The Maximum Degree of Random Planar Graphs

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Abstract

Let P_n denote a graph drawn uniformly at random from the class of all simple planar graphs with n vertices. We show that the maximum degree of a vertex in P_n is with probability $1 - o(1)$ asymptotically equal to $c \log n$, where $c \approx 2.529$ is determined explicitly. A similar result is also true for random 2-connected planar graphs.

Our analysis combines two orthogonal methods that complement each other. First, in order to obtain the upper bound, we resort to exact methods, i.e., to *generating functions* and *analytic combinatorics*. This allows us to obtain fairly precise asymptotic estimates for the expected number of vertices of any given degree in P_n. On the other hand, for the lower bound we use *Boltzmann sampling techniques*. In particular, by tracing the execution of an adequate algorithm that generates a random planar graph, we are able to explicitly find vertices of sufficiently high degree in P_n.

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1 Introduction

A very active research area that requires a deep understanding of several models that generate graphs with structural side constraints is the analysis of social networks. It requires joint efforts from various fields, ranging from sociology over biology and economics all the way to theoretical computer science and discrete mathematics. While the notion of a 'social network' is not precisely defined in a mathematical sense and its properties may differ considerably depending on the application at hand, an underlying key feature is that a social network combines structural – i.e., deterministic – assumptions with randomness. It is this combination that makes their analysis very difficult to handle. In contrast to that, classical random graph theory, as introduced by Erdős and Rényi in the 50's, relies heavily on the assumption that edges appear independently. This assumption breaks down once we add structural constraints. There are various ways to achieve this, for example, by replacing 'global' randomness by 'local' randomness, as it is done in the popular preferential attachment model, cf. [1], and several variations thereof. Another way is to impose a probability distribution, usually the uniform distribution, on a class of graphs that fulfill some structural side constraints. A paradigmatic problem of the later type is the study of planar graphs – a graph class that seemingly misses quite strongly the key property of the Erdős-Rényi model (the independence of the edges) due to long range implications of the planarity condition. Indeed, planar graphs can be defined in terms of excluded minors, and this implies that we do not have any a priori non-trivial upper bounds on the size of the forbidden substructures. It is this problem that we consider in this paper.

The study of random planar graphs was initiated by Denise et al. [3]. McDiarmid et al. [11] showed that a random planar graph P_n in fact has some properties that are quite different compared to their analogues in the classical Erdős-Rényi random graph. They showed that the probability that P_n is connected is, for n tending to infinity, bounded away from 0 and from 1. In contrast, an Erdős-Rényi graph satisfies a 0-1 law for all "natural" properties. While the precise value of this probability is of course given by $\lim_{n\to\infty} |\mathcal{C}_n|/|\mathcal{P}_n|$, where $\mathcal{P}_n$ ($\mathcal{C}_n$) denotes the class of all labeled (connected) planar graphs with n vertices, it took quite a while and required deep methods from combinatorial counting and analytic combinatorics until Giménez and Noy [7], extending earlier work of Bender et al. [2], were able to determine the required values asymptotically. The next challenging problem became the question of the degree distribution. Another question that is extremely simple to answer for the Erdős-Rényi random graph model, but far from obvious for the random planar graph case. Over the last years two research groups independently developed completely different sets of methods and techniques for attacking this problem. On the one hand, Drmota et al. [4] considerably extended the methods from [7] and provided a solution using techniques from and within analytic combinatorics. On the other hand, Panagiotou et al. [12] extended the concept of Boltzmann samplers (originally introduced by Duchon et al. [5] for the uniform *generation* of objects), so that it can be used for analysing the *structure* of random planar graphs.

In this paper we combine our forces (and techniques) to solve the question of determining the value of the maximum degree in a random planar graph. Using combinatorial arguments Reed and McDiarmid [10] showed that the maximum degree is in the order $\Theta(\log n)$. In this paper we solve this question completely and determine the constant. I.e., we show that the maximum degree is $c \log n + O(\log \log n)$ for a constant $c > 0$ that we determine explicitly.

Actually, an additional and perhaps surprising aspect is that we use – and need – both techniques developed over the last years. The algorithmic approach of Boltzmann sampling allows us to show, by tracking the progress of the algorithm, that a vertex of a certain degree will be generated. In contrast, showing that a vertex of a certain degree will not be generated is a much harder task. However, within the analytic counting approach, the proof that a vertex of a certain degree does *not* exist corresponds to a *first moment* argument, an approach that is well known to be much easier compared to the corresponding second order arguments that would be needed for a proof of the existence of vertex with a certain property. We thus are in the fascinating situation that both methods nicely fit together and complement each other.

Let us now describe our results in more detail. Denote by C_n a graph that is drawn uniformly at random from the class $\mathcal{C}_n$ of all connected planar graphs with n vertices and with $\Delta(G)$ the maximum degree of a graph $G = (V, E)$.

Theorem 1.1. *There is a $c > 0$ such that the following is true. With probability $1 - o(1)$ we have $|\Delta(C_n) - c \log n| = O(\log\log n)$. In fact, $c = 1/\log(w_0) \approx 2.529$, where w_0 is given explicitly in (2.3).*

As it is well known that the largest component of a random planar graph P_n has size $n - O(1)$, cf. [7, 9, 11], the above theorem immediately implies the following corollary.

Corollary 1.2. *With probability $1 - o(1)$ we have $|\Delta(P_n) - c \log n| = O(\log\log n)$, where c is as in Theorem 1.1.*

Finally, we can show a similar result for 2-connected planar graphs. We denote by B_n a graph that is drawn uniformly at random from the class of all 2-connected planar graphs with n vertices.

Theorem 1.3. *The conclusion of Theorem 1.1 remains true if we replace C_n by B_n.*

In addition to proving our results we also use this paper as a gentle introduction to the methods and techniques mentioned above, with the hope that both methods can and will find more applications in each other's domain.

1.1 Basic Notation

Let $\mathcal{G}$ be a class of graphs. We denote by $\mathcal{G}_{n,m}$ the subset of $\mathcal{G}$ consisting of all graphs with n vertices and m edges, and we write $g_{n,m} = |\mathcal{G}_{n,m}|$. Moreover, we define $\mathcal{G}_n = \cup_{m\geq 0}\mathcal{G}_{n,m}$ and set $g_n = |\mathcal{G}_n|$. In particular, in the remainder of the paper we will write $\mathcal{C}$ for the class of connected planar graphs, $\mathcal{B}$ for the class of 2-connected planar graphs, and $\mathcal{T}$ for the class of 3-connected planar graphs.

We will frequently use the *pointing* and *derivative* operators, which are used to distinguish specific vertices or edges in the graphs that are contained in the class under consideration. First of all, given a class of graphs $\mathcal{G}$, we define $\mathcal{G}^\bullet = \bigcup_{n\geq 1}\{1, \ldots, n\} \times \mathcal{G}_n$ as the class of *vertex-rooted* graphs. In particular, every graph $G \in \mathcal{G}_n$ is contained n times in $\mathcal{G}_n$, where each copy contains a different distinguished vertex. Similarly, the *vertex-derived* class $\mathcal{G}'_{n-1,m}$ is obtained by removing the label n from every object in $\mathcal{G}_{n,m}$, such that the obtained graphs have $n - 1$ labeled vertices, i.e., vertex n can be considered as a distinguished vertex that does not contribute to the size. Consequently, there is a bijection between the classes $\mathcal{G}'_{n-1}$ and $\mathcal{G}_n$. We set $\mathcal{G}' := \bigcup_{n\geq 0}\mathcal{G}'_n$. It will also be necessary to distinguish edges. To this end, define $\mathcal{G}_e = \bigcup_{n,m\geq 1}\{1, \ldots, m\} \times \mathcal{G}_{n,m}$ as the class of *edge-rooted* graphs, which contains each graph in $\mathcal{G}$ a number of times equal to the number of edges. So, analogously to the case of vertex-rooted graphs, every graph in $\mathcal{G}_e$ has a specific distinguished edge. However, to simplify notation, for technical reasons we will assume that the marked edge does not contribute to the total number of edges in each graph in $\mathcal{G}_e$. In other words, we may think that this edge is removed, but its former endpoints are distinguished, such that the graph can be fully recovered.

The main parameter of study in this paper is the maximum degree of random planar graphs. Let $\mathcal{C}^\bullet_{n,m,k}$ be the class of vertex-rooted planar graphs with n vertices and m edges, such that the degree of the root-vertex is k. Define $\mathcal{B}^\bullet_{n,m,k}$ and $\mathcal{T}^\bullet_{n,m,k}$ similarly. Moreover, for $\mathcal{G} \in \{\mathcal{C}, \mathcal{B}, \mathcal{T}\}$ let

$$G^\bullet(x,y,w) = \sum_{n,m,k\geq 0} \frac{|\mathcal{G}^\bullet_{n,m,k}|}{n!} x^n y^m w^k$$

denote the *exponential generating function (egf)* enumerating the sequence $(|\mathcal{G}^\bullet_{n,m,k}|)_{n,m,k\geq 0}$. We shall omit any of the parameters x, y, w if the corresponding value is equal to one; for example, we write $G^\bullet(x) = G^\bullet(x,1,1)$. Similar to this notation, we will write $G'(x,y,w)$ for the egf enumerating $(|\mathcal{G}'_{n,m,k}|)_{n,m,k\geq 0}$, where $\mathcal{G}'_{n,m,k}$ contains all graphs in $\mathcal{G}'_{n,m}$, whose unlabeled vertex has degree k. Observe that $G^\bullet(x,y,w) = x\frac{\partial}{\partial x}G(x,y,w)$. Finally, note that $G_e(x,y,w) = \frac{\partial}{\partial y}G(x,y,w)$.

As a final remark, let us already mention at this point that all generating functions considered in this work have (at least) one finite dominant non-zero singularity on the real axis. For a generating function G enumerating a graph class $\mathcal{G}$ we will write ρ_G for this singularity.

2 The Lower Bound

Due to space limitations we present in this section only a self-contained proof for the lower bound claimed in Theorem 1.1. This proof highlights the most important ideas in our arguments. The proof of Theorem 1.3 shares many similarities with the proofs presented here; however, it is technically much more involved and requires a deeper study of the structure of 2-connected planar graphs. We refer the reader to the full version for a complete treatment.

This section is structured as follows. In Section 2.1 we exhibit a specific way of generating in an approximate fashion a random planar graph with n vertices. This material is mainly from [12], and is repeated here for the sake of completeness. Then, in Section 2.2, we show how to use this approximate generation to find a lower bound for the maximum degree of a random planar graph that holds with high probability.

2.1 Random Sampling

Let us start with introducing some notation. Let $G \in \mathcal{C}_n$, and denote by $lb(G)$ the size of a largest 2-connected block in G, which we will also refer to, if it is unique, as the *(biconnected) core* of G. Moreover, let us write

$$core(n,m) = \{G \in \mathcal{C}_n \ : \ lb(G) = m\}.$$

Using this notation, the probability that the largest block in a random graph C_n from $\mathcal{C}_n$ has m vertices is

$$\Pr[lb(C_n) = m] = \frac{|core(n,m)|}{|\mathcal{C}_n|}. \qquad (2.1)$$

Theorem 5.4 from [8] immediately implies the following statement about the probability distribution of $lb(C_n)$.

Theorem 2.1. *Let $\varepsilon > 0$. There is a $C = C(\varepsilon) > 0$ such that*

$$\Pr\left[|lb(C_n) - (1-\rho_B B''(\rho_B))n| \geq Cn^{2/3}\right] \leq \varepsilon, \text{ where } 1-\rho_B B''(\rho_B) \approx 0.959.$$

Moreover, uniformly for $|x| \leq C$

$$\Pr\left[lb(C_n) = \lfloor (1-\rho_B B''(\rho_B))n + xn^{2/3} \rfloor\right] = \Theta(n^{-2/3}).$$

An important ingredient of our sampler is a probability distribution that is defined over classes of graphs. Recall that $\mathcal{C}^\bullet$ contains all rooted planar graphs. The *Boltzmann distribution* over $\mathcal{C}^\bullet$ is given through

$$\forall \gamma \in \mathcal{C}^\bullet \;:\; \Pr[\gamma] = \frac{\rho_C^{|\gamma|}}{|\gamma|! \cdot \rho_C C'(\rho_C)}. \tag{2.2}$$

The Boltzmann model can be used to construct a whole *family* of distributions on combinatorial objects, each of which has several distinguished properties. However, the above definition is sufficient for our purposes and we will refer to it for simplicity as the Boltzmann distribution (over $\mathcal{C}^\bullet$). The interested reader can find in [5] a very detailed treatment of the topic.

With the above tools at hand we are ready to describe a random construction procedure for planar graphs.

$S_{\mathcal{C}}(n,\varepsilon)$: $C \to$ the constant given by Theorem 2.1
  $m \to$ a random value according to the distribution (2.1) (∗)
  if $|m - (1-\rho_B B''(\rho_B))n| > Cn^{2/3}$
    return $\perp$
  else
    $B \to$ uniformly at random from $\mathcal{B}_m$ (∗∗)
    repeat
      $\forall v \in B$: choose independently $\gamma_v \in \mathcal{C}^\bullet$ according to (2.2)
    until $(\sum_{v \in B} |\gamma_v| = n)$
    $\forall v \in B$: identify the root of γ_v with v
    partition randomly the set $[n]$ into blocks $(S_v)_{v \in B}$ of size $|S_v| = |\gamma_v|$
    return the resulting graph, with all γ_v relabeled using
      labels from S_v in the canonical way

The following lemma from [12] summarizes the properties of the algorithm that we will exploit.

Lemma 2.2. *Let $\varepsilon > 0$. The following statements are true for sufficiently large n.*

- $\Pr[S_{\mathcal{C}}(n,\varepsilon) = \perp] \leq \varepsilon$.
- *Let $C = C(\varepsilon) > 0$ be the constant guaranteed to exist by Theorem 2.1, and let $G \in \mathcal{C}_n$ be such that $|lb(G) - (1-\rho_B B''(\rho_B))n| \leq Cn^{2/3}$. Then*

$$\Pr[S_{\mathcal{C}}(n,\varepsilon) = G] = |\mathcal{C}_n|^{-1}.$$

Moreover, the following corollary was shown in [12].

Corollary 2.3. *Let $\varepsilon > 0$ and $C = C(\varepsilon) > 0$ be as in Theorem 2.1. Then, uniformly for all m such that $|m - (1 - \rho_B B''(\rho_B))n| \le Cn^{2/3}$ and graphs $\gamma_1, \ldots, \gamma_m$ drawn independently according to the Boltzmann distribution* (2.2)

$$\Pr\left[\sum_{i=1}^{m} |\gamma_i| = n\right] = \Theta(n^{-2/3}).$$

2.2 Bounding the Maximum Degree from Below

Let $G^\bullet \in \mathcal{C}^\bullet$. We will write $rd(G^\bullet)$ for the degree of the root-vertex in $G^\bullet$. The following technical lemma is an important tool in the proof of the lower bound in our main theorem. Its proof, which requires a sophisticated analysis of the generating functions enumerating planar graphs (see also in the next section), is omitted and can be found in the full version.

