the generating functions related to our problem. Then in Section 3 we collect some technical properties that are necessary to apply a saddle point analysis which leads to a proof of Theorem 1.6. A variation of this method leads then in Section 4 to a proof of Theorem 1.7. Finally we comment on some generalizations of Theorem 1.7 concerning missing digits (Section 5.1) and non integer coefficients (Section 5.2).

2. Generating Functions

We first present explicit formulas for

$$S_N(x_0, x_1, \dots, x_{q-1}, y) = \sum_{0<n<N} x_0^{|n|_0} x_1^{|n|_1} \cdots x_{q-1}^{|n|_{q-1}} y^n$$

Lemma 2.1. — *Define $T_{\nu,N}(x_0,x_1,\ldots,x_{q-1},y)$ for $N\leqslant q^\nu$ recursively by*

$$\begin{aligned}
T_{\nu,q^\nu}(x_0,x_1,\ldots,x_{q-1},y) =&\prod_{r<\nu}\left(x_0+x_1y^{q^r}+\cdots+x_{q-1}y^{(q-1)q^r}\right),\\
T_{\nu,\ell q^j}(x_0,x_1,\ldots,x_{q-1},y) =&x_0^{\nu-j-1}\left(x_0+x_1y^{q^j}+\cdots+x_{\ell-1}y^{(\ell-1)q^j}\right)\\
&\times T_{j,q^j}(x_0,x_1,\ldots,x_{q-1},y)\\
&\text{for } 1\leqslant \ell<q \text{ and } j<\nu, \text{ and by}\\
T_{\nu,\ell q^j+N'}(x_0,x_1,\ldots,x_{q-1},y) =&T_{\nu,\ell q^j}(x_0,x_1,\ldots,x_{q-1},y)\\
&+x_0^{\nu-j-1}x_\ell y^{\ell q^j}T_{j,N'}(x_0,x_1,\ldots,x_{q-1},y)\\
&\text{for } 1\leqslant \ell<q \text{ and } N'<q^j.
\end{aligned}$$

Then we have

$$\begin{aligned}
S_{q^\nu}(x_0,x_1,\ldots,x_{q-1},y) =&\sum_{j<\nu}\left(x_1y^{q^j}+\cdots+x_{q-1}y^{(q-1)q^j}\right)\\
&\cdot T_{j,q^j}(x_0,x_1,\ldots,x_{q-1},y),\\
S_{\ell q^\nu}(x_0,x_1,\ldots,x_{q-1},y) =&S_{q^\nu}(x_0,x_1,\ldots,x_{q-1},y)\\
&+\left(x_1y^{q^\nu}+\cdots+x_{\ell-1}y^{(\ell-1)q^\nu}\right)\cdot T_{\nu,q^\nu}(x_0,x_1,\ldots,x_{q-1},y)\\
&\text{for } 2\leqslant \ell<q, \text{ and}\\
S_{\ell q^\nu+N'}(x_0,x_1,\ldots,x_{q-1},y) =&S_{\ell q^\nu}(x_0,x_1,\ldots,x_{q-1},y)\\
&+x_\ell y^{\ell q^j}T_{\nu,N'}(x_0,x_1,\ldots,x_{q-1},y)\\
&\text{for } N'<q^\nu.
\end{aligned}$$

Proof. — First we give an alternate definition for $T_{\nu,N}(x_0,x_1,\ldots,x_{q-1},y)$. Suppose that we consider all number $n<q^\nu$ in the form $n=\varepsilon_0(n)+\varepsilon_1(n)q+\cdots+\varepsilon_{\nu-1}(n)q^{\nu-1}$. Similarly to the above we set

$$|n|_{\nu,\ell}:=\operatorname{card}\{0\leqslant j<\nu:\varepsilon_j(n)=\ell\}.$$

Of course, if $n<q^\nu$ and $\ell\neq 0$ then $|n|_{\nu,\ell}=|n|_\ell$. However, for $\ell=0$ we usually have $|n|_{\nu,0}\neq|n|_0$ since $|n|_{\nu,0}$ takes all zero digits up to $q-1$ into account. Now set (for $N\leqslant q^\nu$)

$$T_{\nu,N}(x_0,x_1,\ldots,x_{q-1},y)=\sum_{n<N}x_0^{|n|_{\nu,0}}x_1^{|n|_{\nu,1}}\cdots x_{q-1}^{|n|_{\nu,q-1}}y^n.$$

With help of this definition the proof of Lemma 2.1 is immediate. □

Corollary 2.2. — *Suppose that $x_0,x_1,\ldots,x_{q-1}$ are complex numbers that are sufficiently close to the positive real axis and $x_0+\cdots+x_{q-1}\neq 1$.*

Then we have

$$\sum_{0<n<N} x_0^{|n|_0} x_1^{|n|_1} \cdots x_{q-1}^{|n|_{q-1}} = G(x_0,\ldots,x_{q-1},\log_q N)\cdot(x_0+\cdots+x_{q-1})^{\log_q N} - \frac{x_1+\cdots+x_{q-1}}{x_0+x_1+\cdots+x_{q-1}-1},$$

where $G(x_0,x_1,\ldots,x_{q-1},t)$ is a function that is analytic in $x_0,x_1,\ldots,x_{q-1}$ and continuous and periodic in t (with period 1).

Furthermore, if I is any closed interval of positive real numbers with $\min I > 1/q$. Then, for every $\varepsilon > 0$ such that there is at least one j with $|\arg(x_j)| \geqslant \varepsilon$, there exists $\delta > 0$ and $C > 0$ such that

(2.1)
$$\left|\sum_{0<n<N} x_0^{|n|_0} x_1^{|n|_1} \cdots x_{q-1}^{|n|_{q-1}}\right| \leqslant C\cdot(|x_0|+|x_1|+\cdots+|x_{q-1}|)^{(1-\eta)\log_q N}$$

uniformly for all x_j with $|x_j|\in I$.

Proof. — We first provide a corresponding representation for $T_{\nu,N'}$. Suppose that the q-adic expansion of N' is given by

$$N' = \ell_1 q^{k_1} + \ell_2 q^{k_2} + \cdots + \ell_L q^{k_L}$$

with digits $0 < \ell_j < q$ and exponents $k_1 > k_2 > \cdots > k_L \geqslant 0$, then it directly follows that

$$\begin{aligned} T_{k_1+1,N'}(x_0,x_2,\ldots,x_{q-1},1) &= (x_0+\cdots+x_{\ell_1-1})X^{k_1} \\ &+ x_0^{k_1-k_2-1}x_{\ell_1}(x_0+\cdots+x_{\ell_2-1})X^{k_2} \\ &+ x_0^{k_1-k_3-2}x_{\ell_1}x_{\ell_2}(x_0+\cdots+x_{\ell_3-1})X^{k_3} \\ &+\cdots \\ &+ x_0^{k_1-k_L-L+1}x_{\ell_1}\cdots x_{\ell_{L-1}}(x_0+\cdots+x_{\ell_L})X^{k_L}, \end{aligned}$$

where X abbreviates $X = x_0+x_1+\cdots+x_{q-1}$. Further we have

$$S_{q^\nu}(x_0,x_2,\ldots,x_{q-1},1) = (X-x_0)\frac{X^\nu-1}{X-1}$$

and (for $\ell \geqslant 2$)

$$\begin{aligned} S_{\ell q^\nu}(x_0,x_2,\ldots,x_{q-1},1) &= (X-x_0)\frac{X^\nu-1}{X-1} + (x_1+\cdots+x_{\ell-1})X^\nu \\ &= (X-x_0)\frac{X^\nu-1}{X-1} - x_0X^\nu + (x_0+\cdots+x_{\ell-1})X^\nu. \end{aligned}$$

Consequently, if N is given by

$$N = \ell_0 q^{k_0} + \ell_1 q^{k_1} + \cdots + \ell_L q^{k_L}$$

then we have

$$
\begin{aligned}
S_N(x_0,x_2,\ldots,x_{q-1},1) &= (X-x_0)\frac{X^{k_0}-1}{X-1} - x_0X^{k_0} \\
&\quad + (x_0+\cdots+x_{\ell_0-1})X^{k_0} \\
&\quad + x_0^{k_0-k_1-1}x_{\ell_0}(x_0+\cdots+x_{\ell_1-1})X^{k_1} \\
&\quad + x_0^{k_0-k_2-2}x_{\ell_0}x_{\ell_1}(x_0+\cdots+x_{\ell_2-1})X^{k_2} \qquad (2.2)\\
&\quad + \cdots \\
&\quad + x_0^{k_0-k_L-L}x_{\ell_0}\cdots x_{\ell_{L-1}}(x_0+\cdots+x_{\ell_L})X^{k_L}.
\end{aligned}
$$

For $0 \leqslant t < 1$ let the q-adic expansion of q^t be given by

$$q^t = \ell_0 + \sum_{j\geqslant 1} \ell_j q^{-k_j}$$

with digits $0 < \ell_j < q$ and exponents $0 < k_1 < k_2 < \cdots$ and set

$$
\begin{aligned}
G(x_0,x_2,\ldots,x_{q-1},t) = X^{-t}\Big(&\frac{X(1-x_0)}{X-1} + (x_0+\cdots+x_{\ell_0-1}) \\
&+ \frac{x_{\ell_0}}{x_0}(x_0+\cdots+x_{\ell_1-1})\left(\frac{X}{x_0}\right)^{-k_1} \\
&+ \frac{x_{\ell_0}}{x_0}\frac{x_{\ell_1}}{x_0}(x_0+\cdots+x_{\ell_2-1})\left(\frac{X}{x_0}\right)^{-k_2} \\
&+ \cdots\Big).
\end{aligned}
$$

It is an easy exercise to show that G is continuous in t and can be periodically extended to a (continuous) function with period 1 provided $x_0,\ldots,x_{q-1}$ are sufficiently close to the positive real line. If fact G is Hölder continuous with a positive exponent depending on $x_0,\ldots,x_{q-1}$ (compare with [9]). Furthermore by definition it follows that

$$S_N(x_0,\ldots,x_{q-1},1) = G(x_0,\ldots,x_{q-1},\log_q N)\cdot X^{\log_q N} - \frac{X-x_0}{X-1}.$$

Finally, if we assume that $|x_j| \in I$ and $|\arg(x_j)| \geqslant \varepsilon$ for some j and for some closed interval I of positive real numbers then the representation (2.2) implies (2.1) almost immediately. Note that $\min I > 1/q$ implies that $|x_0|+\cdots+|x_{q-1}| > 1$. □

Corollary 2.3. — *Set*

$$P_N(z_1,\ldots,z_K,y) = \sum_{n<N}\prod_{k=1}^{K} z_k^{L_k(|n|_0,\ldots,|n|_{q-1})}y^n.$$

Then we have

$$P_N(z_1,\ldots,z_K,y) = S_N\left(\prod_{k=1}^K z_k^{a_{k,0}},\ldots,\prod_{k=1}^K z_k^{a_{k,K}},y\right).$$

Consequently, there exists a function $H(z_1,\ldots,z_k,t)$ *that is analytic in* $z_1,\ldots,z_k$ *(if they are sufficiently close to the positive real axis) and continuous and periodic in* t *(with period* 1*) such that*

$$P_N(z_1,\ldots,z_K,1) = H(z_1,\ldots,z_k,\log_q N)\cdot F(z_1,\ldots,z_k)^{\log_q N} - \frac{F(z_1,\ldots,z_k) - \prod_{k=1}^K z_k^{a_{k,0}}}{F(z_1,\ldots,z_k)-1}, \tag{2.3}$$

where we assume that

$$F(z_1,\ldots,z_k) = \sum_{\ell=0}^{q-1}\prod_{k=1}^K z_k^{a_{k,\ell}} \neq 1.$$

Furthermore, if J *is any closed interval of positive real numbers with the property that* $F(|z_1|,\ldots,|z_K|) > 1$ *for all* z_k *with* $|z_k|\in J$ $(1\leqslant k\leqslant K)$*. Then, for every* $\varepsilon>0$ *such that there is at least one* k *with* $|\arg(z_k)|\geqslant\varepsilon$*, there exists* $\delta>0$ *and* $C>0$ *such that*

$$|P_N(z_1,\ldots,z_K,1)| \leqslant C\cdot F(|z_1|,\ldots,|z_k|)^{(1-\eta)\log_q N} \tag{2.4}$$

uniformly for all z_k *with* $|z_k|\in J$*.*

Proof. — We just have to note that if we set $x_\ell = \prod_{k=1}^K z_k^{a_{k,\ell}}$ then we obtain

$$\prod_{\ell=0}^{q-1} x_\ell^{|n|_\ell} = \prod_{k=1}^K z_k^{\sum_{\ell=0}^{q-1} a_{k,\ell}|n|_\ell}$$

and can apply Corollary 2.2. In particular note that Definition 1.1.(ii) implies that (2.1) translates to (2.4). □

In what follows we will make the assumption that

$$a_{k,0} = 0 \qquad (1\leqslant k\leqslant K). \tag{2.5}$$

This implies that x_0 in $S_N(x_0,\ldots,x_{q-1})$ is substituted by $\prod_{k=1}^K z_k^{a_{k,0}} = 1$. Hence, $F(z_1,\ldots,z_K)$ is of the form

$$F(z_1,\ldots,z_K) = 1 + \sum_{\ell=1}^{q-1}\prod_{k=1}^K z_k^{a_{k,\ell}}.$$

In particular, we always have

$$F(z_1,\ldots,z_K) > 1$$

for all positive real numbers $z_1,\ldots,z_K$.

The assumption (2.5) is no real restriction. If we start with the general linear forms

$$L_k(x_0, x_1, \ldots, x_{q-1}) = \sum_{\ell=0}^{q-1} a_{k\ell} x_\ell,$$

then the slightly modified linear forms

$$\overline{L}_k(x_0, x_1, \ldots, x_{q-1}) = \sum_{\ell=0}^{q-1} (a_{k,\ell} - a_{k,0}) x_\ell = \sum_{\ell=1}^{q-1} (a_{k,\ell} - a_{k,0}) x_\ell$$

have the property that the corresponding coefficients $\overline{a}_{k,\ell} = a_{k,\ell} - a_{k,0}$ satisfy $\overline{a}_{k,0} = 0$ and the condition (1.1) translates to

$$\begin{aligned} \overline{L}_k(|n|_0, |n|_1, \ldots, |n|_{q-1}) &= L_k(|n|_0, |n|_1, \ldots, |n|_{q-1}) - a_{k,0}[\log_q n] \\ &= [\eta_k \log_q n] + \mu_k - a_{k,0}[\log_q n] \\ &= [(\eta_k - a_{k,0}) \log_q n] + \mu_k + O(1), \end{aligned} \tag{2.6}$$

where the $O(1)$-term depends on n and k. This means that if we replace the linear forms L_k by $\overline{L}_k$ then (1.1) is replaces by (2.6) that is almost of the same form. In fact, the following calculations could be worked out, too, by using (2.6) instead of (1.1). However, in this case it would be necessary to keep track of k and n which would make notations even more involved. Therefore we have decided to work with (1.1) and, of course, with (2.5).

3. Estimate of $\mathrm{card}\{n \in \mathcal{N} : n < N\}$: Saddle Point Approximations

Our first goal is to give a precise estimate for the number

$$\mathrm{card}\{n \in \mathcal{N} : n < N\},$$

that is, to prove Theorem 1.6. For this purpose, for every integral multiindex $\mathbf{m} = (m_1, \ldots, m_K)$ we consider the sets

$$V_{\mathbf{m}}(N) = \{n < N : L_k(|n|_0, \ldots, |n|_{q-1}) = m_k,\ 1 \leqslant k \leqslant K\}$$

and their cardinalities $\mathrm{card} V_{\mathbf{m}}(N)$. With help of the generating function $P(z_1, \ldots, z_K, 1)$ we can obtain these numbers by the use of K-fold Cauchy integration:

$$\begin{aligned} \mathrm{card} V_{\mathbf{m}}(N) &= \mathrm{card}\{n < N : L_k(|n|_0, \ldots, |n|_{q-1}) = m_k,\ 1 \leqslant k \leqslant K\} \\ &= \frac{1}{(2\pi i)^K} \int_{\gamma_1} \cdots \int_{\gamma_K} P(z_1, \ldots, z_K, 1) \frac{dz_1}{z_1^{m_1+1}} \cdots \frac{dz_K}{z_K^{m_K+1}}. \end{aligned} \tag{3.1}$$

Since $P(z_1,\ldots,z_K,1)$ can be well approximated by a power $F(z_1,\ldots,z_K)^{\log_q N}$ it is natural to do this with help of a multivariate saddle point method.

We start with a preliminary lemma.