Lemma 2.4. *Let $\varepsilon > 0$. There is a $c > 0$ such that the following is true. Let γ be a random graph drawn from the Boltzmann distribution for $\mathcal{C}^\bullet$. Then*

$$\Pr[rd(\gamma) \ge k] \ge ck^{-5/2} w_0^{-k},$$

where w_0 is given by

$$w_0 = \frac{1}{1-t} \exp\left(\frac{t(t-1)(t+6)}{6t^2+20+6}\right) - 1 \approx 1.48488989 \tag{2.3}$$

and t is the solution of

$$2 = \frac{1+2t}{(1+3t)(1-t)} \exp\left(-\frac{t^2(1-t)(18+36t+5t^2)}{2(3+t)(1+2t)(1+3t)^2}\right). \tag{2.4}$$

With the above fact at hand we can prove the following result about graphs that are generated by $S_{\mathcal{C}}(n, \varepsilon)$.

Lemma 2.5. *Let $G \neq \perp$ be a graph constructed by $S_{\mathcal{C}}(n, \varepsilon)$. Then, with probability $1 - o(1)$, the maximum degree of G is at least $c \log n - O(\log\log n)$, where $c = 1/\log(w_0)$.*

The proof of the lower bound for the maximum degree of random planar graphs can be completed as follows.

Proof of the lower bound in Theorem 1.1. Denote by $\tilde{\mathcal{C}}_n(\varepsilon)$ the set of graphs in $\mathcal{C}_n$ whose 2-connected core has $(1 - \rho_B B''(\rho_B)) \pm C(\varepsilon)n^{2/3}$ vertices, where $C(\varepsilon) > 0$ is the constant from Theorem 2.1. If we write $\Delta(G)$ for the maximum degree of a graph G, by applying Theorem 2.1 we infer that

$$\Pr[\Delta(C_n) \ge \log_{w_0} n - O(\log\log n)] \ge \Pr[\Delta(C_n) \ge \log_{w_0} n - O(\log\log n) \mid C_n \in \tilde{\mathcal{C}}_n(\varepsilon)] - \varepsilon.$$

To estimate the latter probability, note that due to Lemma 2.2 the distributions "$C_n \mid C_n \in \tilde{\mathcal{C}}_n(\varepsilon)$" and "$S_{\mathcal{C}}(n, \varepsilon) \mid S_{\mathcal{C}}(n, \varepsilon) \neq \perp$" coincide. The proof then completes by applying Lemma 2.5. □

Proof of Lemma 2.5. Since $G \neq \perp$, the value m chosen in the line marked with "$(*)$" in the exposition of $S_{\mathcal{C}}(n, \varepsilon)$ satisfies $|m - (1 - \rho_B B''(\rho_B))n| \leq Cn^{2/3}$. Together with Theorem 2.1 this implies for large n that, with plenty of room to spare, $m \geq n/2$.

Recall that G is composed out of a 2-connected graph B with m vertices, where each vertex v is identified with the root of a random graph γ_v, such that $\sum_{v \in B} |\gamma_v| = n$. Thus

$$\Delta(G) \geq \max_{v \in B} rd(\gamma_v).$$

Let us write X_v for the indicator variable for the event that $rd(\gamma_v) \geq (1 - \delta) \log_{w_0} n$, where $\delta = \frac{c' \log \log n}{\log_{w_0} n}$, and $c' > 0$ will be chosen later. Set $X = \sum_{v \in B} X_v$. Then the above inequality says that

$$X > 0 \implies \Delta(G) \geq (1 - \delta) \log_{w_0} n.$$

In the sequel we will bound the probability for $X > 0$. Let us write $\mathcal{E}$ for the event "$\sum_{v \in B} |\gamma_v| = n$" and "$|m - (1 - \rho_B B''(\rho_B))n| \leq Cn^{2/3}$", where the γ_v's are independent graphs drawn from the Boltzmann distribution for $\mathcal{C}^\bullet$. The previous discussion implies that

$$\Pr[\Delta(G) \leq (1 - \delta) \log_{w_0} n \mid G \neq \perp] \leq \Pr[X = 0 \mid \mathcal{E}].$$

Corollary 2.3 guarantees that $\Pr[\mathcal{E}] = \Theta(n^{-2/3})$. Thus,

$$\Pr[\Delta(G) \leq (1 - \delta) \log_{w_0} n \mid G \neq \perp] \leq O(n^{2/3}) \Pr[X = 0]. \tag{2.5}$$

Note that X is binomially distributed with parameters $m \geq n/2$ and success probability equal to the probability that the root-degree of γ_v, for any $v \in B$, is at least $(1 - \delta) \log_{w_0} n$. By applying Lemma 2.4, this is seen to be at least $(\log n)^{-3} n^{-(1-\delta)}$, whenever n is sufficiently large. Thus,

$$\Pr[X = 0] \leq \left(1 - (\log n)^{-3} n^{-1+\delta}\right)^{n/2} \leq e^{-(\log n)^{-3} n^{\delta}/2}.$$

Recall that $\delta = \frac{c' \log \log n}{\log_{w_0} n}$. By setting, say, $c' = 6/\log(w_0)$, the above probability is $o(n^{-1})$. The proof is then completed by applying this bound to the right hand side of (2.5). □

3 The Upper Bound

3.1 Generating Functions and the First Moment Method

In order to obtain an upper bound for the distribution of the maximum degree we use the first moment method. Let $X_{n,k}$ denote the (random) number of vertices of degree k in a 2-connected random graph of size n and let

$$Y_{n,k} = \sum_{\ell > k} X_{n,k}$$

denote the number of vertices of degree larger than k. Obviously, we have

$$\Delta(B_n) > k \quad \Longleftrightarrow \quad Y_{n,k} > 0$$

and consequently

$$\Pr[\Delta(B_n) > k] = \Pr[Y_{n,k} > 0] \leq \mathbb{E}\, Y_{n,k}.$$

Let $d_{n,k}$ denote the probability that the root degree (in a 2-connected graph of size n) equals k, then $\mathbb{E}\, X_{n,k} = np_{n,k}$. Hence, it is sufficient to provide upper bounds of

$$p_{n,k} = \frac{[x^n w^k]\, B'(x,1,w)}{[x^n]\, B'(x)}.$$

It is known that

$$|\mathcal{B}'_n| = [x^n]\, B'(x) \sim c \cdot n^{-5/2} \rho_B^{-n},$$

where $c > 0$ and $\rho_B = 0.03672841...$, see [2, 7]. This follows from a precise analysis of the singularity of $B'(x)$ which is of the form

$$B'(x) = g(x) + h(x)\left(1 - \frac{x}{\rho_B}\right)^{3/2}.$$

Consequently, we just need upper bounds for $[x^n w^k]B'(x,1,w)$. Suppose that $w_0 > 0$ is chosen in a way that $B'(x,1,w_0)$ is a convergent power series. Then we have

$$[x^n w^k]\, B'(x,1,w) \le w_0^{-k} [x^n]\, B'(x,1,w_0).$$

Actually it will turn out that we can choose $w_0 > 1$ in an "optimal way" so that $B'(x,1,w_0)$ has the same radius of convergence ρ_B as $B'(x)$ and also the same kind of singularity.

Lemma 3.1. *Let w_0 be as in Equation* (2.3)*. Then $B'(x,1,w_0)$ has a local representation of the form*

$$B'(x,1,w_0) = \overline{g}(x) + \overline{h}(x)\left(1 - \frac{x}{\rho_B}\right)^{3/2},$$

with functions $\overline{g}(x)$, $\overline{h}(x)$ that are non-zero and analytic at ρ_B. Furthermore

$$[x^n]\, B'(x,1,w_0) \sim \overline{c} \cdot n^{-5/2} \rho_B^{-n}$$

for some constant $\overline{c} > 0$.

Summing up we have

$$\mathbb{E}\, X_{n,k} = O\left(nq^{-k}\right)$$

and consequently

$$\Pr\left[\Delta(B_n) > k\right] = O\left(nq^{-k}\right).$$

Of course, this estimate provides the upper bound in Theorem 1.3. The proof of the lower bound in Theorem 1.1 is precisely the same, we just have to replace "2-connected" by "connected".

Lemma 3.2. *Let w_0 be as in Equation* (2.3)*. Then $C'(x,1,w_0)$ has a local representation of the form*

$$C'(x,1,w_0) = \overline{g}_2(x) + \overline{h}_2(x)\left(1 - \frac{x}{\rho_C}\right)^{3/2},$$

with functions $\overline{g}_2(x)$, $\overline{h}_2(x)$ that are non-zero and analytic at ρ_C. Furthermore

$$[x^n]\, C'(x,1,w_0) \sim \overline{c}_2 \cdot n^{-5/2} \rho_C^{-n}$$

for some constant $\overline{c}_2 > 0$.

3.2 Generating Functions for the Root Degree

Before we sketch the proof of Lemma 3.1 and Lemma 3.2, let us briefly describe the relations that are satisfied by the generating functions $C(x, y, w)$ and $B(x, y, w)$. These relations were discovered in [4], and we refer the reader to this work for further details.

Let us begin with the case of connected planar graphs. A standard decomposition of connected graphs into their maximal 2-connected components implies that any vertex-rooted connected planar graph $C^\bullet$ can be decomposed as a set $\{B'_1, \ldots, B'_\ell\}$ of vertex-derived 2-connected graphs, whose roots are identified into a single vertex, and where each other vertex is substituted by a vertex-rooted connected graph. In other words, the root degree of C' equals the sum of the root degrees of the $(B_i)_{1\leq i\leq \ell}$. This translates immediately to the functional relation

$$C'(x, y, w) = \exp\left(B'\left(C^\bullet(x, y), y, w\right)\right). \tag{3.1}$$

This establishes a relation between $C(x, y, w)$ and $B(x, y, w)$. The authors of [4] studied also the function $B(x, y, w)$ more closely. We omit the details here, and just state the results. The generating functions for $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{T}$ satisfy the relations

$$\frac{\partial B'(x, y, w)}{\partial w} = xy\frac{1 + D(x, y, w)}{1 + yw} \tag{3.2}$$

$$D(x, y, w) = (1 + yw)\exp\left(\frac{xD(x, y, w)D(x, y, 1)}{1 + xD(x, y, 1)} + \overline{T}\left(x, D(x, y, 1), \frac{D(x, y, w)}{D(x, y, 1)}\right)\right) - 1, \tag{3.3}$$

$$\begin{aligned}\overline{T}(x, y, w) = \frac{yw}{2}&\left(\frac{1}{1 + wy} + \frac{1}{1 + xy} - 1-\right. \\ &\left. - \frac{(U + 1)^2\left(-w_1(U, V, w) + (U - w + 1)\sqrt{w_2(U, V, w)}\right)}{2w(Vw + U^2 + 2U + 1)(1 + U + V)^3}\right),\end{aligned} \tag{3.4}$$

with polynomials $w_1 = w_1(U, V, w)$ and $w_2 = w_2(U, V, w)$ given by

$$\begin{aligned} w_1 &= -UVw^2 + w(1 + 4V + 3UV^2 + 5V^2 + U^2 + 2U + 2V^3 + 3U^2V + 7UV) \\ &\quad + (U + 1)^2(U + 2V + 1 + V^2), \\ w_2 &= U^2V^2w^2 - 2wUV(2U^2V + 6UV + 2V^3 + 3UV^2 + 5V^2 + U^2 + 2U \\ &\quad + 4V + 1) + (U + 1)^2(U + 2V + 1 + V^2)^2. \end{aligned}$$

Further, it is possible to integrate $\frac{\partial B'(x,y,w)}{\partial w}$ to obtains an explicit expression in terms of T. The details are omitted, and we refer again the reader to the full version.

3.3 Singular Functional Equations

First we have a closer look at (3.3). If we set $w = 1$ then it reduces to an equation for $D(x, y, 1)$ which is precisely the equation enumerating so-called planar networks. In order to avoid conflicts with the notation we set $E(x, y) := D(x, y, 1)$. From [2, 7] we know the analytic behaviour of $E(x, y)$ around its dominant singularity:

$$E(x, y) = E_0(y) + E_2(y)X^2 + E_3(y)X^3 + O(X^4), \tag{3.5}$$

where $X = \sqrt{1 - \frac{x}{\rho_D(y)}}$. Recall that the coefficient of the squareroot term X vanishes. Since we are not interested in the number of edges we will set $y = 1$ in (most of) the following calculations. The most important step in our analysis is the discussion of the equation (3.3). First, we rewrite it to

$$D + 1 = \exp\left(G(x, D, w, E, U, V) + H(x, D, E, U, V)\sqrt{J(D, E, U, V)}\right),$$

where

$$\begin{aligned}
G &= \log(1 + w) + \frac{xDE}{1 + xE} \\
&\quad + \frac{D}{2}\left(\frac{1}{1 + D} + \frac{1}{1 + xE} - 1 + \frac{(U + 1)^2 w_1(U, V, D/E)}{2D/E(VD/E + U^2 + 2U + 1)(1 + U + V)^3}\right), \\
H &= -\frac{(U + 1)^2 D(U - D/E + 1)}{4D/E(VD/E + U^2 + 2U + 1)(1 + U + V)^3}, \\
J &= w_2(U, V, D/E).
\end{aligned}$$

In the following analysis we will consider first E, U, V as *new variables*. Finally, we will substitute them by $E = E(x, 1)$, $U = U(x, 1)$, $V = V(x, 1)$. Set

$$t_0 = t(1) \approx 0.626, \qquad x_0 = \rho_D(1) = \frac{(3t_0 + 1)(1 - t_0)^3}{16t_0^3} \approx 0.038,$$

$$D_0 = \frac{t_0}{1 - t_0} \approx 1.676, \qquad w_0 = \frac{1}{1 - t_0}\exp\left(\frac{t_0(t_0 - 1)(t_0 + 6)}{6t_0^2 + 20t_0 + 6}\right) - 1 \approx 1.484$$

$$E_0 = E(x_0, 1, 1) = \frac{3t_0^2}{(1 - t_0)(3t_0 + 1)} \approx 1.094, \quad U_0 = \frac{1}{3t_0} \approx 0.532, \quad V_0 = \frac{1 + 3t_0}{3(1 - t_0)} \approx 2.568.$$

Then we actually have

$$H(x_0, D_0, E_0, U_0, V_0) = J(D_0, E_0, U_0, V_0) = 0,$$

which can easily be checked by writing $H(x_0, D_0, w_0, E_0, U_0, V_0)$ and $J(D_0, E_0, U_0, V_0)$ in terms of t_0. Hence, we are in a situation, where the following lemma applies.

Lemma 3.3. *Let $\mathbf{v} = (v_1, \ldots, v_d)$ be a d-dimensional complex vector and let $y = y(\mathbf{v})$ be a function with $y(\mathbf{v}_0) = y_0$ that satisfies a functional equation*

$$y = \exp\left(G(y, \mathbf{v}) + H(y, \mathbf{v})\sqrt{J(y, \mathbf{v})}\right), \tag{3.6}$$

where G, H, and J are analytic functions at $(y_0, \mathbf{v}_0)$ such that

$$H(y_0, \mathbf{v}_0) = J(y_0, \mathbf{v}_0) = 0 \quad \text{and} \quad y_0 G_y(y_0, \mathbf{v}_0) \neq 1.$$

Then, $y(\mathbf{v})$ has a local representation of the form

$$y(\mathbf{v}) = P(\mathbf{v}) + \sqrt{Q(\mathbf{v})}, \tag{3.7}$$

where P and Q are analytic at $(\mathbf{v}_0)$, the evaluation of P at $(\mathbf{v}_0)$ is y_0, and Q and all its partial derivatives up to order 2 are zero at $(\mathbf{v}_0)$. Furthermore, the evaluation of Q_{xxx} at $(\mathbf{v}_0)$, for any variable x in $\mathbf{v}$, is

$$Q_{xxx}(\mathbf{v}_0) = \frac{6\left(y_0 H_y G_x - H_x(y_0 G_y - 1)\right)^2 \left(y_0 J_y G_x - J_x(y_0 G_y - 1)\right) y_0^2}{(y_0 G_y - 1)^5},$$

with $y_0 = y(\mathbf{v}_0)$.