LEMMA 3.1. — *Suppose that the system of equations*

$$\sum_{\ell=0}^{q-1} a_{k\ell}x_\ell = \eta_k \qquad (1 \leqslant k \leqslant K), \tag{3.2}$$

$$\sum_{\ell=0}^{q-1} x_\ell = 1 \tag{3.3}$$

has a positive solution $x_0 > 0, x_1 > 0, \ldots, x_{q-1} > 0$*. Then there uniquely exist* $z_1 > 0, \ldots, z_K > 0$ *with*

$$\sum_{\ell=0}^{q-1} a_{k\ell} \prod_{r=1}^{K} z_r^{a_{r\ell}} = \eta_k \sum_{\ell=0}^{q-1} \prod_{r=1}^{K} z_r^{a_{r\ell}} \qquad (1 \leqslant k \leqslant K). \tag{3.4}$$

Proof. — Let Z denote the set of solution $(x_0,\ldots,x_{q-1})$ of (3.4) and (3.3) with positive coordinates. By assumption Z is not empty, in particular, it either consists of exactly one point (if $q = K+1$) or it is the intersection of a $(q-K-1)$-dimensional hyperplane with the half spaces $x_j > 0$, and, thus, can be considered as an open set in a $(q-K-1)$-dimensional space. Next consider the function

$$f(x_0,\ldots,x_{q-1}) = -\sum_{\ell=0}^{q-1} x_j \log x_j, \quad (x_0,\ldots,x_{q-1}) \in Z.$$

Observe that f is a strictly concave positive function with unbounded derivative if one of the x_j goes to 0. Hence f attains its (only) maximum at some point $(x_0^\circ,\ldots,x_{q-1}^\circ) \in Z$.

Alternatively, this maximum can be calculated with help of Lagrange multipliers. Set

$$\tilde f(x_0,\ldots,x_{q-1},\lambda_0,\ldots,\lambda_K) = -\sum_{\ell=0}^{q-1} x_j \log x_j + \lambda_0 \left(\sum_{\ell=0}^{q-1} x_\ell - 1\right) + \sum_{k=1}^{K} \lambda_k \left(\sum_{\ell=0}^{q-1} a_{k,\ell}x_\ell - \eta_k\right).$$

Then by Lagrange's theorem there exists $\lambda_0^\circ,\ldots,\lambda_K^\circ$ such that $(x_0^\circ,\ldots,x_{q-1}^\circ)$ satisfies the system of equations

$$\frac{\partial \tilde f}{\partial x_j} = -\log x_j^\circ + 1 + \lambda_0^\circ + \sum_{k=1}^{K} \lambda_k^\circ a_{k,j} = 0 \qquad (0 \leqslant j < q).$$

Hence, if we set $z_k = e^{\lambda_k^\circ}$, we have

$$x_j^\circ = e^{1+\lambda_0^\circ} \prod_{k=1}^{K} z_j^{a_{k,j}},$$

and since (3.4) and (3.3) imply that

$$\sum_{\ell=0}^{q-1} a_{k\ell} x_\ell^\circ = \eta_k \sum_{\ell=0}^{q-1} x_\ell^\circ,$$

it directly follows that (3.4) is satisfied for $z_k = e^{\lambda_k^\circ}$. This is also the unique solution since every solution of (3.4) can be reinterpreted as a maximum of f on Z. □

In what follows, we will denote by Ω the (open) set of $(\eta_1, \ldots, \eta_K)$ for which (3.4) has a unique solution $z_k(\eta_1, \ldots, \eta_K)$ $(1 \leqslant k \leqslant K)$ in the above sense. In fact, this is also a multivariate saddle point as the proof of the following theorem shows.

Recall that we always assume that $a_{k,0} = 0$, which implies that $F(z_1, \ldots, z_K) > 1$ for all positive real numbers $z_1, \ldots, z_K$.

THEOREM 3.2. — *Suppose that E is a compact subset of Ω. Then uniformly for all integer vectors $\mathbf{m} = (m_0, \ldots, m_{q-1}) \in \mathbb{Z}^q$ with*

$$\left(\frac{m_0}{\log_q N}, \ldots, \frac{m_{q-1}}{\log_q N} \right) \in E$$

and as $N \to \infty$ we have

(3.5)
$$\mathrm{card} V_{\mathbf{m}}(N) = \frac{H(\tilde{z}_1, \ldots, \tilde{z}_K, \log_q N)}{(2\pi \log_q N)^{K/2} \tilde{\Delta}^{1/2}} F\left(\tilde{z}_1, \ldots, \tilde{z}_K\right)^{\log_q N} \tilde{z}_1^{-m_1} \cdots \tilde{z}_K^{-m_K} \times \left(1 + O\left(\frac{1}{\log N} \right) \right)$$

where

$$\tilde{z}_k = z_k \left(\frac{m_0}{\log_q N}, \ldots, \frac{m_{q-1}}{\log_q N} \right) \qquad (1 \leqslant k \leqslant K)$$

and

$$\tilde{\Delta} = \det \left(\left. \frac{\partial^2 \log F(\tilde{z}_1 e^{t_1}, \ldots, \tilde{z}_K e^{t_K})}{\partial t_i \partial t_j} \right|_{t_1=0,\ldots,t_K=0} \right)_{1 \leqslant i,j \leqslant K}.$$

Proof. — Our starting point is the representation (3.1), where we will use the circles of integration

$$\gamma_k = \{ z_k : |z_k| = \tilde{z}_k \} \qquad (1 \leqslant k \leqslant K).$$

Due to the upper bound (2.4) we thus get an upper bound for those parts of the integral where $|\arg(z_k)| \geqslant \varepsilon$ (for some k) of the form

$$C \cdot F(\tilde{z}_1, \ldots, \tilde{z}_K)^{(1-\eta)\log_q N}.$$

Hence, these parts of the integral can be neglected.

For the remaining parts we use standard saddle point approximation on powers of functions (see [8]). Note that $(\tilde{z}_1, \ldots, \tilde{z}_K)$ is the saddle point of the function

$$\begin{aligned}(z_1, \ldots, z_K) \ &\mapsto F(z_1, \ldots, z_K)^{\log_q N} z_1^{-m_1} \cdots z_K^{-m_K} \\ &= \exp\left(\log_q N \, \log\left(F(z_1, \ldots, z_K)\right) - \sum_{k=1}^{K} m_k \log z_k\right).\end{aligned}$$

Hence, we directly obtain (3.5). □

Remark 3.3. — Theorem 3.2 has a slight extension. We also have

(3.6)

$$\begin{aligned}\mathrm{card} V_{\mathbf{m}}(N) = \frac{H(\tilde{z}_1, \ldots, \tilde{z}_K, \log_q N)}{(2\pi \log_q N)^{K/2} \tilde{\Delta}^{1/2}} \, F\left(\tilde{z}_1, \ldots, \tilde{z}_K\right)^{\log_q N} \tilde{z}_1^{-m_1} \cdots \tilde{z}_K^{-m_K} \\ \times \left(1 + O\left(\frac{1}{\log N}\right)\right)\end{aligned}$$

where

$$\tilde{z}_k = z_k\left(\eta_1, \ldots, \eta_K\right) \qquad (1 \leqslant k \leqslant K)$$

and $m_k - \eta_k \log_q N = O(1)$. This means that we can vary m_k a little bit without changing the saddle points $\tilde{z}_k$, that only depends on $\eta_1, \ldots, \eta_K$. This property will be frequently used in the sequel.

Remark 3.4. — If we do not use the saddle point $(\tilde{z}_1, \ldots, \tilde{z}_K)$ but any point $(\zeta_1, \ldots, \zeta_K)$ of positive real numbers we get an upper bound of the form

(3.7)

$$\begin{aligned}\mathrm{card} V_{\mathbf{m}}(N) \leqslant \frac{H(\zeta_1, \ldots, \zeta_K, \log_q N)}{(2\pi \log_q N)^{K/2} \Delta^{1/2}} \, F\left(\zeta_1, \ldots, \zeta_K\right)^{\log_q N} \zeta_1^{-m_1} \cdots \zeta_K^{-m_K} \\ \times \left(1 + O\left(\frac{1}{\log N}\right)\right),\end{aligned}$$

with

$$\Delta = \det\left(\left.\frac{\partial^2 \log F(\zeta_1 e^{t_1}, \ldots, \zeta_K e^{t_K})}{\partial t_i \partial t_j}\right|_{t_1=0,\ldots,t_K=0}\right)_{1\leqslant i,j\leqslant K}.$$

This follows from the fact that the absolute value of $F(\zeta_1 e^{it_1},\ldots,\zeta_K e^{it_K})$ can be estimated by
(3.8)
$$\left|F(\zeta_1 e^{it_1},\ldots,\zeta_K e^{it_K})\right| \leqslant F(\zeta_1,\ldots,\zeta_K)\exp\left(-\frac12\sum_{i,j=1}^K \Delta_{ij}t_jt_j + O\left(\sum_{i=1}^K |t_i|^3\right)\right),$$
where
$$\Delta_{ij} = \left.\frac{\partial^2 \log F(\zeta_1 e^{t_1},\ldots,\zeta_K e^{t_K})}{\partial t_i\partial t_j}\right|_{t_1=0,\ldots,t_K=0}.$$
Of course, the constant implied by the term $O(1/\log N)$ depends (continuously) on $\zeta_1,\ldots,\zeta_K$.