The proof is omitted from this extended abstract. By applying Lemma 3.3 with $y = D+1$ and $\mathbf{v} = (x, w, E, U, V)$ we obtain a representation of D as a function of x, w, E, U, V of the form

$$D = P(x, w, E, U, V) + \sqrt{Q(x, w, E, U, V)}, \tag{3.8}$$

where Q and all partial derivatives of Q up to order 2 vanish. In particular if we substitute $E = E(x, 1)$ etc. we see that $Q(x, w, E(x), U(x), V(x))$ can be represented as

$$Q(x, w, E(x, 1), U(x, 1), V(x, 1)) = X^3 h_1(X) + X^2 W h_2(X, W) + X W^2 h_3(W) + W^3 h_4(W),$$

where $W = 1 - w/w_0$, $X = \sqrt{1 - x/x_0}$ and $h_1, \ldots, h_4$ are proper convergent power series. A simple (but tedious) computation provides

$$\begin{aligned} h_1(0) &\approx 0.009976, & h_2(0) &\approx -0.039447, \\ h_3(0) &= 0, & h_4(0) &\approx 0.091370. \end{aligned}$$

It should be remarked that $h_1(0) > 0$, $h_4(0) > 0$, and $h_3(0) = 0$.[1] This shows that $D(x, 1, w_0)$ has a singular behavior of the form

$$D(x, 1, w_0) = \overline{g}(x) + \overline{h}(x) X^3 \tag{3.9}$$

with $X = \sqrt{1 - x/x_0}$ and where $h(x_0) > 0$.

3.4 Proof of Lemma 3.1 and Lemma 3.2

With all the above facts at hand it is now not very difficult to provide the proof of Lemma 3.1. We use the explicit representation of B and apply the local expansion (3.5) for $E(x, 1)$ and (3.9) for $D(x, 1, w_0)$ (and also those of $u = U(x, E(x, 1))$ and $v = V(x, E(x, 1))$). This leads directly to a singular representation of $B'(x, 1, w_0)$ of the following type:

$$B'(x, 1, w_0) = \overline{g}_1(x) + \overline{h}_1(x) X^3. \tag{3.10}$$

Note that we definitely have $h_1(x_0) \neq 0$ and hence $h_1(x_0) > 0$. Namely if $h_1(x_0) = 0$ then we would have $[x^n]\, B'(x, 1, w_0) = O\left(x_0^{-n} n^{-7/2}\right)$ which is impossible. Thus by applying the transfer lemma of Flajolet and Odlyzko [6] we obtain

$$[x^n]\, B'(x, 1, w_0) \sim c_1 x_0^{-n} n^{-5/2},$$

which completes the proof of the Lemma 3.1. Lemma 3.2 can be shown analogously – the details are omitted.

[1] Actually we have $h_3(W) = 0$ which can be shown without doing any numerical calculations. If $h_3 \neq 0$ it would follow that the dominant singularity of $D(x, 1, w)$ would have a singular behavior of the form $XW^{\ell - 1/2}$ for some integer $\ell \geq 0$ which would lead to an asymptotic leading term of the coefficient of $x^n w^k$ of the squareroot part of the form $c\, x_0^{-n} w_0^{-k} n^{-3/2} k^{-\ell-1/2}$ Similarly if $P(x, w, E(x), U(x), V(x))$ has a factor X in its expansion then the dominant behavior in n would be of the form $x_0^{-n} n^{-3/2}$. In both cases this contradicts the asymptotic expansion of for the coefficient $[x^n]\, D(x, 1, 1) \sim c_1\, x_0^{-n} n^{-5/2}$.

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The Maximum Degree of Random Planar Graphs

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Abstract

Let P_n denote a graph drawn uniformly at random from the class of all simple planar graphs with n vertices. We show that the maximum degree of a vertex in P_n is with probability $1 - o(1)$ asymptotically equal to $c \log n$, where $c \approx 2.529$ is determined explicitly. A similar result is also true for random 2-connected planar graphs.

Our analysis combines two orthogonal methods that complement each other. First, in order to obtain the upper bound, we resort to exact methods, i.e., to *generating functions* and *analytic combinatorics*. This allows us to obtain fairly precise asymptotic estimates for the expected number of vertices of any given degree in P_n. On the other hand, for the lower bound we use *Boltzmann sampling techniques*. In particular, by tracing the execution of an adequate algorithm that generates a random planar graph, we are able to explicitly find vertices of sufficiently high degree in P_n.

1 Introduction

A very active research area that requires a deep understanding of several models that generate graphs with structural side constraints is the analysis of social networks. It requires joint efforts from various fields, ranging from sociology over biology and economics all the way to theoretical computer science and discrete mathematics. While the notion of a 'social network' is not precisely defined in a mathematical sense and its properties may differ considerably depending on the application at hand, an underlying key feature is that a social network combines structural – i.e., deterministic – assumptions with randomness. It is this combination that makes their analysis very difficult to handle. In contrast to that, classical random graph theory, as introduced by Erdős and Rényi in the 50's, relies heavily on the assumption that edges appear independently. This assumption breaks down once we add structural constraints. There are various ways to achieve this, for example, by replacing 'global' randomness by 'local' randomness, as it is done in the popular preferential attachment model, cf. [1], and several variations thereof. Another way is to impose a probability distribution, usually the uniform distribution, on a class of graphs that fulfill some structural side constraints. A paradigmatic problem of the later type is the study of planar graphs – a graph class that seemingly misses quite strongly the key property of the Erdős-Rényi model (the independence of the edges) due to long range implications of the planarity condition. Indeed, planar graphs can be defined in terms of excluded minors, and this implies that we do not have any a priori non-trivial upper bounds on the size of the forbidden substructures. It is this problem that we consider in this paper.

The study of random planar graphs was initiated by Denise et al. [6]. McDiarmid et al. [18] showed that a random planar graph P_n in fact has some properties that are quite different compared to their analogues in the classical Erdős-Rényi random graph. They showed that the probability that P_n is connected is, for n tending to infinity, bounded away from 0 and from 1. In contrast, an Erdős-Rényi graph satisfies a 0-1 law for all "natural" properties. While the precise value of this probability is of course given by $\lim_{n\to\infty} |\mathcal{C}_n|/|\mathcal{P}_n|$, where $\mathcal{P}_n$ ($\mathcal{C}_n$) denotes the class of all labeled (connected) planar graphs with n vertices, it took quite a while and required deep methods from combinatorial counting and analytic combinatorics until Giménez and Noy [14], extending earlier work of Bender et al. [2], were able to determine the required values asymptotically. The next challenging problem became the question of the degree distribution. Another question that is extremely simple to answer for the Erdős-Rényi random graph model, but far from obvious for the random planar graph case. Over the last years two research groups independently developed completely different sets of methods and techniques for attacking this problem. On the one hand, Drmota et al. [7] considerably extended the methods from [14] and provided a solution using techniques from and within analytic combinatorics. On the other hand, Panagiotou et al. [21] extended the concept of Boltzmann samplers (originally introduced by Duchon et al. [9] for the uniform *generation* of objects), so that it can be used for analysing the *structure* of random planar graphs.

In this paper we combine our forces (and techniques) to solve the question of determining the value of the maximum degree in a random planar graph. Using combinatorial arguments Reed and McDiarmid [17] showed that the maximum degree is in the order $\Theta(\log n)$. In this paper we solve this question completely and determine the constant. I.e., we show that the maximum degree is $c \log n + O(\log\log n)$ for a constant $c > 0$ that we determine explicitly.

Actually, an additional and perhaps surprising aspect is that we use – and need – both techniques developed over the last years. The algorithmic approach of Boltzmann sampling allows us to show, by tracking the progress of the algorithm, that a vertex of a certain degree will be generated. In contrast, showing that a vertex of a certain degree will not be generated is a much harder task. However, within the analytic counting approach, the proof that a vertex of a certain degree does *not* exist corresponds to a *first moment* argument, an approach that is well known to be much easier compared to the corresponding second order arguments that would be needed for a proof of the existence of vertex with a certain property. We thus are in the fascinating situation that both methods nicely fit together and complement each other.

Let us now describe our results in more detail. Denote by C_n a graph that is drawn uniformly at random from the class $\mathcal{C}_n$ of all connected planar graphs with n vertices and with $\Delta(G)$ the maximum degree of a graph $G = (V, E)$.

Theorem 1.1. *There is a $c > 0$ such that the following is true. With probability $1 - o(1)$ we have $|\Delta(C_n) - c \log n| = O(\log\log n)$. In fact, $c = 1/\log(w_0) \approx 2.529$, where w_0 is given explicitly in (3.1).*

As it is well known that the largest component of a random planar graph P_n has size $n - O(1)$, cf. [14, 16, 18], the above theorem immediately implies the following corollary.

Corollary 1.2. *With probability $1 - o(1)$ we have $|\Delta(P_n) - c \log n| = O(\log\log n)$, where c is as in Theorem 1.1.*

Finally, we can show a similar result for 2-connected planar graphs. We denote by B_n a graph that is drawn uniformly at random from the class of all 2-connected planar graphs with n vertices.

Theorem 1.3. *The conclusion of Theorem 1.1 remains true if we replace C_n by B_n.*

In addition to proving our results we also use this paper as a gentle introduction to the methods and techniques mentioned above, with the hope that both methods can and will find more applications in each other's domain.

2 Tools & Techniques

2.1 Basic Notation

Let $\mathcal{G}$ be a class of graphs. We denote by $\mathcal{G}_{n,m}$ the subset of $\mathcal{G}$ consisting of all graphs with n vertices and m edges, and we write $g_{n,m} = |\mathcal{G}_{n,m}|$. Moreover, we define $\mathcal{G}_n = \cup_{m\geq 0}\mathcal{G}_{n,m}$ and set $g_n = |\mathcal{G}_n|$. In particular, in the remainder of the paper we will write $\mathcal{C}$ for the class of connected planar graphs, $\mathcal{B}$ for the class of 2-connected planar graphs, and $\mathcal{T}$ for the class of 3-connected planar graphs.

We will frequently use the *pointing* and *derivative* operators, which are used to distinguish specific vertices or edges in the graphs that are contained in the class under consideration. First of all, given a class of graphs $\mathcal{G}$, we define $\mathcal{G}^\bullet = \bigcup_{n\geq 1}\{1,\ldots,n\} \times \mathcal{G}_n$ as the class of *vertex-rooted* graphs. In particular, every graph $G \in \mathcal{G}_n$ is contained n times in $\mathcal{G}_n$, where each copy contains a different distinguished vertex. Similarly, the *vertex-derived* class $\mathcal{G}'_{n-1,m}$ is obtained by removing the label n from every object in $\mathcal{G}_{n,m}$, such that the obtained graphs have $n-1$ labeled vertices, i.e., vertex n can be considered as a distinguished vertex that does not contribute to the size. Consequently, there is a bijection between the classes $\mathcal{G}'_{n-1}$ and $\mathcal{G}_n$. We set $\mathcal{G}' := \bigcup_{n\geq 0}\mathcal{G}'_n$. It will also be necessary to distinguish edges. To this end, define $\mathcal{G}_e = \bigcup_{n,m\geq 1}\{1,\ldots,m\} \times \mathcal{G}_{n,m}$ as the class of *edge-rooted* graphs, which contains each graph in $\mathcal{G}$ a number of times equal to the number of edges. So, analogously to the case of vertex-rooted graphs, every graph in $\mathcal{G}_e$ has a specific distinguished edge. However, to simplify notation, for technical reasons we will assume that the marked edge does not contribute to the total number of edges in each graph in $\mathcal{G}_e$. In other words, we may think that this edge is removed, but its former endpoints are distinguished, such that the graph can be fully recovered.

The main parameter of study in this paper is the maximum degree of random planar graphs. Let $\mathcal{C}^\bullet_{n,m,k}$ be the class of vertex-rooted planar graphs with n vertices and m edges, such that the degree of the root-vertex is k. Define $\mathcal{B}^\bullet_{n,m,k}$ and $\mathcal{T}^\bullet_{n,m,k}$ similarly. Moreover, for $\mathcal{G} \in \{\mathcal{C},\mathcal{B},\mathcal{T}\}$ let

$$G^\bullet(x,y,w) = \sum_{n,m,k\geq 0} \frac{|\mathcal{G}^\bullet_{n,m,k}|}{n!} x^n y^m w^k$$

denote the *exponential generating function (egf)* enumerating the sequence $(|\mathcal{G}^\bullet_{n,m,k}|)_{n,m,k\geq 0}$. We shall omit any of the parameters x, y, w if the corresponding value is equal to one; for example, we write $G^\bullet(x) = G^\bullet(x,1,1)$. Similar to this notation, we will write $G'(x,y,w)$ for the egf enumerating $(|\mathcal{G}'_{n,m,k}|)_{n,m,k\geq 0}$, where $\mathcal{G}'_{n,m,k}$ contains all graphs in $\mathcal{G}'_{n,m}$, whose unlabeled vertex has degree k. Observe that $G^\bullet(x,y,w) = x\frac{\partial}{\partial x}G(x,y,w)$. Finally, note that $G_e(x,y,w) = \frac{\partial}{\partial y}G(x,y,w)$.

As a final remark, let us already mention at this point that all generating functions considered in this work have (at least) one finite dominant non-zero singularity on the real axis. For a generating function G enumerating a graph class $\mathcal{G}$ we will write ρ_G for this singularity.

2.2 Combinatorial Constructions, Generating Functions and Boltzmann Samplers

The goal of this section is to give a concise and self-contained overview of the three basic tools that will be used extensively in our analysis: universal combinatorial constructions, relations of generating functions, and Boltzmann samplers. From today's point of view, there is a vast amount of literature dealing with these subjects, which is by far not limited to the study of planar graphs. It is beyond the scope of this article to review all this work, and we will restrict ourselves to summarizing only the most relevant facts. We refer the interested reader to the excellent book by Flajolet and Sedgewick [10], which contains a general treatment of combinatorial constructions and the associated analysis of generating functions. Moreover, the article [9] by Duchon, Flajolet, Louchard and Schaeffer contains an excellent introduction to the topic of Boltzmann sampling, and the article [13] by Fusy extends the techniques to the context of planar graphs.

In this section we describe a collection of five universal constructions (disjoint union, product, set, vertex- and edge-substitution), together with the associated relations for the generating functions and the resulting Boltzmann sampling algorithms, that are used subsequently to formulate a decomposition for the class of all connected planar graphs. Before we proceed, let us define the concept of *Boltzmann samplers.* Let $\mathcal{G}$ be a class of combinatorial objects (in our case graphs, where possibly vertices or/and edges might be distinguished), enumerated by the function $G(x, y)$. A Boltzmann sampler is a randomized algorithm that draws graphs from $\mathcal{G}$ under a certain probability distribution that is spread over the whole class. More precisely, suppose that x, y are such that $G(x, y)$ exists. Then, the Boltzmann distribution with parameters x, y assigns to each $\gamma \in \mathcal{G}$ the weight

$$\Pr[\gamma] = \frac{x^{v(\gamma)} y^{e(\gamma)}}{v(\gamma)! G(x, y)}, \tag{2.1}$$

where $v(\gamma)$ denotes the number of labeled vertices in γ, and $e(\gamma)$ denotes the number of edges of γ. A Boltzmann sampler $\Gamma G(x, y)$ is an algorithm that generates graphs according to the distribution in (2.1).

Note that Boltzmann samplers are not a priori suited for studying the distribution of graphs that are drawn uniformly at random from $\mathcal{G}_n$, as (2.1) defines a distribution over the whole of $\mathcal{G}$. However, observe that if we set $y = 1$ in (2.1), then the Boltzmann distribution is actually the uniform distribution over any given size of graphs. More precisely, if we denote by G_n a graph drawn uniformly at random from $\mathcal{G}_n$ and abbreviate $\gamma = \Gamma G(x, 1)$, then for any $\mathcal{P} \subseteq \mathcal{G}$ we have

$$\Pr[G_n \in \mathcal{P}] = \Pr[\gamma \in \mathcal{P} \mid \gamma \in \mathcal{G}_n] = \Pr[\gamma \in \mathcal{P} \text{ and } \gamma \in \mathcal{G}_n] \cdot \Pr[\gamma \in \mathcal{G}_n]^{-1}. \tag{2.2}$$

As we shall see below, Boltzmann samplers can be constructed explicitly, and provide essentially "recipies", which translate sequences of independent and identically distributed (iid) random variables into graphs. So, if the Boltzmann probability of getting a desired graph size n is not too small, then the study of random graphs boils down with (2.2) to studying properties of sequences of iid random variables. Such approaches were used for example in [3, 20, 12], and will be also very useful in Section 4.

Let us proceed with the definition of the combinatorial constructions and the associated generating functions and Boltzmann samplers. The proofs for the given relations of the generating functions and the validity of the Boltzmann samplers, if omitted here, can all be found in [13]. The basic class in all our constructions is denoted by $\mathcal{X}$, and contains a single graph that contains just one vertex. Using the notation from Section 2.1, the egf enumerating $\mathcal{X}$ is given by x. The combinatorial constructions that we will exploit are described below.

Disjoint Union: We will say that a class $\mathcal{G}$ is the disjoint union of two classes $\mathcal{A}$ and $\mathcal{B}$, in symbols $\mathcal{G} = \mathcal{A} + \mathcal{B}$, if any object in $\mathcal{G}$ is either contained in $\mathcal{A}$ or $\mathcal{B}$. To guarantee the uniqueness of the decomposition, it is necessarily assumed that $\mathcal{A}$ and $\mathcal{B}$ are disjoint. Using this notation, the egf enumerating $\mathcal{G}$ is obviously given by $G(x, y) = A(x, y) + B(x, y)$. Moreover, a Boltzmann sampler ΓG for $\mathcal{G}$ can be described in terms of Boltzmann samplers for $\mathcal{A}$ and $\mathcal{B}$, where we denote by $\mathsf{Be}(p)$ a Bernulli random variable with success probability p.

$\Gamma G(x, y)$: $b \leftarrow \mathsf{Be}(\frac{A(x,y)}{G(x,y)})$
if $b = 1$ **return** $\Gamma A(x, y)$
else return $\Gamma B(x, y)$

In other words, the Boltzmann sampler for $\mathcal{G}$ first makes a Bernulli choice between $\mathcal{A}$ and $\mathcal{B}$, and then resorts to the Boltzmann sampler for the chosen class. Let us replicate the proof of correctness from [9, 13], as it is simple and gives the main idea behind Boltzmann sampling principles. Indeed, suppose that $g \in \mathcal{A}$. Since $\mathcal{A}$ and $\mathcal{B}$ are disjoint, the probability that ΓG generates g is equal to the probability that simultaneously "$b = 1$" and "$\Gamma A(x, y)$ generates g". Since these two events are independent and $\Gamma A(x, y)$ is a Boltzmann sampler for $\mathcal{A}$ we obtain that

$$\Pr[\Gamma G(x, y) = g] = \frac{A(x, y)}{G(x, y)} \cdot \frac{x^{v(g)} y^{e(g)}}{v(g)! A(x, y)} = \frac{x^{v(g)} y^{e(g)}}{v(g)! G(x, y)},$$

which agrees with (2.1). Moreover, a similar calculation shows that the output distribution is also correct when $g \in \mathcal{B}$, thus completing the proof.

Product: The product of two classes $\mathcal{A}$ and $\mathcal{B}$, in symbols $\mathcal{G} = \mathcal{A} \times \mathcal{B}$, is obtained by taking any pair of graphs $a \in \mathcal{A}_n$ and $b \in \mathcal{B}_{n'}$ and relabeling arbitrarily the vertices, such that the vertices in the resulting graph have labels from $\{1, \ldots, n + n'\}$. In this case the egf enumerating $\mathcal{G}$ is given by $G(x, y) = A(x, y)B(x, y)$. A Boltzmann sampler ΓG for $\mathcal{G}$ can be described concisely in terms of Boltzmann samplers for $\mathcal{A}$ and $\mathcal{B}$ as follows. We will use in the sequel an algorithm *RandomLabels*(G), which assigns random labels to the vertices of G from the set $\{1, \ldots, v(G)\}$.

$\Gamma G(x, y)$: $\gamma_A \leftarrow \Gamma A(x, y)$
$\gamma_B \leftarrow \Gamma B(x, y)$
return RandomLabels$((\gamma_A, \gamma_B))$

Note that the Boltzmann sampler performs independent calls to the samplers associated to $\mathcal{A}$ and $\mathcal{B}$, and composes a graph from $\mathcal{G}$ by assembling them and distributing randomly labels.

Set: Let $\mathcal{A}$ be a class of graphs that contain at least one labeled vertex[1]. We will say that the class $\mathcal{G}$ is a set of $\mathcal{A}$, in symbols $\mathcal{G} = \mathrm{Set}(\mathcal{A})$, if every graph $\mathcal{G}$ is composed out of a finite

[1]Note that the derivative operator from Section 2.1 allows us to construct classes in which all graphs bear no labels, e.g. $\mathcal{X}'$. Such classes are not allowed to be used within the Set construction

set of graphs from $\mathcal{A}$, relabeled arbitrarily so that all vertices have distinct labels. Moreover, for any non-negative integer k we will write $\mathcal{G} = \mathrm{Set}_{\geq k}(\mathcal{A})$, if only sets of size at least k are allowed in the previous definition. Using this notation, the egf enumerating $\mathcal{G}$ is given by

$$G(x,y) = e^{A(x,y)} - \sum_{i=0}^{k-1} \frac{A(x,y)^i}{i!}.$$

Let $\mathsf{Po}_{\geq k}(\lambda)$ be a Poisson distributed random variable with expectation λ, conditioned on being at least k. In other words, for $j \geq k$

$$\Pr[\mathsf{Po}_{\geq k}(\lambda) = j] = e^{-\lambda}\frac{\lambda^j}{j!} \cdot \left(1 - \sum_{i=0}^{k-1} e^{-\lambda}\frac{\lambda^i}{i!}\right)^{-1}.$$

A Boltzmann sampler for the set construction is given by the following algorithm.

$\Gamma G(x,y)$: $j \leftarrow \mathsf{Po}_{\geq k}(A(x,y))$
for $\ell = 1 \ldots j$ **do** $\gamma_i \leftarrow \Gamma A(x,y)$
return RandomLabels$((\gamma_1, \ldots, \gamma_j))$

In other words, the number of "components" in a (Boltzmann) random graph from $\mathcal{G}$ is Poisson distributed with parameter $A(x,y)$; this can be easily verified by observing that the subclass of $\mathcal{G}$ containing all graphs with exactly j components from $\mathcal{A}$ is enumerated by $A(x,y)^j/j!$, from which the correctness of the sampler follows immediately.

Vertex Substitution: Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two classes such that all graphs in $\mathcal{B}$ have at least one labeled vertex. Then the class $\mathcal{G}$ is obtained by vertex substitution from the core class $\mathcal{A}$, and the replacement class $\mathcal{B}$, in symbols $\mathcal{G} = \mathcal{A} \circ \mathcal{B}$, as follows. We begin with any $a \in \mathcal{A}$, substitute each labeled vertex $v \in V(a)$ in a unique fashion with any graph from $b_v \in \mathcal{B}$, and relabel arbitrarily all labeled vertices in $(b_v)_{v\in V(a)}$ such that they bear distinct labels. The egf enumerating $\mathcal{G}$ is given by $G(x,y) = A(B(x,y),y)$, i.e., the vertex substitution corresponds formally to the substitution of the variable marking the labeled vertices by $B(x,y)$.

The Boltzmann sampler for $\mathcal{G}$ proceeds along the lines of the definition of the class. In particular, it first samples a core object from the Boltzmann distribution for $\mathcal{A}$, and then replaces independently each vertex with a random graph from $\mathcal{B}$ as follows:

$\Gamma G(x,y)$: $\gamma \leftarrow \Gamma A(B(x,y),y)$
for each vertex $v \in V(\gamma)$ **do**
  $\gamma_v \leftarrow \Gamma B(x,y)$
  replace v by γ_v in γ
return RandomLabels(γ)

Edge Substitution: As in the case of vertex substitution, let $\mathcal{A}$ and $\mathcal{B}$ be two classes such that all graphs in $\mathcal{B}$ have at least one labeled vertex. Then $\mathcal{G}$ is obtained by edge substitution from the core class $\mathcal{A}$, and the replacement class $\mathcal{B}$, in symbols $\mathcal{G} = \mathcal{A} \tilde{\circ} \mathcal{B}$, as follows. Take a graph $a \in \mathcal{A}$, substitute every edge $e \in E(a)$ in a unique way by a graph $b_e \in \mathcal{B}$, and relabel arbitrarily all vertices in a and $(b_e)_{e\in E(a)}$ such that they bear distinct labels. With this notation, the egf enumerating $\mathcal{G}$ is given by $G(x,y) = A(x, B(x,y))$. Observe that edge substitution corresponds formally to the substitution of the variable marking the edges by $B(x,y)$. Moreover, as before, the Boltzmann sampler for $\mathcal{G}$ proceeds analogouesly:

$$
\begin{aligned}
\Gamma G(x,y): \quad & \gamma \leftarrow \Gamma A(x, B(x,y)) \\
& \textbf{for } \text{each edge } e \in E(\gamma) \textbf{ do} \\
& \quad \gamma_e \leftarrow \Gamma B(x,y) \\
& \quad \text{replace } e \text{ by } \gamma_e \text{ in } \gamma \\
& \textbf{return } \text{RandomLabels}(\gamma)
\end{aligned}
$$

2.3 Grammars and Generating Functions for Planar Graphs

In his classical book "Connectivity in graphs" Tutte [23] describes a canonical way of decomposing graphs into components of higher connectivity. There, any connected graph is described in a unique way in terms of its 2-connected components, each of which is further decomposed into 3-connected components. We will repeat this decomposition here, mostly tailored to the specific setting of planar graphs and using the notation from the previous section. A modern and much more detailed exposition can be found in the article [5] by Chapuy, Fusy, Kang and Shoilekova.

We begin with the well-known decomposition of a graph into 2-connected components. Given any vertex-derived connected graph $C' \in \mathcal{C}'$, we say that a block is a maximal 2-connected induced subgraph of C'. To decompose C', note that the distinguished vertex gives a starting point for a recursive description. Indeed, any vertex-derived connected graph can be obtained as follows. Start with a set $\{B'_1, \ldots, B'_\ell\}$ of vertex-derived 2-connected graphs whose distinguished vertices are identified with single vertex (this is the root of C'), and substitute every other vertex in $B'_1, \ldots, B'_\ell$ with a vertex-rooted connected graph. Note that in this description the 2-connected graphs correspond in the resulting graph to the blocks incident to the distinguished vertex of C'. This description gives us the combinatorial relation

$$\mathcal{C}' = \mathrm{Set}(\mathcal{B}' \circ \mathcal{C}^\bullet), \tag{2.3}$$

which, using the relations described in the previous sections, immediately translates to the following relation for the generating functions:

$$\frac{\partial C(x,y)}{\partial x} = \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right). \tag{2.4}$$

The decomposition of 2-connected planar graphs into 3-connected components is more involved. We will describe it in sufficient detail, as it is crucial for our further analysis. Let us introduce an auxiliary graph class that plays an important role in the subsequent discussion. Following Trakhtenbrot [22] and Tutte [23] we define a *(planar) network* as a connected graph with two "special" vertices, called the left pole and the right pole, such that after adding the edge between the poles and ignoring any possibly created multiple edges, the resulting graph is a 2-connected planar graph. The poles do not bear labels, and thus in the egf enumerating networks the variable x marks the number of non-pole vertices only. The above description provides us with an explicit relation between the class $\mathcal{B}$ and the class of networks $\mathcal{D}$, which we shall revise for completeness. Note that every edge-rooted 2-connected planar graph $B_e \in \mathcal{B}_e$, except of the graph that is a single edge, gives rise to two networks with $n-2$ labeled vertices: one is obtained by removing the labels from the endpoints of the root-edge (and relabeling the remaining vertices canonically such that only labels in $\{1, \ldots, n-2\}$ appear), and the other is obtained by adding the root-edge to B_e, removing the labels from the endpoints of the former root edge, and performing a relabeling as before. Note that in this fashion we

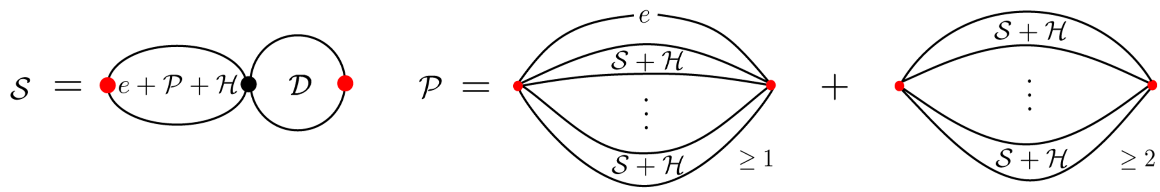

Figure 2.1: The decomposition of series and parallel networks.

have not constructed the network that consists of a single edge only. Thus, if we write e for the network consisting of a single edge, and e' for the graph in B_e that is a single edge, then $\mathcal{D}$ and $\mathcal{B}$ are related through

$$(\mathcal{D}-e)\times\mathcal{X}^2=(1+e)\times(\mathcal{B}_e-e').$$

On a generating function level, by using the relations described in the previous section, this translates to the relation

$$\frac{\partial B(x,y)}{\partial y}=\frac{x^2}{2}\frac{1+D(x,y)}{1+y}. \tag{2.5}$$