The case $\eta_1=\cdots=\eta_K=0$ is now easy to deal with. The corresponding asympototic formula for the numbers $\mathrm{card}\{n\in\mathcal{N} : n<N\}$ is an immediate corollary of the above remark.

Corollary 3.5. — *Suppose that $\eta_1=\cdots=\eta_K=0$ and let $\mu_1,\ldots,\mu_K$ be given (fixed) integers. Then $(0,\ldots,0)\in\Omega$ and we have*
$$\begin{aligned}\mathrm{card}\{n\in\mathcal{N} : n<N\} &= \mathrm{card}\{n<N : L_k(|n|_0,\ldots,|n|_{q-1})=\mu_k,\ 1\leqslant k\leqslant K\}\\ &= \frac{H(\tilde z_1,\ldots,\tilde z_K,\log_q N)}{(2\pi\log_q N)^{K/2}\Delta^{1/2}}\, F(\tilde z_1,\ldots,\tilde z_K)^{\log_q N}\,\tilde z_1^{-\mu_1}\cdots\tilde z_K^{-\mu_k}\left(1+O\left(\frac{1}{\log N}\right)\right),\end{aligned}$$
where $\tilde z_k = z_k(0,\ldots,0)>0$ satisfy
$$\sum_{\ell=0}^{q-1} a_{k\ell}\prod_{r=1}^K \tilde z_r^{a_{r\ell}} = 0, \qquad (1\leqslant k\leqslant K).$$

The next step is a little bit more involved. Suppose that there exist k with $\eta_k>0$ and consider the set
$$S = \bigcup_{k\,:\,\eta_k\neq 0}\left\{q^{(m-\mu_k)/\eta_k} : m\in\mathbb{Z}, (m-\mu_k)/\eta_k>0\right\}$$
that is the union of geometric sequences.

Let $s_0<s_1<\cdots$ be a an ordered version of the elements of $S=\{s_0,s_1,\ldots\}$. Observe that if $n>0$ is an integer with $s_j\leqslant n<s_{j+1}$ then for all k with $\eta_k>0$ there exists $m_{j,k}\in\mathbb{Z}$ with
$$q^{\frac{m_{j,k}-\mu_k}{\eta_k}} \leqslant n < q^{\frac{m_{j,k}+1-\mu_k}{\eta_k}}.$$
For those k with $\eta_k=0$ we set $m_{j,k}=\mu_k$. If fact, this means that for all $k\in\{1,2,\ldots,K\}$ and $n\in\{s_j,s_j+1,\ldots,s_{j+1}-1\}$ we have
$$[\eta_k\log_q n]+\mu_k = m_{j,k}.$$

Let $\mathbf{m}_j = (m_{j,1}, \ldots, m_{j,K})$ denote the multiindex that collects these $m_{j,k}$. More precisely this shows that

$$\mathrm{card}\{n \in \mathcal{N} : s_j \leqslant n < s_{j+1}\} = \mathrm{card} V_{\mathbf{m}_j}(s_{j+1}) - \mathrm{card} V_{\mathbf{m}_j}(s_j).$$

Thus, we have proved the following lemma.

Lemma 3.6. — *Assume that there exists k with $\eta_k > 0$ and suppose that N is a positive integer with $N = [s_J]$ for some $s_J \in S$. Then we have*

$$\mathrm{card}\{n \in \mathcal{N} : n < N\} = \sum_{j<J} \left(\mathrm{card} V_{\mathbf{m}_j}(s_{j+1}) - \mathrm{card} V_{\mathbf{m}_j}(s_j)\right).$$

Proof of Theorem 1.6

Lemma 3.6 can be used to determine the asymptotic order of magnitude of the numbers $\mathrm{card}\{n \in \mathcal{N} : n < N\}$. We will actually prove that there are two positive constants C_1, C_2 with

$$\frac{C_1}{(\log_q N)^{K/2}} \left(\frac{F(\tilde{z}_1, \ldots, \tilde{z}_K)}{\tilde{z}_1^{\eta_1} \cdots \tilde{z}_K^{\eta_K}} \right)^{\log_q N} \leqslant \mathrm{card}\{n \in \mathcal{N} : n < N\} \leqslant \frac{C_2}{(\log_q N)^{K/2}} \left(\frac{F(\tilde{z}_1, \ldots, \tilde{z}_K)}{\tilde{z}_1^{\eta_1} \cdots \tilde{z}_K^{\eta_K}} \right)^{\log_q N},$$

where

$$\tilde{z}_k = z_k(\eta_1, \ldots, \eta_K) \qquad (1 \leqslant k \leqslant K).$$

Thus, γ from Theorem 1.6 is explicitly given by

$$\gamma = \log_q \left(\frac{F(\tilde{z}_1, \ldots, \tilde{z}_K)}{\tilde{z}_1^{\eta_1} \cdots \tilde{z}_K^{\eta_K}} \right).$$

Proof. — If $\eta_1 = \ldots = \eta_K = 0$ then this estimate follows from Corollary 3.5.

If there is k with $\eta_k > 0$ then by Lemma 3.6 and (3.6) we get the upper bound:

$$\begin{aligned}
\mathrm{card}\{n \in \mathcal{N} : n < N\} &\leqslant \sum_{j : s_{j-1} < N} \mathrm{card}\left(V_{\mathbf{m}_j}(s_{j+1})\right) \\
&\leqslant \sum_{k:\eta_k>0} \sum_{m \leqslant \eta_k \log_q N + \mu_k} \mathrm{card}\left(V_{([(m-\mu_k)\eta_\ell/\eta_k]+\mu_\ell)_{1\leqslant \ell \leqslant K}}(q^{(m-\mu_k)/\eta_k})\right) \\
&\ll \sum_{k:\eta_k>0} \sum_{m \leqslant \eta_k \log_q N + \mu_k} \frac{1}{m^{K/2}} \left(\frac{F(\tilde{z}_1, \ldots, \tilde{z}_K)}{\tilde{z}_1^{\eta_1} \cdots \tilde{z}_K^{\eta_K}} \right)^{(m-\mu_k)/\eta_k} \\
&\ll \frac{1}{(\log_q N)^{K/2}} \left(\frac{F(\tilde{z}_1, \ldots, \tilde{z}_K)}{\tilde{z}_1^{\eta_1} \cdots \tilde{z}_K^{\eta_K}} \right)^{\log_q N}.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}\mathrm{card}\{n \in \mathcal{N} : n < N\} \geqslant \mathrm{card}\big(V_{\mathbf{m}_j}(s_{j+1})\big) - \mathrm{card}\big(V_{\mathbf{m}_j}(s_j)\big) \\ + \mathrm{card}\big(V_{\mathbf{m}_{j+1}}(s_{j+2})\big) - \mathrm{card}\big(V_{\mathbf{m}_{j+1}}(s_{j+1})\big)\end{aligned}$$

for every j with $s_{j+2} < N$. Let k_0 be chosen such that η_{k_0} is largest. By obvious reasoning there exists a constant C such that $s_j = q^{(m-\mu_{k_0})/\eta_{k_0}}$, $s_{j+1} = q^{(m+1-\mu_{k_0})/\eta_{k_0}}$, and $s_j \geqslant N/C$. Further we have $m_{j,k} = [(m - \mu_{k_0})\eta_k/\eta_{k_0}] + \mu_k$. Hence we can use the saddle point $\tilde{z}_k = z_k(\eta_1, \dots, \eta_K)$ $(1 \leqslant k \leqslant K)$ and obtain by (3.6):

$$\begin{aligned}&\mathrm{card}\big(V_{\mathbf{m}_j}(s_{j+1})\big) - \mathrm{card}\big(V_{\mathbf{m}_j}(s_j)\big) \\ &\quad = \frac{H(\tilde{z}_1, \dots, \tilde{z}_K, \log_q s_{j+1})}{(2\pi \log_q s_{j+1})^{K/2}\tilde{\Delta}^{1/2}} \frac{F(\tilde{z}_1, \dots, \tilde{z}_K)^{\log_q s_{j+1}}}{\tilde{z}_1^{m_1} \cdots \tilde{z}_K^{m_K}} \left(1 + O\left(\frac{1}{\log N}\right)\right) \\ &\quad - \frac{H(\tilde{z}_1, \dots, \tilde{z}_K, \log_q s_j)}{(2\pi \log_q s_j)^{K/2}\tilde{\Delta}^{1/2}} \frac{F(\tilde{z}_1, \dots, \tilde{z}_K)^{\log_q s_j}}{\tilde{z}_1^{m_1} \cdots \tilde{z}_K^{m_K}} \left(1 + O\left(\frac{1}{\log N}\right)\right) \\ &\quad \gg \frac{1}{(\log_q N)^{K/2}} \left(\frac{F(\tilde{z}_1, \dots, \tilde{z}_K)}{\tilde{z}_1^{\eta_1} \cdots \tilde{z}_K^{\eta_K}}\right)^{\log_q N} \\ &\qquad \times \Big(H(\tilde{z}_1, \dots, \tilde{z}_K, \log_q s_{j+1})\, F(\tilde{z}_1, \dots, \tilde{z}_K)^{\log_q s_{j+1} - \log_q s_j} \\ &\qquad\qquad - H(\tilde{z}_1, \dots, \tilde{z}_K, \log_q s_j) + O\Big(\frac{1}{\log N}\Big)\Big).\end{aligned}$$

Similarly we get

$$\begin{aligned}&\mathrm{card}\big(V_{\mathbf{m}_{j+1}}(s_{j+2})\big) - \mathrm{card}\big(V_{\mathbf{m}_{j+1}}(s_{j+1})\big) \\ &\qquad \gg \frac{1}{(\log_q N)^{K/2}} \left(\frac{F(\tilde{z}_1, \dots, \tilde{z}_K)}{\tilde{z}_1^{\eta_1} \cdots \tilde{z}_K^{\eta_K}}\right)^{\log_q N} \\ &\qquad\quad \times \Big(H(\tilde{z}_1, \dots, \tilde{z}_K, \log_q s_{j+2})\, F(\tilde{z}_1, \dots, \tilde{z}_K)^{\log_q s_{j+2} - \log_q s_{j+1}} \\ &\qquad\qquad - H(\tilde{z}_1, \dots, \tilde{z}_K, \log_q s_{j+1}) + O\Big(\frac{1}{\log N}\Big)\Big).\end{aligned}$$

Since $\mathrm{card}\big(V_{\mathbf{m}_j}(s_{j+1})\big) - \mathrm{card}\big(V_{\mathbf{m}_j}(s_j)\big) \geqslant 0$ and $\mathrm{card}\big(V_{\mathbf{m}_{j-1}}(s_{j+2})\big) - \mathrm{card}\big(V_{\mathbf{m}_{j+1}}(s_{j+1})\big) \geqslant 0$, it follows that

$$\begin{aligned}H(\tilde{z}_1, \dots, \tilde{z}_K, \log_q s_{j+2})\, F(\tilde{z}_1, \dots, \tilde{z}_K)^{\log_q s_{j+2} - \log_q s_{j+1}} - H(\tilde{z}_1, \dots, \tilde{z}_K, \log_q s_j) \\ \geqslant -\frac{C'}{\log N}\end{aligned}$$

for some constant $C' > 0$. Hence we get

$$\begin{aligned}\mathrm{card}\{n \in \mathcal{N} : n < N\} \gg \frac{1}{(\log_q N)^{K/2}} &\left(\frac{F(\tilde{z}_1, \ldots, \tilde{z}_K)}{\tilde{z}_1^{\eta_1} \cdots \tilde{z}_K^{\eta_K}}\right)^{\log_q N} \\ &\times \left(\left(F(\tilde{z}_1, \ldots, \tilde{z}_K)^{1/\eta_{k_0}} - 1\right) + O\left(\frac{1}{\log N}\right)\right) \\ \gg \frac{1}{(\log_q N)^{K/2}} &\left(\frac{F(\tilde{z}_1, \ldots, \tilde{z}_K)}{\tilde{z}_1^{\eta_1} \cdots \tilde{z}_K^{\eta_K}}\right)^{\log_q N}.\end{aligned}$$

This completes the proof of the Theorem 1.6. □

4. Uniform Distribution Modulo 1 of the Sequence $(\alpha n)_{n\in\mathcal{N}}$

As we remark at the end of section 1 proving Theorem 1.7 is equivalent to prove that for any irrational α we have

$$W = \sum_{n\in\mathcal{N}, n<N} e(\alpha n) = o\left(\mathrm{card}\{n \in \mathcal{N} : n < N\}\right).$$

For this purpose we introduce the function

$$U(x_0, \ldots, x_{q-1}; t_0, \ldots, t_{q-1}) = x_0 e(t_0) + \cdots + x_{q-1} e(t_{q-1}).$$

We will also use the short hand notation $U(\mathbf{x}, \mathbf{t})$.

Lemma 4.1. — *Suppose that $x_0, \ldots, x_{q-1}$ are positive real numbers Then there exists a constant $c > 0$ that depends continuously on $\mathbf{x} = (x_0, \ldots, x_{q-1})$ such that for all real vectors $\mathbf{t} = (t_0, \ldots, t_{q-1})$ and $\mathbf{t}_0 = (t_{0,0}, \ldots, t_{0,q-1})$*

$$|U(\mathbf{x}, \mathbf{t}) U(\mathbf{x}, \mathbf{t} + \mathbf{t}_0)| \leqslant U(\mathbf{x}, \mathbf{0})^2 \exp\left(-c \sum_{0\leqslant i<j<q} \|t_{0,i} - t_{0,j}\|^2\right),$$

where $\|x\| = \min_{k\in\mathbb{Z}} |x - k|$ denotes the distance to the nearest integer.

Proof. — We first consider $|U(\mathbf{x},\mathbf{t})|^2$. By using the inequality $|\sin(\pi t)| \geqslant \frac{2}{\pi}\|t\|$ we obtain

$$\begin{aligned}|U(\mathbf{x},\mathbf{t})|^2 &= U(\mathbf{x},\mathbf{t})U(\mathbf{x},-\mathbf{t})\\ &= \sum_{j=0}^{q-1} x_j^2 + 2\sum_{0\leqslant i<j<q} x_i x_j \cos(2\pi(t_i-t_j))\\ &= U(\mathbf{x},\mathbf{0})^2 - 4\sum_{0\leqslant i<j<q} x_i x_j \sin(\pi(t_i-t_j))^2\\ &\leqslant U(\mathbf{x},\mathbf{0})^2 - \frac{16}{\pi^2}\sum_{0\leqslant i<j<q} x_i x_j \|t_i-t_j\|^2\\ &\leqslant U(\mathbf{x},\mathbf{0})^2 \exp\left(-c_1\sum_{0\leqslant i<j<q}\|t_i-t_j\|^2\right),\end{aligned}$$

where

$$c_1 = \frac{16}{\pi^2}\,\frac{\min\limits_{0\leqslant i<j<q} x_i x_j}{U(\mathbf{x},\mathbf{0})^2}$$

is a positive constant depending continuously on $\mathbf{x}$. Since $\|t\|^2+\|t+t'\|^2 \geqslant \frac{1}{2}\|t'\|^2$ for any real numbers t and t' it immediately follows that

$$\begin{aligned}&|U(\mathbf{x},\mathbf{t})U(\mathbf{x},\mathbf{t}+\mathbf{t}_0)|\\ &\quad\leqslant U(\mathbf{x},\mathbf{0})^2\exp\left(-\frac{c_1}{2}\sum_{0\leqslant i<j<q}\left(\|t_i-t_j\|^2+\|t_i-t_j+t_{0,i}-t_{0,j}\|^2\right)\right)\\ &\quad\leqslant U(\mathbf{x},\mathbf{0})^2\exp\left(-\frac{c_1}{4}\sum_{0\leqslant i<j<q}\|t_{0,i}-t_{0,j}\|^2\right).\end{aligned}$$

This proves the lemma for $c = c_1/4$. □

Next we set

$$\overline{U}(z_1,\ldots,z_K;s_1,\ldots,s_K;s) :=$$
$$U\left(\prod_{k=0}^{K} z_k^{a_{k,0}},\ldots,\prod_{k=0}^{K} z_k^{a_{k,q-1}};\sum_{k=0}^{K} a_{k,0}s_k,\sum_{k=0}^{K} a_{k,1}s_k+s,\ldots,\sum_{k=0}^{K} a_{k,q-1}s_k+(q-1)s\right),$$

that is, we substitute $x_\ell = \prod_{k=0}^{K} z_k^{a_{k,\ell}}$ and $t_\ell = \sum_{k=0}^{K} a_{k,\ell}s_k + \ell s$ $(1\leqslant \ell \leqslant K)$.