We now proceed with describing the decomposition for the class of networks in terms of 3-connected planar graphs. Following Tutte [23], any network is either an edge, whose end-vertices are the poles, or is in the class $\mathcal{S}$ (*series network*), or it is in $\mathcal{P}$ (*parallel network*), or is in class $\mathcal{H}$ (*core network*). The latter classes are disjoint, and thus we obtain the combinatorial composition

$$\mathcal{D}=e+\mathcal{S}+\mathcal{P}+\mathcal{H}. \tag{2.6}$$

Moreover, writing $S(x,y)$ for the egf enumerating $\mathcal{S}$, $P(x,y)$ for the egf enumerating $\mathcal{P}$ and $H(x,y)$ for the egf enumerating $\mathcal{H}$ we obtain the relation

$$D(x,y)=y+S(x,y)+P(x,y)+H(x,y). \tag{2.7}$$

The decomposition of series networks is as follows, see also Figure 2.1 (which is taken from [12]). Any network in $\mathcal{S}$ consists of two networks D_1 and D_2, such that the right pole of D_1 is identified with the left pole of D_2. Here, D_1 is restricted to be either an edge, or a network in $\mathcal{P}$ or in $\mathcal{H}$, and $D_2\in\mathcal{D}$. Hence,

$$\mathcal{S}=(e+\mathcal{P}+\mathcal{H})\times\mathcal{X}\times\mathcal{D} \quad\text{and}\quad S(x,y)=x(y+P(x,y)+H(x,y))D(x,y). \tag{2.8}$$

Any parallel network, see also Figure 2.1, consists either of an edge and a non-empty set of networks, either in $\mathcal{S}$ or in $\mathcal{H}$, where their right poles (left poles) are identified into a single right pole (left pole), or a set of networks with at least 2 two elements, either in $\mathcal{S}$ or in $\mathcal{H}$ where the identification of the poles is as before. Thus,

$$\mathcal{P}=e\times\mathrm{Set}_{\geq 1}(\mathcal{S}+\mathcal{H})+\mathrm{Set}_{\geq 2}(\mathcal{S}+\mathcal{H}), \tag{2.9}$$

and consequently

$$P(x,y)=y(e^{S(x,y)+H(x,y)}-1)+(e^{S(x,y)+H(x,y)}-S(x,y)-H(x,y)-1). \tag{2.10}$$

In order to complete the description of the decomposition, let us define the class of core networks. Recall that $\mathcal{T}$ denotes the class of all 3-connected planar graphs, and let $\bar{\mathcal{T}}$ be the class of networks that are created by taking any graph in $\mathcal{T}$, deleting an edge, and then turning its (former) endvertices into poles. Any network in $\mathcal{H}$, see also Figure 2.2, consists of a network from $\bar{\mathcal{T}}$, where each edge is replaced by a network whose poles are identified in a unique way with the endvertices of the edges. We thus obtain the relations

$$\mathcal{H} = \overline{\mathcal{T}} \,\tilde{\circ}\, \mathcal{D} \quad \text{and} \quad H(x,y) = \overline{T}(x, D(x,y)). \tag{2.11}$$

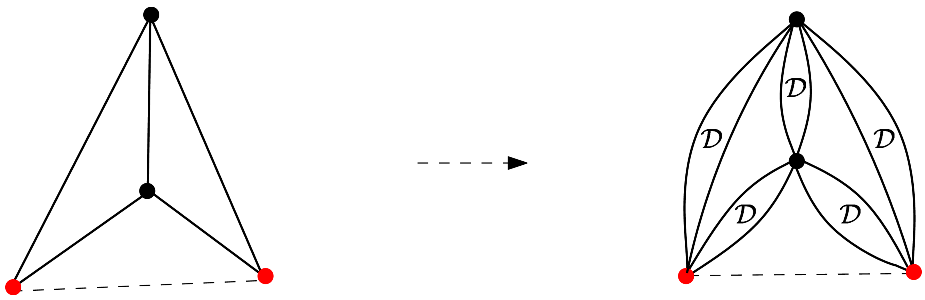

Figure 2.2: The decomposition of core networks.

This concludes the definition of the networks and the setup of the associated generating function. By a simple elimination procedure, see also [22], the Equations (2.7) - (2.11) can be reduced to one equation for $D(x,y)$:

$$D(x,y) = (1+y)\exp\left(\frac{xD(x,y)^2}{1+xD(x,y)} + \overline{T}(x, D(x,y))\right) - 1. \tag{2.12}$$

It is also known (see [14] and [5]) that $B(x,y)$ can be explicitly computed in terms of $D(x,y)$, that is, the integration in (2.5) can be made explicit. In particular, by abbreviating $D = D(x,y)$ we obtain that

$$B(x,y) = T(x,D) - \frac{xD}{2} + \frac{1}{2}\log(1+xD) + \frac{x^2}{2}\left(D + \frac{D^2}{2} + (1+D)\log\left(\frac{1+y}{1+D}\right)\right). \tag{2.13}$$

The last step that is necessary to complete the decomposition of the class of all connected planar graphs is to specify the class $\mathcal{T}$. We will not describe the decomposition here, as it is not needed for our further analysis. More details can be found in the work [19] of Mullin and Schellenberg, and the paper [4] by Bodirsky, Gröpl, Johannsen and Kang. However, we will use the associated generating functions, which satisfy the following equations. The generating function $\overline{T}(x,y)$ is given by

$$\overline{T}(x,y) = \frac{y}{2}\left(\frac{1}{1+xy} + \frac{1}{1+y} - 1 - \frac{(1+U(x,y))^2(1+V(x,y))^2}{(1+U(x,y)+V(x,y))^3}\right), \tag{2.14}$$

where $U(x,y)$ and $V(x,y)$ are defined by

$$U(x,y) = xy(1+V(x,y))^2 \quad \text{and} \quad V(x,y) = y(1+U(x,y))^2. \tag{2.15}$$

Similarly there is an explicit expression for $T(x,y)$ in terms of $U(x,y)$ of $V(x,y)$, see [5].

2.4 Singular Expansions and Asymptotics

A main feature of the use of generating functions in the asymptotic analysis of discrete structures is the fact that analytic properties, in particular the local behavior around singularities of $y(x) = \sum_n y_n x^n$, can be usually translated into asymptotic expansions for the coefficients $y_n = [x^n]y(x)$.

In this paper we will use the so-called Transfer Lemma by Flajolet and Odlyzko [11]. Let x_0 be a non-zero complex number, and ϵ and δ positive (real) numbers. Then the region

$$\Delta = \Delta(x_0, \epsilon, \delta) = \{x \in \mathbb{C} : |x| < x_0 + \epsilon,\ |\arg(x/x_0 - 1)| > \delta\}$$

is called a Δ-region. Suppose that a function $y(x)$ is analytic in a Δ-region $\Delta(x_0, \epsilon, \delta)$ and satisfies

$$y(x) = C\,(1 - x/x_0)^{\alpha} + O\left((1 - x/x_0)^{\beta}\right), \qquad x \in \Delta(x_0, \epsilon, \delta),$$

where $\beta > \alpha$ and α is a not a non-negative integer. Then we have

$$[x^n]\,y(x) = C\frac{n^{-\alpha-1}}{\Gamma(-\alpha)}x_0^{-n} + O\left(x_0^{-n}n^{\max\{-\alpha-2,-\beta-1\}}\right). \tag{2.16}$$

It is an important additional observation that the implicit constants are also effective which means that the O-constant in the expansion of $y(x)$ provides explicitly an O-constant for the expansion for $[x^n]\,y(x)$, see [11]. In particular it follows that singular expansions that are uniform in some parameter also translate into asymptotic expansions of the form (2.16) with a uniform error term.

A typical situation, where the Transfer Lemma applies, is a generating function with a so-called *squareroot singularity*, that is, we have a representation of the form

$$y(x) = g(x) - h(x)\sqrt{1 - \frac{x}{x_0}} \tag{2.17}$$

that holds in a (complex) neighborhood U of x_0 with $x_0 \neq 0$ (we only have to cut off the half lines $\{x \in \mathbb{C} : \arg(x/x_0 - 1) = 0\}$ in order to have an unambiguous value of the square root). The functions $g(x)$ and $h(x)$ are analytic in U. We also assume that $y(x)$ has an analytic continuation to the region $\{x \in \mathbb{C} : |x| < x_0 + \varepsilon\} \setminus U$ for some $\varepsilon > 0$. These assumptions imply that

$$y(x) = g(x_0) - h(x_0)\sqrt{1 - \frac{x}{x_0}} - x_0 g'(x_0)\left(1 - \frac{x}{x_0}\right) + O\left(\left|1 - \frac{x}{x_0}\right|^{3/2}\right)$$

uniformly in a Δ-region. Hence we have

$$y_n = [x^n]\,y(x) = \frac{h(x_0)}{2\sqrt{\pi}}n^{-3/2}x_0^{-n} + O\left(n^{-5/2}x_0^{-n}\right). \tag{2.18}$$

Note that if a function $y(x)$ of the form (2.17) then we can also represent it as

$$y(x) = \sum_{\ell \geq 0} a_\ell \left(1 - \frac{x}{x_0}\right)^{\ell/2} = \sum_{\ell \geq 0} a_\ell X^\ell, \tag{2.19}$$

where X abbreviates $X = \sqrt{1 - x/x_0}$ and the power series

$$\sum_{\ell \ge 0} a_\ell X^\ell$$

converges for $|X| < r$ (for some properly chosen $r > 0$), that is, it represents an analytic function in X. It is also clear that a representation of the form (2.19) can be rewritten into (2.17). We will call both representations as *singular expansions* of $y(x)$. If we are only interested in the first few terms then we write

$$y(x) = a_0 + a_1 X + a_2 X^2 + a_3 X^3 + O(X^4)$$

We will also encounter several situations where the coefficient $a_1 = 0$. Then $y(x)$ can be represented as

$$y(x) = \overline{g}(x) + \overline{h}(x) X^3 = \overline{g}(x) + \overline{h}(x) \left(1 - \frac{x}{x_0}\right)^{3/2}.$$

In this case the corresponding asymptotic expansion for the coefficients is of the form

$$y_n = \frac{3h(x_0)}{4\sqrt{\pi}} n^{-5/2} x_0^{-n} + O(n^{-7/2} x_0^{-n}).$$

Functions $y(x)$ with a squareroot singularity appear naturally as solutions of functional equations $\Phi(x, y(x)) = 0$, where $\Phi(x, y)$ is an analytic function (see [8]). More precisely if we know that there is x_0 and $y_0 = y(x_0)$ such that (x_0, y_0) is a regular point of $\Phi(x, y)$ with

$$\Phi(x_0, y_0) = 0 \quad \text{and} \quad \Phi_y(x_0, y_0) = 0 \tag{2.20}$$

and the conditions

$$\Phi_x(x_0, y_0) \ne 0 \quad \text{and} \quad \Phi_{yy}(x_0, y_0) \ne 0, \tag{2.21}$$

then x_0 is a singularity of $y(x)$ and there is a local representation of the form (2.17) with $g(x_0) = y_0$ and $h(x_0) = \sqrt{2x_0 \Phi_x(x_0, y_0)/\Phi_{yy}(x_0, y_0)}$.

Usually it is easy to very that $y(x)$ has an analytic continuation to a Δ-region. For example, if $\Phi(x, y)$ is of the form $\Phi(x, y) = y - F(x, y)$, where $F(0, y) = 0$ and $F(x, y) = \sum_{i,j} f_{ij} x^i y^j$ has non-negative coefficients f_{ij}, and where the power series solution $y(x) = \sum_n y_n z^n$ of $y = F(x, y)$ with $y(0) = 0$ has (at least) two non-zero coefficients y_{n_1}, y_{n_2} with $\gcd(n_1, n_2) = 1$, then there exist uniquely real positive x_0, y_0 with (2.20) and (2.21). Furthermore the gcd-conditions ensures that $|F_y(x, y(x))| < F_y(|x|, y(|x|))$ if x is not real and positive. Consequently it is impossible that $F_y(x, y(x)) = 1 = F(x_0, y_0)$ for $|x| \le x_0$ and $x \ne x_0$. Hence the implicit function theorem implies that there are no singularities in this range, and thus there is an analytic continuation to a Δ-region. Similar properties hold for solutions $\mathbf{y}(x) = (y_1(x), \ldots, y_d(x))$ of a system of equations $\mathbf{y}(x) = \mathbf{F}(x, \mathbf{y}(x))$, where $\mathbf{F}$ is *positive* and *strongly connected*. For details see [8].

If the functional equation has an additional analytic *parameter* u, that is, $y = y(x, u)$ satisfies $\Phi(x, u, y) = 0$, then we are in a situation that is relevant in this paper (the additional parameter will typically mark the number of edges and/or the degree of the root vertex). Then we (usually) have a local representation of the form

$$f(x, u) = g(x, u) - h(x, u) \sqrt{1 - \frac{x}{\rho(u)}} \tag{2.22}$$

that holds in a (complex) neighborhood U of (x_0, u_0) with $x_0 \neq 0$, $u_0 \neq 0$ and with $\rho(u_0) = x_0$ (we only have to cut off the half lines $\{x \in \mathbb{C} : \arg(x - \rho(u)) = 0\}$). The functions $g(x, u)$ and $h(x, u)$ are analytic in U and $\rho(u)$ is analytic in a neighborhood of u_0. As above it is usually easy to establish that $f(x, u)$ has an analytic continuation to the region $\{(x, u) \in \mathbb{C}^2 : |x| < x_0 + \varepsilon,\ |u| < u_0 + \varepsilon\} \setminus U$ for some $\varepsilon > 0$. Moreover, in complete analogy to the case without the additional parameter, a function $f(x, u)$ of the form (2.22) can be represented as

$$f(x, u) = \sum_{\ell \geq 0} a_\ell(u) \left(1 - \frac{x}{\rho(u)}\right)^{\ell/2} = \sum_{\ell \geq 0} a_\ell(u) X^\ell, \tag{2.23}$$

where $X = \sqrt{1 - x/\rho(u)}$ and where the coefficients $a_\ell(u)$ are analytic in u (for u close to u_0).

We recall that squareroot singularities appear if we consider solutions $y(x)$ with $y(x_0) = y_0$ of a functional equation $\Phi(x, y) = 0$, where (x_0, y_0) is a regular point of $\Phi(x, y)$. Of course, this will not remain true if (x_0, y_0) is a singularity of $\Phi(x, y)$. Nevertheless we will encounter several situations, where a special singular structure appears. The following lemma is [8, Theorem 2.31].

Theorem 2.1. *Suppose that $F(x, y, u)$ has a local representation of the form*

$$F(x, y, u) = g(x, y, u) + h(x, y, u) \left(1 - \frac{y}{r(x, u)}\right)^{3/2} \tag{2.24}$$

with functions $g(x, y, u)$, $h(x, y, u)$, $r(x, u)$ that are analytic around (x_0, y_0, u_0) and satisfy $g_y(x_0, y_0, u_0) \neq 1$, $h(x_0, y_0, u_0) \neq 0$, $r(x_0, u_0) \neq 0$ and $r_x(x_0, u_0) \neq g_x(x_0, y_0, u_0)$. Furthermore, suppose that $y = y(x, u)$ is a solution of the functional equation

$$y = F(x, y, u)$$

with $y(x_0, u_0) = y_0$. Then $y(x, u)$ has a local representation of the form

$$y(x, u) = g_1(x, u) + h_1(x, u) \left(1 - \frac{x}{\rho(u)}\right)^{3/2}, \tag{2.25}$$

where $g_1(x, u)$, $h_1(x, u)$ and $\rho(u)$ are analytic at (x_0, u_0) and satisfy $h_1(x_0, u_0) \neq 0$ and $\rho(u_0) = x_0$.

2.5 Asymptotics for the Number of Planar Graphs

The analysis of the system of equations for the generating functions $B(x, y)$ and $C(x, y)$, as described in Section 2.3, can be used to obtain asymptotic formulas for the numbers b_n and c_n of 2-connected and connected planar graphs, see the paper by [14]. Since we will use some of the proof methods in the analysis of the root degree we also give a short proof.