Note that

$$\overline{U}(z_1,\ldots,z_K;s_1,\ldots,s_K;0) = F(z_1e(s_1),\ldots,z_ke(s_K)). \tag{4.1}$$

Furthermore, we have the following upper bound.

LEMMA 4.2. — *Suppose that* $x_0, \ldots, x_{q-1}$ *are positive real numbers and* $t_0, \ldots, t_{q-1}$ *and* α *are real numbers. Then there exists a constant* $C_3 > 0$ *(that depends continuously on* $x_0, \ldots, x_{q-1}$*) with*

$$|S_N(x_0 e^{it_1}, \ldots, x_{q-1}e^{it_{q-1}}, e^{i\alpha})| \leqslant C_3 \sum_{\ell \leqslant \log_q N} \left| \prod_{j<\ell} U(x_0, \ldots, x_{q-1}; t_0, t_1 + \alpha q^j, \ldots, t_{q-1} + (q-1)\alpha q^j) \right|.$$

Consequently, we have for positive real numbers $z_1, \ldots, z_K$ *and real numbers* $s_1, \ldots, s_K$ *and* α

$$|P_N(z_1 e^{is_1}, \ldots, z_K e^{is_K}, e^{i\alpha})| \leqslant C_3 \sum_{\ell \leqslant \log_q N} \left| \prod_{j<\ell} \overline{U}(z_1, \ldots, z_K; s_1, \ldots, s_K; \alpha q^j) \right|.$$

Proof. — The estimate for S_N follows immediately from the representations given in Lemma 2.1. The upper bound for P_N is just a rewritten version of the upper bound for S_N. □

This estimate shows that if we are interested in upper bounds for

$$W_{\mathbf{m}}(N) = \sum_{n \in V_{\mathbf{m}}(N)} e(\alpha n) = \int_0^1 \cdots \int_0^1 P_N(z_1 e^{is_1}, \ldots, z_K e^{is_K}, e^{i\alpha})\,(z_1 e^{is_1})^{-m_1} \cdots (z_K e^{is_K})^{-m_K}\, ds_1 \cdots ds_K,$$

then it is sufficient to get proper upper bounds for integrals of the form

$$\int_0^1 \cdots \int_0^1 \prod_{j<\nu} \left| \overline{U}(z_1, \ldots, z_K; s_1, \ldots, s_K; \alpha q^j) \right| \, ds_1 \cdots ds_K. \tag{4.2}$$

Following this idea we prove upper bounds for (4.2) in Lemma 4.3 and 4.5 which will lead to upper bounds for $W_{\mathbf{m}}(N)$ in Lemma 4.6.

We have the following estimates.

LEMMA 4.3. — *Suppose that* $z_1, \ldots, z_K$ *are positive real numbers. Then there exists a constant* $C_4 > 0$ *(that depends continuously on* $z_1, \ldots, z_K$*) such that for all integers* $\nu \geqslant 1$ *and all real numbers* α

$$\int_0^1 \cdots \int_0^1 \left| \overline{U}(z_1, \ldots, z_K; s_1, \ldots, s_K; \alpha) \right|^\nu \, ds_1 \cdots ds_K \leqslant \frac{C_4}{\nu^{K/2}} F(z_1, \ldots, z_K)^\nu.$$

Proof. — Observe that Lemma 4.1 also implies that

$$|U(\mathbf{x},\mathbf{t})| \leqslant U(\mathbf{x},\mathbf{0}) \exp\left(-c \sum_{0\leqslant i<j<q} \|t_i - t_j\|^2\right).$$

Hence, we also get

$$|\overline{U}(z_1,\dots,z_K;s_1,\dots,s_K;\alpha)|$$
$$\leqslant \overline{U}(z_1,\dots,z_K;0,\dots,0;0) \exp\left(-c \sum_{0\leqslant i<j<q} \left\|\sum_{k=1}^{K}(a_{k,i}-a_{k,j})s_k + (i-j)\alpha\right\|^2\right).$$

By the linear independence assumption on the forms L_k and by Definition 1.1.(ii) there exist $j_0 < j_1 < \cdots < j_K$ such the matrix $C = (a_{k,j_\ell} - a_{k,j_0})_{1\leqslant k,\ell\leqslant K}$ is regular and, thus, has determinant $\det C = d \neq 0$. Further, there exist δ_k with

$$\sum_{k=1}^{K}(a_{k,j_\ell} - a_{k,j_0})\delta_k = (j_\ell - j_0)\alpha \quad (1 \leqslant \ell \leqslant K).$$

Hence, there exit integers $d_{j\ell}$ with

$$d(s_j+\delta_j) = \sum_{\ell=1}^{K} d_{j\ell}\left(\sum_{k=1}^{K}(a_{k,j_\ell} - a_{k,j_0})(s_k+\delta_k)\right).$$

Hence, for all j we have

$$\begin{aligned}\|d(s_j+\delta_j)\|^2 &\ll \sum_{\ell=1}^{K}\left\|\sum_{k=1}^{K}(a_{k,j_\ell} - a_{k,j_0})(s_k+\delta_k)\right\|^2\\ &\leqslant \sum_{0\leqslant i<j<q}\left\|\sum_{k=1}^{K}(a_{k,i}-a_{k,j})s_k + (i-j)\alpha\right\|^2.\end{aligned}$$

Consequently, there exists a constant $c' > 0$ with

$$\begin{aligned}&|\overline{U}(z_1,\dots,z_K;s_1,\dots,s_K;\alpha)|\\ &\leqslant \overline{U}(z_1,\dots,z_K;0,\dots,0;0)\exp\left(-c'\sum_{k=1}^{K}\|d(s_k+\delta_k)\|^2\right)\\ &F(z_1,\dots,z_k)\exp\left(-c'\sum_{k=1}^{K}\|d(s_k+\delta_k)\|^2\right),\end{aligned}$$

so that

$$\begin{aligned}
&\int_0^1 \cdots \int_0^1 \left|\overline{U}(z_1,\ldots,z_K;s_1,\ldots,s_K;\alpha)\right|^\nu \, ds_1\cdots ds_K \\
&\quad\leqslant F(z_1,\ldots,z_k)^\nu \int_0^1 \cdots \int_0^1 \exp\left(-c'\nu \sum_{k=1}^K \|d(s_k+\delta_k)\|^2\right) ds_1\cdots ds_K \\
&\quad\leqslant F(z_1,\ldots,z_k)^\nu \left(\int_0^1 e^{-c'\nu\|ds\|^2}\, ds\right)^K \\
&\quad\leqslant \frac{C_4}{\nu^{K/2}} F(z_1,\ldots,z_k)^\nu.
\end{aligned}$$

This completes the proof of the lemma. □

Remark 4.4. — Alternatively we can prove Lemma 4.3 by using property (4.1) and previous estimates for F. Namely, by using (3.8) for $|s_j|\leqslant\varepsilon$ (where $\varepsilon>0$ is chosen sufficiently small) and the property that

$$|F(z_1e(s_1),\ldots,z_ke(s_K))| \leqslant F(z_1,\ldots,z_K)^{1-\eta}$$

for some $\eta>0$ if there is some j with $|s_j|\geqslant\varepsilon$ (compare with (2.4)) the upper bound follows.

The next lemma is crucial for proving upper bound on Weyl sums.