Lemma 2.2. *The generating functions $B(x)$ and $C(x)$ for planar graphs have finite radii of convergence ρ_B and ρ_C, respectively, and have local representations of the forms*

$$B(x) = g_2(x) + h_2(x) \left(1 - \frac{x}{\rho_B}\right)^{5/2} \quad \textit{and} \quad C(x) = g_4(x) + h_4(x) \left(1 - \frac{x}{\rho_C}\right)^{5/2}$$

with functions $g_2(x)$, $h_2(x)$ and $g_4(x)$, $h_4(x)$ that are non-zero and analytic at ρ_B and ρ_C, respectivley, and $B(x)$ and $C(x)$ have analytic continuations to proper Δ-regions. In particular, if $t(y)$ is given by the equation

$$y = \frac{1+2t}{(1+3t)(1-t)} \exp\left(-\frac{t^2(1-t)(18+36t+5t^2)}{2(3+t)(1+2t)(1+3t)^2}\right) - 1 \qquad (2.26)$$

then $\rho_B = (1+3t(1))(1-t(1))^3/(16t(1)^3)$ and $\rho_C = \rho_B e^{-B'(\rho_B,1)}$.

Consequently we have

$$b_n = b \cdot n^{-7/2} \rho_B^{-n}\, n! \left(1 + O\left(\frac{1}{n}\right)\right) \quad \text{and} \quad c_n = c \cdot n^{-7/2} \rho_C^{-n}\, n! \left(1 + O\left(\frac{1}{n}\right)\right),$$

where b, c are positive constants.

Proof. The main part of the proof is to characterize the kind of singularities of the generating functions. The analytic continuation to proper Δ-regions is always straightforward to establish (see also [2]).

First, it follows from the fact that U and V satisfy a positive systems of equations (see [8]) that U and V have a singular expansion of the form[2]

$$U(x,z) = u_0(x) + u_1(x)Z + u_2(x)Z^2 + u_3(x)Z^3 + O(Z^3),$$
$$V(x,z) = v_0(x) + v_1(x)Z + v_2(x)Z^2 + v_2(x)Z^3 + O(Z^3),$$

where $Z = \sqrt{1 - \frac{z}{\tau(x)}}$. Moreover, it follows that the function $u_0(x)$ is the solution of the equation

$$x = \frac{(1+u)(3u-1)^3}{16u}$$

and $\tau(x)$ is given by

$$\tau(x) = \frac{1}{(4x^2(1+u_0(x)))^{2/3}}.$$

The functions $u_j(x)$ and $v_j(x)$ are also analytic and can be explicitly given in terms of $u = u_0(x)$. With the help of these expansions it follows that there is a cancellation of the coefficient of Z in the expansion of

$$\frac{(1+U)^2(1+V)^2}{(1+U+V)^3} = E_0 + E_2 Z^2 + E_3 Z^3 + O(Z^4),$$

Thus, $\overline{T}(x,z)$ can be represented as

$$\overline{T}(x,z) = T_0(x) + T_2(x)Z^2 + T_3(x)Z^3 + O(Z^4).$$

Next we use (2.12)

$$D = (1+y)\exp\left(\frac{xD^2}{1+xD} + \overline{T}(x,D)\right) - 1 = \Phi(x,y,D)$$

[2] Actually the two equations for U and V can be reduced to single equation – e.g. $U = xy(1+y(1+U)^2)$ – so that we can apply directly the methods explained in Section 2.4. In particular it follows that U and V have analytic extensions to proper Δ-regions.

and suppose that y equals 1 (or is very close to 1). Due to the singular structure of the right hand side we can apply Theorem 2.1 and obtain a local expansion for $D = D(x, y)$ of the form

$$D(x, y) = D_0(y) + D_2(y)X^2 + D_3(y)X^3 + O(X^4), \tag{2.27}$$

where

$$X = \sqrt{1 - \frac{x}{\rho_D(y)}}$$

for some function $\rho_D(y)$. In fact, we can be much more precise. Let $t = t(y)$ be defined by (2.26) that exists in a suitable neighborhood of $y = 1$. Then $\rho_D(y)$ is given by

$$\rho_D(y) = \frac{(1 + 3t(y))(1 - t(y))^3}{16t(y)^3},$$

in particular $\rho_D = \rho_D(1) = 0.038191...$. There are several ways to show that $D(x, y)$ extends analytically to a Δ-region. One way is to rewrite the system of equations (2.8), (2.10), (2.11) explicitly into one equation of the form $f(x, y) = F(x, y, f(x, y))$ for the function $f(x, y) = S(x, y) + H(x, y)$, where F has non-negative coefficients. It is easy to check that $F_f(x_0, 1, f(x_0, 1)) < 1$ which implies that $|F_f(x, y, f(x, y))| < 1$ for $|x| \le x_0$ and $|y| \le 1$. By the implicit function theorem there is an analytic continuation to a proper Δ-region for $f(x, y) = S(x, y)+H(x, y)$ and consequently for $D(x, y) = y+f(x, y)+y(e^{f(x,y)}-1)+e^{f(x,y)}-1-f(x, y)$.

The representation (2.27) provides a local expansion for $\frac{\partial B(x,y)}{\partial y}$ of the form

$$\begin{aligned}\frac{\partial B(x, y)}{\partial y} &= \overline{B}_0(y) + \overline{B}_2(y)X + \overline{B}_3(y)X^3 + O(X^4) \\ &= g_1(x, y) + h_1(x, y)X^3\end{aligned}$$

with certain analytic functions $g_1(x, y)$ and $h_1(x, y)$. Hence, by integration (see [8]) or by using the representation (2.13), where one has to check that the coefficients of X and X^3 disappear, $B(x, y)$ and consequently $\frac{\partial B(x,y)}{\partial x}$ have an expansions of the form

$$\begin{aligned}B(x, y) &= g_2(x, y) + h_2(x, y)X^5, \\ \frac{\partial B(x, y)}{\partial x} &= g_3(x, y) + h_3(x, y)X^3\end{aligned}$$

with certain analytic functions $g_2(x, y)$, $g_3(x, y)$ and $h_2(x, y)$, $h_2(x, y)$. Note that $\rho_B = \rho_D$ and the analytic continuation property of $D(x, y)$ implies a corresponding property for $B(x, y)$.

Finally we have to solve (2.4). For simplicity set $y = 1$. Since $\rho_D\, B''(\rho_D) \approx 0.0402624 < 1$, the singularity of the right-hand-side induces the singular behaviour of the solution $xC'(x)$. Actually we just have to apply Theorem 2.1 and obtain a local expansion for $C'(x)$ of the form

$$C'(x) = g_3(x) + h_3(x)\left(1 - \frac{x}{\rho_C}\right)^{3/2}, \tag{2.28}$$

where $\rho_C = \rho_B e^{-B'(\rho_B)} = 0.0367284...$, and consequently we obtain corresponding representations for

$$C(x) = g_4(x) + h_4(x)\left(1 - \frac{x}{\rho_C}\right)^{5/2}.$$

Note that the condition $\rho_D\, B''(\rho_D) \approx 0.0402624 < 1$ also ensures that $C'(x)$ (and also $C(x)$) has no other singularity for $|x| \le \rho_C$ which implies that $C'(x)$ (and $C(x)$) has an analytic continuation to a Δ-region.

Using these representations the asymptotic expansion for b_n and c_n follow immediately by the Transfer Lemma of Flajolet and Odlyzko [11].

□

2.6 Generating Functions for the Root Degree

In this section we extend the results from Section 2.3 to incorporate into the generating functions also the root degree. Let us begin with the case of connected planar graphs. Recall (2.3), which says that any vertex-derived connected planar graph C' can be decomposed as a set $\{B'_1, \ldots, B'_\ell\}$ of vertex-derived 2-connected graphs, whose roots are identified into a single vertex, and where each other vertex is substituted by a vertex-rooted connected graph. In other words, the root degree of C' equals the sum of the root degrees of the $(B_i)_{1\le i\le \ell}$. This translates immediately to the functional relation

$$C'(x,y,w) = \exp\left(B'\left(C^\bullet(x,y),y,w\right)\right). \tag{2.29}$$

In [7] it was shown that also the remaining steps of the decomposition can be translated into corresponding relations for the generating functions that also take into account the root degree. We omit the details here, and just state the results. The generating functions for $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{T}$ satisfy the relations

$$\frac{\partial B'(x,y,w)}{\partial w} = xy\frac{1+D(x,y,w)}{1+yw} \tag{2.30}$$

$$D(x,y,w) = (1+yw)\exp\left(\frac{xD(x,y,w)D(x,y,1)}{1+xD(x,y,1)} + \overline{T}\left(x, D(x,y,1), \frac{D(x,y,w)}{D(x,y,1)}\right)\right) - 1, \tag{2.31}$$

$$\overline{T}(x,y,w) = \frac{yw}{2}\left(\frac{1}{1+wy} + \frac{1}{1+xy} - 1 - \right. \tag{2.32}$$
$$\left. - \frac{(U+1)^2\left(-w_1(U,V,w) + (U-w+1)\sqrt{w_2(U,V,w)}\right)}{2w(Vw+U^2+2U+1)(1+U+V)^3}\right),$$

with polynomials $w_1 = w_1(U,V,w)$ and $w_2 = w_2(U,V,w)$ given by

$$\begin{aligned} w_1 &= -UVw^2 + w(1+4V+3UV^2+5V^2+U^2+2U+2V^3+3U^2V+7UV) \\ &\quad + (U+1)^2(U+2V+1+V^2), \\ w_2 &= U^2V^2w^2 - 2wUV(2U^2V+6UV+2V^3+3UV^2+5V^2+U^2+2U \\ &\quad + 4V+1) + (U+1)^2(U+2V+1+V^2)^2. \end{aligned}$$

Again it is possible to integrate $\frac{\partial B'(x,y,w)}{\partial w}$ and one obtains the following expression (see [7]):

$$B'(x,y,w) = x\left(D - \frac{xED}{1+xE}\left(1+\frac{D}{2}\right)\right) - x(1+D)\overline{T}(x,E,D/E) + x\int_0^D \overline{T}'(x,E,t/E)\,dt, \tag{2.33}$$

where for simplicity we let $D = D(x, y, w)$ and $E = E(x, y) = D(x, y, 1)$ and the remaining integral can be computed with the help of the following lengthy relation:

$$\int_0^D \overline{T}(x, E, t/E)\, dt = -\frac{(xED^2 - 2D - 2xED + (2 + 2xE)\log(1 + D))}{4(1 + xE)}$$
$$-\frac{uv}{2x(1 + u + v)^3}\Bigg(\frac{D/E(2u^3 + (6v + 6)u^2 + (6v^2 - vD/E + 14v + 6)u + 4v^3 + 10v^2 + 8v + 2)}{4v(v + 1)^2}$$
$$+\frac{(1 + u)(1 + u + 2v + v^2)(2u^3 + (4v + 5)u^2 + (3v^2 + 8v + 4)u + 2v^3 + 5v^2 + 4v + 1)}{4uv^2(v + 1)^2}$$
$$-\frac{\sqrt{Q}(2u^3 + (4v + 5)u^2 + (3v^2 - vD/E + 8v + 4)u + 5v^2 + 2v^3 + 4v + 1)}{4uv^2(v + 1)^2}$$
$$+\frac{(1 + u)^2(1 + u + v)^3 \log(Q_1)}{2v^2(1 + v)^2}$$
$$+\frac{(u^3 + 2u^2 + u - 2v^3 - 4v^2 - 2v)(1 + u + v)^3 \log(Q_2)}{2v^2(1 + v)^2 u}\Bigg),$$

where the expressions Q, Q_1 and Q_2 are given by

$$Q = u^2v^2D^2/E^2 - 2uvD/E(u^2(2v + 1) + u(3v^2 + 6v + 2) + 2v^3 + 5v^2 + 4v + 1)$$
$$+ (1 + u)^2(u + (v + 1)^2)^2$$
$$Q_1 = \frac{1}{2(Dv/E + (u + 1)^2)^2(v + 1)(u^2 + u(v + 2) + (v + 1)^2)}\big(-uvD/E(u^2 + u(v + 2) + 2v^2 + 3v + 1)$$
$$+(u + 1)(u + v + 1)\sqrt{Q} + (u + 1)^2(2u^2(v + 1) + u(v^2 + 3v + 2) + v^3 + 3v^2 + 3v + 1\big)$$
$$Q_2 = \frac{-Duv/E + u^2(2v - 1) + u(3v^2 + 6v + 2) + 2v^3 + 5v^2 + 4v + 1 - \sqrt{Q}}{2v(u^2 + u(v + 2) + (v + 1)^2)}$$

and u and v abbreviate $u = U(x, E)$ and $v = V(x, E)$.

2.7 A Boltzmann Sampler for Networks

In this section we describe a Boltzmann sampler for the class of planar networks, which plays a central role in our study of the maximum degree of random planar graphs, see Section 4. This sampler was already developed in [12] for general classes that can be decomposed into 3-connected components. We repeat here the exposition, tailored to our case, as several details are important in our proofs.

Let us start with the Boltzmann sampler for the class $\mathcal{D}$ of all networks. Recall (2.6), which says that $\mathcal{D}$ is the disjoint union of the classes e (single edge), $\mathcal{S}$ (series networks), $\mathcal{P}$ (parallel networks), and $\mathcal{H}$ (core networks). By applying the construction rule for Boltzmann samplers from Section 2.2 for the disjoint union of classes, we see that a Boltzmann sampler for $\mathcal{D}$ will call a sampler for a subclass with a probability proportional to the value of the generating function of this subclass. More precisely, we say that a variable X is *network-distributed* with parameters x and y, $X \sim \mathtt{Net}(x, y)$, if its domain is the set of symbols $\Omega_{\mathtt{Net}} = \{e, S, P, H\}$ and for any $s \in \Omega_{\mathtt{Net}}$ it holds $\Pr[X = s] = \frac{s(x,y)}{N(x,y)}$. Then the sampler $\Gamma D(x, y)$ with parameters x, y for $\mathcal{D}$ can be described concisely as follows, where $\Gamma e, \Gamma S, \Gamma P$, and ΓH are (yet to be defined) Boltzmann samplers for the classes $e, \mathcal{S}, \mathcal{P}$, and $\mathcal{H}$.

$\Gamma D(x,y):\quad s \leftarrow \texttt{Net}(x,y)$
$\qquad\qquad\quad$ **return** $\Gamma s(x,y)$

Next we describe the sampler for $\mathcal{S}$. The combinatorial relation in (2.8), see also Figure 2.1, implies that $\mathcal{S} = \mathcal{A} \times \mathcal{X} \times \mathcal{D}$, where $\mathcal{A} = e + \mathcal{P} + \mathcal{H}$. By combining the construction rules for Boltzmann samplers from Section 2.2 for the disjoint union and the product of classes, we conclude that a Boltzmann sampler for $\mathcal{S}$ proceeds in the following way. It first samples a network from $\mathcal{A}$, by making a "three-way" Bernulli choice among e, $\mathcal{P}$, and $\mathcal{H}$ with the appropriate probabilities, and generates a Boltzmann distributed object N_1 from the chosen class. Then, it generates a network N_2 that is Boltzmann distributed from $\mathcal{D}$. Finally, it creates and returns a network (N_1, N_2) such that the right pole of N_1 is identified with the left pole of N_2, and in which the labels are distributed randomly. More formally, we say that a variable X is *series-distributed* with parameters x and y, $X \sim \texttt{Ser}(x,y)$, if its domain is the set of symbols $\Omega_{\texttt{Ser}} = \{e, P, H\}$ and for any $s \in \Omega_{\texttt{Ser}}$ it holds $\Pr[X = s] = \frac{s(x,y)}{S(x,y)}$. Then $\Gamma S(x,y)$ can be described concisely as follows:

$\Gamma S(x,y):\quad s \leftarrow \texttt{Ser}(x,y)$
$\qquad\qquad\quad N_1 \leftarrow \Gamma s(x,y)$
$\qquad\qquad\quad N_2 \leftarrow \Gamma D(x,y)$
$\qquad\qquad\quad$ **return** (N_1, N_2), relabeling randomly its non-pole vertices

We proceed with the description of the sampler for $\mathcal{P}$. The combinatorial relation 2.9, see also Figure 2.1, guarantees that $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$, where $\mathcal{P}_1 = e \times \mathrm{Set}_{\geq 1}(\mathcal{S}+\mathcal{H})$ and $\mathcal{P}_2 = \mathrm{Set}_{\geq 2}(\mathcal{S}+\mathcal{H})$. Together with the rules for Boltzmann samplers from Section 2.2 for disjoint union and set, this implies that $\Gamma P(x,y)$ first makes a Bernulli choice between $\mathcal{P}_1$ and $\mathcal{P}_2$, and then samples a set (with a given lower bound on the number of elements) of networks from $\mathcal{S}$ or $\mathcal{H}$ according to the Boltzmann distribution.

Let us introduce some notation before we describe formally the sampler. We say that a variable X is *parallel-distributed* with parameters x and y, and write $X \sim \texttt{Par}(x,y)$, if $X \sim 1 + \texttt{Be}(\frac{e^{S(x,y)+H(x,y)} - 1 - S(x,y) - H(x,y)}{P(x,y)})$. Moreover, we say that a variable is *sh-distributed* with parameters x and y, $X \sim \texttt{sh}(x,y)$, if its domain is the set of symbols $\Omega_{\texttt{sh}} = \{S, H\}$ and for $s \in \Omega_{\texttt{sh}}$ it holds $\mathbb{P}(X = s) = \frac{s(x,y)}{S(x,y)+H(x,y)}$. In other words, X "distinguishes" between the two possibilities in the definition a parallel network. Using $\texttt{Po}_{\geq p}(\lambda)$ to denote a Poisson distributed random variable with parameter λ conditioned on being at least p, the Boltzmann sampler ΓP works as follows.

$\Gamma P(x,y):\quad p \leftarrow \texttt{Par}(x,y)$
$\qquad\qquad\quad k \leftarrow \texttt{Po}_{\geq p}(S(x,y) + H(x,y))$
$\qquad\qquad\quad$ **for** $i = 1 \ldots k$
$\qquad\qquad\qquad b_i \leftarrow \texttt{sh}(x,y)$
$\qquad\qquad\qquad p_i \leftarrow \Gamma b_i(x,y)$
$\qquad\qquad\quad$ construct a network P by identifying the left and right poles of $p_1, \ldots, p_k$
$\qquad\qquad\quad$ relabel randomly the non-pole vertices of P
$\qquad\qquad\quad$ **if** $p = 1$ **then return** P, where the poles are joined by an edge
$\qquad\qquad\quad$ **else return** P

Finally, we describe the sampler for $\mathcal{H}$. Recall (2.11), which guarantees that a $\mathcal{H}$-network is obtained by substituting the edges of some network from $\bar{\mathcal{T}}$ by graphs from $\mathcal{D}$. Here we will assume that we have an auxiliary sampler $\Gamma\bar{T}(x,y)$, which samples graphs from $\bar{\mathcal{T}}$ according to the Boltzmann distribution. Then the sampler for $\mathcal{H}$ can be described as follows.

$\Gamma H(x,y):$ $T \leftarrow \Gamma\bar{T}(x, N(x,y))$
foreach edge e of T
  $\gamma_e \leftarrow \Gamma N(x,y)$
replace every e in T by γ_e
return T, relabeling randomly its non-pole vertices

This completes the description of the samplers. The next lemma was shown in [12], and it can be proved in the present case directly by using the asymptotic enumeration results for 2-connected planar graphs, as obtained by Bender, Gao and Wormald [2], or by using Lemma 2.2. The proof is included for completeness.

Lemma 2.3. *Let $x, y \geq 0$ be such that $D(x,y) < \infty$. Then $\Gamma D(x,y)$ is a Boltzmann sampler with parameters x and y for $\mathcal{D}$. Moreover,*

$$\Pr[\Gamma D(\rho_D, 1) \in \mathcal{D}_n] = \Theta(n^{-5/2}),$$

where $\rho_D = \rho_B$ denotes the singularity of $D(x,1)$ and ρ_B is gi ven in Lemma 2.2.

Proof. Recall Equation (2.27), which says that

$$D(x) = D_0 + D_2\,(1 - x/\rho_D) + D_3\,(1 - x/\rho_D)^{3/2} + O\left((1 - x/\rho_D)^2\right).$$

Moreover, the discussion after (2.27) guarantees that $D(x)$ is analytic in an appropriate Δ-domain. Thus, the Tranfer Lemma applies, implying that

$$|\mathcal{D}_n| = n!\,[x^n]D(x) = \Theta(1) \cdot n^{-5/2}\,\rho_D^{-n}\,n!.$$

The definition of the Boltzmann model then implies that

$$\Pr[\Gamma D(\rho_D, 1) \in \mathcal{D}_n] = |\mathcal{D}_n| \cdot \frac{\rho_D^n}{n!\,D(\rho_D, 1)} = \Theta(n^{-5/2}),$$

as claimed. □

In other words, if we choose $(x,y) = (\rho_B, 1)$, then $\Gamma N(x,y)$ has a polynomially small probability of generating a network of a given size n. This important fact will be used in Section 4.

3 The Upper Bound

3.1 Generating Functions and the First Moment Method

In order to obtain an upper bound for the distribution of the maximum degree we use the first moment method. Let $X_{n,k}$ denote the (random) number of vertices of degree k in a 2-connected random planar graph of size n and let

$$Y_{n,k} = \sum_{\ell > k} X_{n,k}$$

denote the number of vertices of degree larger than k. Obviously, we have

$$\Delta(B_n) > k \quad \Longleftrightarrow \quad Y_{n,k} > 0$$

and consequently

$$\Pr\left[\Delta(B_n) > k\right] = \Pr\left[Y_{n,k} > 0\right] \le \mathbb{E}\, Y_{n,k}.$$

Let $d_{n,k}$ denote the probability that the root degree (in a 2-connected graph of size n) equals k, then $\mathbb{E}\, X_{n,k} = np_{n,k}$. Hence, it is sufficient to provide upper bounds of

$$p_{n,k} = \frac{[x^n w^k]\, B'(x,1,w)}{[x^n]\, B'(x)}.$$

The asymptotic expansion of

$$[x^n]\, B'(x) \sim c \cdot n^{-5/2} \rho_B^{-n}$$

is known, where $c > 0$ and $\rho_B = 0.03672841...$, see [2, 14] or Section 2.5. This follows from a precise analysis of the singularity of $B'(x)$ which is of the form

$$B'(x) = g(x) + h(x) \left(1 - \frac{x}{\rho_B}\right)^{3/2}.$$

Consequently, we just need upper bounds for $[x^n w^k] B'(x,1,w)$. Suppose that $w_0 > 0$ is chosen in a way that $B'(x,1,w_0)$ is a convergent power series. Then we have

$$[x^n w^k]\, B'(x,1,w) \le w_0^{-k} [x^n]\, B'(x,1,w_0).$$

Actually it will turn out that we can choose $w_0 > 1$ in an "optimal way" so that $B'(x,1,w_0)$ has the same radius of convergence ρ_B as $B'(x)$ and also the same kind of singularity.

Lemma 3.1. *Let $t(y)$ be given by (2.26) and set*

$$w_0 = \frac{1}{1-t(1)} \exp\left(\frac{t(1)(t(1)-1)(t(1)+6)}{6t(1)^2+20t(1)+6}\right) - 1 \approx 1.48488989 \tag{3.1}$$

Then $B'(x,1,w_0)$ has a local representation of the form

$$B'(x,1,w_0) = \overline{g}(x) + \overline{h}(x) \left(1 - \frac{x}{\rho_B}\right)^{3/2},$$

with functions $\overline{g}(x)$, $\overline{h}(x)$ that are non-zero and analytic at ρ_B. Furthermore

$$[x^n]\, B'(x,1,w_0) \sim \overline{c} \cdot n^{-5/2} \rho_B^{-n}$$

for some constant $\overline{c} > 0$.

We recall that $q = w_0$ is the radius of convergence of the generating function $\sum_{k\ge1} d_k w^k$ of the limiting degree distribution of 2-connected planar graphs (see [7]). Summing up we have

$$\mathbb{E}\, X_{n,k} = O\left(nq^{-k}\right)$$

and consequently

$$\Pr\left[\Delta(B_n) > k\right] = O\left(nq^{-k}\right).$$

Of course, this estimate provides the upper bound in Theorem 1.3. The proof of the lower bound in Theorem 1.1 is precisely the same, we just have to replace "2-connected" by "connected".

Lemma 3.2. *Let w_0 be the same constant as in Lemma 3.1. Then $C'(x, 1, w_0)$ has a local representation of the form*

$$C'(x, 1, w_0) = \overline{g}_2(x) + \overline{h}_2(x) \left(1 - \frac{x}{\rho_C}\right)^{3/2},$$

with functions $\overline{g}_2(x)$, $\overline{h}_2(x)$ that are non-zero and analytic at ρ_C. Furthermore

$$[x^n]\, C'(x, 1, w_0) \sim \overline{c}_2 \cdot n^{-5/2} \rho_C^{-n}$$

for some constant $\overline{c}_2 > 0$.

3.2 Singular Functional Equations

First we have a closer look at the equation (2.31). If we set $w = 1$ then it reduces to an equation for $D(x, y, 1)$ which is precisely the same equation as (2.12). In order to avoid conflicts with the notation we set $E(x, y) := D(x, y, 1)$. From (2.27) we know the analytic behaviour of $E(x, y)$ around its dominant singularity:

$$E(x, y) = E_0(y) + E_2(y)X^2 + E_3(y)X^3 + O(X^4), \tag{3.2}$$

where

$$X = \sqrt{1 - \frac{x}{\rho_D(y)}}$$

Recall that the coefficient of the squareroot term X vanishes. Since we are not interested in the number of edges we will set $y = 1$ in (most of) the following calculations. The most important step in our analysis is the discussion of the equation (2.31). First, we rewrite it to

$$D + 1 = \exp\left(G(x, D, w, E, U, V) + H(x, D, E, U, V)\sqrt{J(D, E, U, V)}\right),$$

where

$$\begin{aligned} G &= \log(1 + w) + \frac{xDE}{1 + xE} \\ &\quad + \frac{D}{2}\left(\frac{1}{1 + D} + \frac{1}{1 + xE} - 1 + \frac{(U + 1)^2 w_1(U, V, D/E)}{2D/E(VD/E + U^2 + 2U + 1)(1 + U + V)^3}\right), \\ H &= -\frac{(U + 1)^2 D(U - D/E + 1)}{4D/E(VD/E + U^2 + 2U + 1)(1 + U + V)^3}, \\ J &= w_2(U, V, D/E). \end{aligned}$$

In the following analysis we will consider first E, U, V as *new variables*, in particular when we apply Lemma 3.3. Finally, we will substitute them by $E = E(x, 1)$, $U = U(x, 1)$, $V = V(x, 1)$.

Set

$$
\begin{aligned}
t_0 &= t(1) \approx 0.626371, \\
x_0 &= \rho_D(1) = \frac{(3t_0+1)(1-t_0)^3}{16t_0^3} \approx 0.038191, \\
D_0 &= \frac{t_0}{1-t_0} \approx 1.676457 \\
w_0 &= \frac{1}{1-t_0} \exp\left(\frac{t_0(t_0-1)(t_0+6)}{6t_0^2+20t_0+6}\right) - 1 \approx 1.48488989 \\
E_0 &= E(x_0,1,1) = \frac{3t_0^2}{(1-t_0)(3t_0+1)} \approx 1.094175, \\
U_0 &= \frac{1}{3t_0} \approx 0.532166, \\
V_0 &= \frac{1+3t_0}{3(1-t_0)} \approx 2.568609.
\end{aligned}
$$

Then we actually have

$$H(x_0, D_0, E_0, U_0, V_0) = J(D_0, E_0, U_0, V_0) = 0,$$

which can easily be checked by writing $H(x_0, D_0, w_0, E_0, U_0, V_0)$ and $J(D_0, E_0, U_0, V_0)$ in terms of t_0. Hence, we are in a situation, where the following Lemma 3.3 applies.

Lemma 3.3. *Let $\mathbf{v} = (v_1, \dots, v_d)$ be a d-dimensional complex vector and let $y = y(\mathbf{v})$ be a function with $y(\mathbf{v}_0) = y_0$ that satisfies a functional equation*

$$R(y,\mathbf{v})^2 + S(y,\mathbf{v}) = 0, \tag{3.3}$$

where $R(y,\mathbf{v})$ and $S(y,\mathbf{v})$ are analytic functions at $(y_0,\mathbf{v}_0)$ such that

$$R(y_0,\mathbf{v}_0) = S(y_0,\mathbf{v}_0) = 0$$

and, in addition, all the partial derivatives of S up to order 2 are zero at $(y_0,\mathbf{v}_0)$, and $R_y(y_0,\mathbf{v}_0) \neq 0$. Then, $y(\mathbf{v})$ has a local representation of the form

$$y(\mathbf{v}) = P(\mathbf{v}) + \sqrt{Q(\mathbf{v})}, \tag{3.4}$$

where P and Q are analytic at $\mathbf{v}_0$ with $P(\mathbf{v}_0) = Q(\mathbf{v}_0) = 0$, and Q and all its partial derivatives up to order 2 are zero at $\mathbf{v}_0$. Furthermore, the evaluations of the partial derivatives Q_{xxx}, Q_{xxw} and Q_{xwz} at $(\mathbf{v}_0)$ for any variables x, w, z of $\mathbf{v}$ are

$$
\begin{aligned}
Q_{xxx} &= \frac{R_x^3 S_{yyy} - 3R_x^2 R_y S_{xyy} + 3R_x R_y^2 S_{xxy} - R_y^3 S_{xxx}}{R_y^5}, \\
Q_{xxw} &= \frac{1}{R_y^5}\left(R_x^2 R_w S_{yyy} - 2R_x R_w R_y S_{xyy} + 2R_x R_y^2 S_{xwy}\right. \\
&\quad \left. - R_x^2 R_y S_{wyy} + R_w R_y^2 S_{xxy} - R_y^3 S_{xxw}\right), \\
Q_{xwz} &= \frac{1}{R_y^5}\left(R_x R_w R_z S_{yyy} - R_w R_z R_y S_{xyy} - R_x R_z R_y S_{wyy}\right. \\
&\quad \left. - R_x R_w R_y S_{zyy} + R_w R_y^2 S_{xzy} + R_z R_y^2 S_{xwy} + R_x R_y^2 S_{wzy} - R_y^3 S_{xwz}\right).
\end{aligned}
$$

Proof. Set

$$F(y, \mathbf{v}) := R(y, \mathbf{v})^2 + S(y, \mathbf{v}). \tag{3.5}$$

By the assumptions we have

$$\begin{aligned} F(y_0, \mathbf{v}_0) &= 0, \\ F_y(y_0, \mathbf{v}_0) &= 0, \\ F_{yy}(y_0, \mathbf{v}_0) &= 2R_y(y_0, \mathbf{v}_0)^2 \neq 0. \end{aligned}$$

Hence, by the Weierstrass preparation theorem, there exist analytic functions $p = p(\mathbf{v})$, $q = q(\mathbf{v})$, and $K = K(y, \mathbf{v})$ with $p(\mathbf{v}_0) = q(\mathbf{v}_0) = 0$ and $K(y_0, \mathbf{v}_0) \neq 0$ such that

$$F(y, \mathbf{v}) = K(y, \mathbf{v}) \left((y - y_0)^2 + p(\mathbf{v})(y - y_0) + q(\mathbf{v}) \right). \tag{3.6}$$

Consequently, the original equation (3.3) is equivalent to

$$(y - y_0)^2 + p(\mathbf{v})(y - y_0) + q(\mathbf{v}) = 0$$

and, thus, we obtain Equation (3.4) with

$$P(\mathbf{v}) = y_0 - \frac{p(\mathbf{v})}{2} \quad \text{and} \quad Q(\mathbf{v}) = \frac{p(\mathbf{v})^2}{4} - q(\mathbf{v}).$$

We now compute the partial derivatives of $Q(\mathbf{v})$. The basic idea is to differenciate both Equations (3.5) and (3.6), and to rewrite the partial derivatives of $p(\mathbf{v})$ and $q(\mathbf{v})$ in terms of those of $R(y, \mathbf{v})$ and $S(y, \mathbf{v})$. In what follows, all functions are evaluated at $(y_0, \mathbf{v}_0)$ or $(\mathbf{v}_0)$, and the symbols x, w, z denote any three variables of $\mathbf{v}$.