LEMMA 4.5. — *Suppose that $z_1,\ldots,z_K$ are positive real numbers. Then there exists a constant $C_5>0$ (that depends continuously on $z_1,\ldots,z_K$) such that for all integers $\nu\geqslant1$ and all real number α*

$$\begin{aligned}
&\int_0^1 \cdots \int_0^1 \prod_{j<\nu} \left|\overline{U}(z_1,\ldots,z_K;s_1,\ldots,s_K;\alpha q^j)\right| \, ds_1\cdots ds_K \\
&\quad\leqslant \frac{C_5}{\nu^{K/2}} F(z_1,\ldots,z_K)^\nu \exp\left(-\frac{c}{4}\sum_{j<\nu}\|\alpha(q-1)q^j\|^2\right).
\end{aligned}$$

Proof. — For simplicity we assume that ν is a multiple of 4. The other cases can be handled in the same way.

We split the product of the integrand into two part:

$$\begin{aligned}
&\prod_{j<\nu,\, j\equiv 0 \bmod 4} \left|\overline{U}(z_1,\ldots,z_K;s_1,\ldots,s_K;\alpha q^j)\overline{U}(z_1,\ldots,z_K;s_1,\ldots,s_K;\alpha q^{j+1})\right| \\
&\qquad\times \prod_{j<\nu,\, j\equiv 2,3 \bmod 4} \left|\overline{U}(z_1,\ldots,z_K;s_1,\ldots,s_K;\alpha q^j)\right|.
\end{aligned}$$

By applying Lemma 4.1 with $t_{0,\ell} = \ell(q-1)q^j\alpha$ we get

$$\prod_{j<\nu, j\equiv 0 \bmod 4} \left|\overline{U}(z_1,\ldots,z_K;s_1,\ldots,s_K;\alpha q^j)\overline{U}(z_1,\ldots,z_K;s_1,\ldots,s_K;\alpha q^{j+1})\right|$$
$$\leqslant F(z_1,\ldots,z_K)^{\nu/2}\exp\left(-c\sum_{j<\nu,j\equiv 0 \bmod 4}\ \sum_{0<i<j\leqslant q}\|(i-j)(q-1)q^j\alpha\|^2\right)$$
$$\leqslant F(z_1,\ldots,z_K)^{\nu/2}\exp\left(-c\sum_{j<\nu,j\equiv 0 \bmod 4}\|(q-1)q^j\alpha\|^2\right).$$

Furthermore, by applying the inequality

$$|v_1\cdots v_m| \leqslant \frac{|v_1|^m+\cdots+|v_1|^m}{m},$$

we obtain (by applying Lemma 4.3):

$$\int_0^1\cdots\int_0^1 \prod_{j<\nu,j\equiv 2,3 \bmod 4}\left|\overline{U}(z_1,\ldots,z_K;s_1,\ldots,s_K;\alpha q^j)\right|\,ds_1\cdots ds_K$$
$$\leqslant \frac{1}{\nu/2}\sum_{j<\nu,j\equiv 2,3 \bmod 4}\int_0^1\cdots\int_0^1\left|\overline{U}(z_1,\ldots,z_K;s_1,\ldots,s_K;\alpha q^j)\right|^{\nu/2}\,ds_1\cdots ds_K$$
$$\ll \frac{1}{\nu^{K/2}}F(z_1,\ldots,z_K)^{\nu/2}.$$

Hence,

$$\int_0^1\cdots\int_0^1\prod_{j<\nu}\left|\overline{U}(z_1,\ldots,z_K;s_1,\ldots,s_K;\alpha q^j)\right|\,ds_1\cdots ds_K$$
$$\leqslant \frac{C_5}{\nu^{K/2}}F(z_1,\ldots,z_K)^{\nu}\exp\left(-c\sum_{j<\nu,j\equiv 0 \bmod 4}\|\alpha(q-1)q^j\|^2\right).$$

Similarly we can deal with the other residue classes modulo 4. Combining these 4 estimates finally proves the lemma. □

We set

$$E(\alpha,\nu) = \frac{1}{4}\sum_{j<\nu}\|\alpha(q-1)q^j\|^2.$$

For irrational α it is clear that $E(\alpha,\nu)\to\infty$ as $\nu\to\infty$ (compare with [12]).

Next we prove an upper bound for

$$W_{\mathbf{m}}(N) = \sum_{n\in V_{\mathbf{m}}(N)} e(\alpha n).$$

LEMMA 4.6. — *Suppose that* $m_k = [\eta_k \log_q N] + \mu_k$. *Then, there exists a constants* $C_6 > 0$ *and* $C_7 > 0$ *(that depends on* $\eta_1, \ldots, \eta_K$ *and on* $\mu_1, \ldots, \mu_K$*) such that as* $N \to \infty$

$$|W_{\mathbf{m}}(N)| \leqslant C_6 \cdot \mathrm{card}\big(V_{\mathbf{m}}(N)\big) \cdot \Big(e^{-c\,E(\alpha,[\log_q N]/2)} + e^{-C_7 \log_q N}\Big).$$

Proof. — We first fix positive numbers $z_1, \ldots, z_K$. Recall that $F(z_1, \ldots, z_K) > 1$. Hence, by Lemma 4.2 and Lemma 4.5 we have

$$\begin{aligned}
|W_{\mathbf{m}}(N)| &= \left| \sum_{n \in V_{\mathbf{m}}(N)} e(\alpha n) \right| \\
&= \left| \int_0^1 \cdots \int_0^1 P_N(z_1 e^{is_1}, \ldots, z_K e^{is_K}, e^{i\alpha})\, (z_1 e^{is_1})^{-m_1} \cdots (z_K e^{is_K})^{-m_K}\, ds_1 \cdots ds_K \right| \\
&\leqslant \int_0^1 \cdots \int_0^1 |P_N(z_1 e^{is_1}, \ldots, z_K e^{is_K}, e^{i\alpha})|\, z_1^{-m_1} \cdots z_K^{-m_K}\, ds_1 \cdots ds_K \\
&\ll \sum_{\ell \leqslant \log_q N} \int_0^1 \cdots \int_0^1 \prod_{j<\ell} \left|\overline{U}(z_1, \ldots, z_K; s_1, \ldots, s_K; \alpha q^j)\right|\, ds_1 \cdots ds_K \\
&\ll \sum_{\ell \leqslant \log_q N} \frac{1}{\ell^{K/2}} F(z_1, \ldots, z_K)^\ell z_1^{-m_1} \cdots z_K^{-m_K} \exp\left(-cE((q-1)\alpha, \ell)\right) \\
&\ll \frac{1}{(\log N)^{K/2}} F(z_1, \ldots, z_K)^{\log_q N} z_1^{-m_1} \cdots z_K^{-m_K} \\
&\qquad \cdot \exp\big(-cE((q-1)\alpha, [\log_q N]/2)\big) \\
&\quad + \frac{1}{(\log N)^{K/2}} F(z_1, \ldots, z_K)^{\frac12 \log_q N} z_1^{-m_1} \cdots z_K^{-m_K} \\
&\ll \frac{1}{(\log N)^{K/2}} F(z_1, \ldots, z_K)^{\log_q N} z_1^{-m_1} \cdots z_K^{-m_K} \\
&\qquad \cdot \Big(e^{-c\,E(\alpha,[\log_q N]/2)} + e^{-C_7 \log_q N}\Big),
\end{aligned}$$

where $C_7 > 0$ depends on $z_1, \ldots, z_K$. Now, if we choose $z_k = \tilde{z}_k = z_k(\eta_1, \ldots, \eta_K)$ we, thus, obtain the proposed estimate. □

Proof of Theorem 1.7

We are now ready to prove the final step of Theorem 1.7, i.e., that for all irrational numbers α we have, as $N \to \infty$,

$$\sum_{n\in\mathcal{N}, n\leqslant N} e(\alpha n) = o\big(\mathrm{card}\{n \in \mathcal{N} : n \leqslant N\}\big).$$

Proof. — First assume that $\eta_1 = \cdots = \eta_K = 0$. Here we have with $\mathbf{m} = (\mu_1, \ldots, \mu_K)$

$$W = \sum_{n\in\mathcal{N}, n<N} e(\alpha n) = W_{\mathbf{m}}(N)$$

and we can directly apply Lemma 4.6.

Now suppose that there exists k with $\eta_k > 0$. With the same reasoning as in Lemma 3.6 we have (if $N = [s_J]$ for some $s_J \in S$)

$$\begin{aligned} W &= \sum_{n\in\mathcal{N}, n<N} e(\alpha n) \\ &= \sum_{j\in J} \left(W_{\mathbf{m}_j}(s_{j+1}) - W_{\mathbf{m}_j}(s_j)\right) \end{aligned}$$

and consequently

$$|W| \leqslant \sum_{j<J} \left(|W_{\mathbf{m}_j}(s_{j+1})| + |W_{\mathbf{m}_j}(s_j)|\right).$$

Now, with help of Lemma 4.6 we get the upper bound

$$\begin{aligned} \sum_{j<J} \left|W_{\mathbf{m}_j}(s_{j+1})\right| &\leqslant \sum_{k:\eta_k>0} \sum_{m\leqslant \eta_k \log_q N+\mu_k} \left|W_{([(m-\mu_k)\eta_\ell/\eta_k]+\mu_\ell)_{1\leqslant\ell\leqslant K}}\left(q^{(m-\mu_k)/\eta_k}\right)\right| \\ &\ll \sum_{k:\eta_k>0} \sum_{m\leqslant \eta_k \log_q N+\mu_k} \frac{1}{m^{K/2}} \left(\frac{F(\tilde z_1, \ldots, \tilde z_K)}{\tilde z_1^{\eta_1}\cdots \tilde z_K^{\eta_K}}\right)^{(m-\mu_k)/\eta_k} \\ &\qquad \times \left(e^{-cE(\alpha,(m-\mu_k)/(2\eta_k))} + e^{-C_7 m/\eta_k}\right) \\ &= o\left(\frac{1}{(\log_q N)^{K/2}} \left(\frac{F(\tilde z_1, \ldots, \tilde z_K)}{\tilde z_1^{\eta_1}\cdots \tilde z_K^{\eta_K}}\right)^{\log_q N}\right) \\ &= o\left(\mathrm{card}\{n \in \mathcal{N} : n \leqslant N\}\right). \end{aligned}$$

Similarly we can estimate the second sum $\sum_{j<J} W_{\mathbf{m}_j}(s_j)$. This proves the lemma if $N = [s_J]$ for some $s_J \in S$. If N is not of that form we just have to add

$$W_{\mathbf{m}_J}(N) - W_{\mathbf{m}_J}(s_J)$$

which can be handled with help of Lemma 4.6. □

5. Generalizations

5.1. Missing digits

A first generalization of Theorem 1.7 is to assume that some digits $\mathcal{D} \subseteq \{0, 1 \ldots, q-1\}$ do not appear, that is, we additionally assume that

$$|n|_\ell = 0 \quad \text{for } \ell \in \mathcal{D} \tag{5.1}$$

(compare also with [19]). Formally this condition could be included into (1.1) without any change of notation. However, then there is no positive solution of the corresponding system of equations $L_k(x_0, \ldots, x_{q-1}) = \eta_k$ $(1 \leqslant k \leqslant K)$ since (5.1) forces $x_\ell = 0$ for all $\ell \in \mathcal{D}$. Nevertheless, we can work with in the missing-digit-case almost in the same way as above.

First, it is clear that the generating function

$$S_N^{\mathcal{D}}((x_j)_{j \notin \mathcal{D}}, y) = \sum_{n<N} \prod_{j \notin \mathcal{D}} x_j^{|n|_j} \, y^n$$

is just obtained by using $S_N(x_1, \ldots, x_N, y)$ and setting $x_\ell = 0$ for $\ell \in \mathcal{D}$. In particular, we directly use Lemma 2.1 and, hence, all subsequent considerations directly transfer.

After all we get precisely the same as Theorem 1.7. The only difference is that we have to consider linear forms in the *remaining variables* x_j, $j \notin \mathcal{D}$. More precisely we have to assume that the system $\mathcal{L}^{\mathcal{D}} = (L_k^{\mathcal{D}})_{k=1,\ldots,K}$

$$L_k^{\mathcal{D}} = \sum_{\ell \notin \mathcal{D}} a_{k,\ell} x_\ell$$

is complete (compare with Definition 1.1) and that the system

$$\sum_{\ell \notin \mathcal{D}} a_{k,\ell} x_\ell = \eta_k \qquad k = 1, \ldots, K$$
$$\sum_{\ell \notin \mathcal{D}} x_\ell = 1$$

has a positive solution. Then the sequence $(\alpha n)_{n \in \mathcal{N}}$ is uniformly distributed modulo 1, where $\mathcal{N}$ is the set of positive integers with $|n|_j = 0$ for $j \in \mathcal{D}$ and

$$\sum_{\ell \notin \mathcal{D}} a_{k,\ell} |n|_\ell = [\eta_k \log_q n] + \mu_k, \qquad k = 1, \ldots, K.$$

5.2. Non-integer coefficients

The restriction that the coefficients $a_{k,\ell}$ of the linear forms L_k are integers was natural in the context of Theorem 1.7. Nevertheless we can also consider general linear forms

$$L_k(x_0,\ldots,x_{q-1}) = \sum_{\ell=0}^{q-1} a_{k,\ell}x_\ell \qquad (1 \leqslant k \leqslant K)$$

and fix intervals $I_1 = [a_1,b_1],\ldots,I_K = [a_K,b_K]$ contained in the positive real line. A corresponding set $\mathcal{N}$ can be then defined by the set of non-negative integers n with

$$L_k(|n|_0,\ldots,|n|_{q-1}) - \eta_k \log_q n \in I_k \qquad (1 \leqslant k \leqslant K),$$

where $\eta_1,\ldots,\eta_K$ are given real numbers.

Instead of Cauchy's formula we can then use the inverse Laplace transform. For example, if we set (similarly to the above)

$$V_{\mathbf{m}}(N) = \{n < N : L_k(|n|_0,\ldots,|n|_{q-1}) - m_k \in I_k\},$$

where $\mathbf{m} = (m_1,\ldots,m_K)$ is any vector of real numbers, then we have for all real numbers $s_{0,1},\ldots,s_{0,K}$

$$\sum_{n\in V_{\mathbf{m}}(N)} y^n = \frac{1}{(2\pi i)^K} \lim_{T_1\to\infty} \int_{s_{0,1}-iT_1}^{s_{0,1}-iT_1} \cdots \lim_{T_K\to\infty} \int_{s_{0,K}-iT_K}^{s_{0,K}-iT_K}$$
$$\times P_N(e^{s_1},\ldots,e^{s_K},y)e^{-m_1s_1-\cdots-m_Ks_K} \prod_{k=1}^K \frac{e^{-a_ks_k}-e^{-b_ks_k}}{s_k}\, ds_1\cdots ds_K.$$

In particular, we can use $s_{0,k} = \log \tilde{z}_k$, where $\tilde{z}_k = z_k(m_1/\log_q N,\ldots,m_K/\log_q N)$ are the saddle points from above. Then these integrals can be asymptotically evaluated by a usual saddle point approximation, in particular if $y=1$ and also if $y = e(\alpha)$.

Of course, there are some technical difficulties that might occur. First note that the above integrals are not absolutely convergent. This is due to the factor $1/s_k$ of the Laplace transform

$$\int_{a_k}^{b_k} e^{-sx}\, dx = \frac{e^{-a_ks_k}-e^{-b_ks_k}}{s_k}.$$

As usual, this can be handled by *smoothing* the characteristic functions of the itervals $I_k = [a_k,b_k]$.

Second, if there are rational relations between the coefficients of the linear forms L_k then we have to deal with infinitely many saddle points on the lines $\Re(s_k) = s_{0,k}$. For example, if the coefficients $a_{k,\ell}$ and m_k are integers then

$$P_N(e^{s_1+2r_1\pi i}, \ldots, e^{s_K+2r_K\pi i}, y)e^{-m_1(s_1+2r_1\pi i)-\cdots-m_K(s_K+2r_K\pi i)}$$
$$= P_N(e^{s_1}, \ldots, e^{s_K}, y)e^{-m_1s_1-\cdots-m_Ks_K}$$

for all integers $r_1, \ldots, r_K$. However, it is possible to deal with all these kinds of problems. We again observe that the sequence $(\alpha n)_{n\in\mathcal{N}}$ is uniformly distributed modulo 1.

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Michael DRMOTA
Vienna University of Technology
Institute of Discrete Mathematics and Geometry
Wiedner Hauptstrasse 8-10
A-1040 Vienna (Austria)
michael.drmota@tuwien.ac.at
http://www.dmg.tuwien.ac.at/drmota/

Christian MAUDUIT
Université de la Méditerranée
Institut de Mathématiques de Luminy
CNRS UMR 6206
Campus de Luminy, Case 907
13288 Marseille Cedex 9 (France)
mauduit@iml.univ-mrs.fr