First observe that, due to Equation (3.5) and the fact that $R = S = 0$, the first partial derivatives of $F(y, \mathbf{v})$ vanish,

$$F_y = 0, \qquad F_x = 0, \tag{3.7}$$

and that the second derivatives of $F(y, \mathbf{v})$ are given by

$$\begin{aligned} F_{yy} &= 2R_y^2, & F_{xy} &= 2R_x R_y, \\ F_{xx} &= 2R_x^2, & F_{xw} &= 2R_x R_w. \end{aligned} \tag{3.8}$$

Next, by using Equation (3.6) and $p = q = 0$, we obtain that

$$\begin{aligned} F_y &= 0, & F_x &= K q_x, \\ F_{yy} &= 2K, & F_{xy} &= K_y q_x + K p_x, \\ F_{xx} &= 2K_x q_x + K q_{xx}, & F_{xw} &= K_x q_w + K_w q_x + K q_{xw}. \end{aligned} \tag{3.9}$$

Hence from Equations (3.7), (3.8) and (3.9) we derive that $K = R_y^2$, and that

$$\begin{aligned} q_x &= 0, & p_x &= 2\frac{R_x}{R_y}, \\ q_{xx} &= 2\frac{R_x^2}{R_y^2}, & q_{xw} &= 2\frac{R_x R_w}{R_y^2}. \end{aligned}$$

Consequently,

$$\begin{aligned}
Q_x &= \frac{pp_x}{2} - q_x = 0, \\
Q_{xx} &= \frac{pp_{xx} + p_x^2}{2} - q_{xx} = 2\frac{R_x^2}{R_y^2} - 2\frac{R_x^2}{R_y^2} = 0, \\
Q_{xw} &= \frac{pp_{xw} + p_x p_w}{2} - q_{xw} = 2\frac{R_x R_w}{R_y^2} - 2\frac{R_x R_w}{R_y^2} = 0,
\end{aligned}$$

as claimed. Finally, it remains to obtain the values of Q_{xxx}, Q_{xxw} and Q_{xwz} in terms of the partial derivatives of H and J. To compute

$$Q_{xxx} = \frac{3}{2}p_x p_{xx} - q_{xxx},$$

observe that, on the one hand, by differenciating Equation (3.6) we obtain

$$\begin{aligned}
F_{xxx} &= 3K_x q_{xx} + K q_{xxx}, \\
F_{xxy} &= K_y q_{xx} + 2K_x p_x + K p_{xx}, \\
F_{xyy} &= 2K_y p_x + 2K_x, \\
F_{yyy} &= 6K_y,
\end{aligned}$$

and that, on the other hand, from Equation (3.5) we obtain

$$\begin{aligned}
F_{xxx} &= 6R_x R_{xx} + S_{xxx}, \\
F_{xxy} &= 4R_x R_{xy} + 2R_y R_{xx} + S_{xxy} \\
F_{xyy} &= 4R_y R_{xy} + 2R_x R_{yy} + S_{xyy} \\
F_{yyy} &= 6R_y R_{yy} + S_{yyy}.
\end{aligned}$$

It is just a matter of computation to derive that

$$Q_{xxx} = \frac{R_x^3 S_{yyy} - 3R_x^2 R_y S_{xyy} + 3R_x R_y^2 S_{xxy} - R_y^3 S_{xxx}}{R_y^5},$$

as claimed. In a completely analogous way, and by considering the other third derivatives of F, one obtains the expressions for Q_{xxw} and Q_{xwz}. □

Corollary 3.4. *Let* $\mathbf{v} = (v_1, \ldots, v_d)$ *be a* d*-dimensional complex vector and let* $y = y(\mathbf{v})$ *be a function with* $y(\mathbf{v}_0) = y_0$ *that satisfies a functional equation*

$$y = \exp\left(G(y, \mathbf{v}) + H(y, \mathbf{v})\sqrt{J(y, \mathbf{v})}\right), \tag{3.10}$$

where G*,* H*, and* J *are analytic functions at* $(y_0, \mathbf{v}_0)$ *such that*

$$H(y_0, \mathbf{v}_0) = J(y_0, \mathbf{v}_0) = 0$$

and

$$y_0 G_y(y_0, \mathbf{v}_0) \neq 1.$$

Then, $y(\mathbf{v})$ has a local representation of the same form as in Lemma 3.3, that is,

$$y(\mathbf{v}) = P(\mathbf{v}) + \sqrt{Q(\mathbf{v})}, \tag{3.11}$$

where P and Q are analytic at $(\mathbf{v}_0)$, the evaluation of P at $(\mathbf{v}_0)$ is y_0, and Q and all its partial derivatives up to order 2 are zero at $(\mathbf{v}_0)$. Furthermore, the evaluation of Q_{xxx} at $(\mathbf{v}_0)$, for any variable x in $\mathbf{v}$, is

$$Q_{xxx}(\mathbf{v}_0) = \frac{6\,(y_0 H_y G_x - H_x(y_0 G_y - 1))^2\,(y_0 J_y G_x - J_x(y_0 G_y - 1))\,y_0^2}{(y_0 G_y - 1)^5},$$

with $y_0 = y(\mathbf{v}_0)$.

Proof. Just set

$$R(y,\mathbf{v}) := \log y - G(y,\mathbf{v}), \qquad S(y,\mathbf{v}) := -H(y,\mathbf{v})^2 J(y,\mathbf{v}),$$

and apply Lemma 3.3. Of course, by rewriting the derivatives of R and S in terms of the derivatives of G, H and J we obtain the proposed representation for Q_{xxx}. □

As noted above we can apply Corollary 3.4 with $y = D + 1$ and $\mathbf{v} = (x, w, E, U, V)$ and obtain a representation of D as a function of x, w, E, U, V of the form

$$D = P(x, w, E, U, V) + \sqrt{Q(x, w, E, U, V)}, \tag{3.12}$$

where Q and all partial derivatives of Q up to order 2 vanish. In particular if we substitute $E = E(x, 1)$ etc. we see that $Q(x, w, E(x), U(x), V(x))$ can be represented as

$$\begin{aligned} &Q(x, w, E(x,1), U(x,1), V(x,1)) \\ &\quad = X^3 h_1(X) + X^2 W h_2(X, W) + X W^2 h_3(W) + W^3 h_4(W), \end{aligned} \tag{3.13}$$

where $W = 1 - w/w_0$, $X = \sqrt{1 - x/x_0}$ and $h_1, \ldots, h_4$ are proper convergent power series. A simple (but tedious) computation provides

$$\begin{aligned} h_1(0) &\approx 0.009976458560, \\ h_2(0) &\approx -0.03944762502, \\ h_3(0) &= 0, \\ h_4(0) &\approx 0.09137050078. \end{aligned}$$

It should be remarked that $h_1(0) > 0$, $h_4(0) > 0$, and $h_3(0) = 0$.[3] This shows that $D(x, 1, w_0)$ has a singular behavior of the form

$$D(x, 1, w_0) = \overline{g}(x) + \overline{h}(x) X^3 \tag{3.14}$$

[3] Actually we have $h_3(W) = 0$ which can be shown without doing any numerical calculations. If $h_3 \neq 0$ it would follow that the dominant singularity of $D(x, 1, w)$ would have a singular behavior of the form $XW^{\ell-1/2}$ for some integer $\ell \geq 0$ which would lead to an asymptotic leading term of the coefficient of $x^n w^k$ of the squareroot part of the form $c\,x_0^{-n} w_0^{-k} n^{-3/2} k^{-\ell-1/2}$ Similarly if $P(x, w, E(x), U(x), V(x))$ has a factor X in its expansion then the dominant behavior in n would be of the form $x_0^{-n} n^{-3/2}$. In both cases this contradicts the asymptotic expansion of for the coefficient $[x^n]\,D(x, 1, 1) \sim c_1\, x_0^{-n} n^{-5/2}$.

with $X = \sqrt{1 - x/x_0}$ and where $h(x_0) > 0$.

It is also not difficult to show that $D(x, 1, w_0)$ has an analytic continuation to a Δ-region. For this purpose we can proceed similarly as for the function $D(x, y) = D(x, y, 1)$. For technical reasons it is preferable to work with $f(x, y, w) = S(x, y, w) + H(x, y, w)$ that satisfies a functional equation of the form $f = F(x, y, w, f)$, where F has non-negative coefficients. The point $(x_0, 1, w_0, f(x_0, 1, w_0)$ has the property that $F_f(x_0, 1, w_0, f(x_0, 1, w_0)) = 1$. Consequently we have $|F_f(x, 1, w_0, f(x, 1, w_0))| < 1$ for $|x| \le x_0$ and $x \ne x_0$. Hence the implicit function theorem implies that $f(x, 1, w_0)$ can be continued analytically to a Δ-region. Consequently the same holds for $D(x, 1, w_0)$.

3.3 Proof of Lemma 3.1

With all the above facts at hand it is now not very difficult to provide the proof of Lemma 3.1 (which leads to the proof of the upper bound of Theorem 1.3 as it was discussed in Section 3.1). We use the explicit representation (2.33) and apply the local expansion (3.2) for $E(x, 1)$ and (3.14) for $D(x, 1, w_0)$ (and also those of $u = U(x, E(x, 1))$ and $v = V(x, E(x, 1)))$. This leads directly to a singular representation of $B'(x, 1, w_0)$ of the following type:

$$B'(x, 1, w_0) = \overline{g}_1(x) + \overline{h}_1(x) X^3. \tag{3.15}$$

Note that we definitely have $h_1(x_0) \ne 0$ and hence $h_1(x_0) > 0$. Namely if $h_1(x_0) = 0$ then we would have $[x^n]\, B'(x, 1, w_0) = O\left(x_0^{-n} n^{-7/2}\right)$ which is impossible. Thus by applying the transfer lemma of Flajolet and Odlyzko [11] we obtain

$$[x^n]\, B'(x, 1, w_0) \sim c_1 x_0^{-n} n^{-5/2},$$

which completes the proof of the Lemma 3.1

3.4 Proof of Lemma 3.2

By using (2.29) and the local expansions (2.28) and (3.16) it follows that

$$C^{\bullet}(x, 1, w_0) = \overline{g}_2(x) + \overline{h}_2(x) \left(1 - \frac{x}{\rho_C}\right)^{3/2}. \tag{3.16}$$

Now we proceed as is the 2-connected case. We recall that the proof of the upper bound in Theorem 1.1 is precisely the same as that of the upper bound in Theorem 1.3.

4 The Lower Bound

This section is structured as follows. In the next subsection we collect some basic facts and tools that will be useful in our arguments. Then, in Section 4.2 we give the full proof of the lower bound in Theorem 1.3, i.e., we show a lower bound for the maximum degree in random 2-connected planar graphs that holds with high probability. Finally, in Section 4.3 we demonstrate that the lower bound Theorem 1.1 is a simple corollary of the lower bound in Theorem 1.3.

4.1 Networks & Boltzmann Sampling

Before we investigate the maximum degree of graph that is drawn uniformly at random from the class of 2-connected planar graphs, let us mention an auxiliary result that reduces the analysis to the study of random networks. The following lemma is from [12].

Lemma 4.1. *Let B_n be a uniform random graph from $\mathcal{B}_n$, and D_n a network that is drawn uniformly at random from $\mathcal{D}_n$. Suppose that $\Pr[D_{n-2} \in \mathcal{P}] \geq 1 - f(n-2)$, where $\mathcal{P}$ is any property of graphs that is closed under automorphisms. Then $\Pr[B_n \in \mathcal{P}] \geq 1 - 6f(n-2)$.*

Therefore, it is sufficient to show a lower bound for the maximum degree of a random network.

Recall the decomposition of networks that is described in Section 2.3, see (2.6)–(2.11). In particular, (2.6) guarantees that a network is either an edge, or a series-network, or a parallel network, or a core network. Except of the former case, in all other cases the classes of networks are described recursively. We will say that a network D *has a (3-connected) core of size s*, if the largest graph from $\mathcal{T}$ that was used in the decomposition of D has s vertices. Note that a network can have an empty core, in which case it consists only of series and parallel connections. However, in [12, 15] it was shown that a "typical" network has a very large core; here we present a simplified version of that result that is sufficient for our purposes.

Theorem 4.2. *There is a constant $c > 1/2$ such that the following is true. Let $\varepsilon > 0$ and denote by $C(D_n)$ the size of the largest core in a random network D_n from $\mathcal{D}_n$. Then, with probability $1 - o(1)$, we have that $C(D_n) > cn$.*

The Pole Degree in the Boltzmann Model In the sequel we will write $rd(N)$ for the degree of the left pole of a network N. The following technical lemma is an important tool in the proof of the lower bound of the maximum degree of random networks.

Lemma 4.3. *Let γ be a random network drawn from the Boltzmann distribution for $\mathcal{D}$ with parameters $x = \rho_D$ and $y = 1$. Then*

$$\Pr[rd(\gamma) \geq k] \sim ck^{-5/2}w_0^{-k},$$

for some constant $c > 0$, where w_0 is given in (3.1).

Proof. Let $\ell \geq 1$. The definition of the Boltzmann model implies that

$$\Pr[rd(\gamma) = \ell] = \frac{1}{D(\rho_D, 1)} \sum_{D \in \mathcal{D}:\ rd(D)=\ell} \frac{\rho_D^{v(D)}}{v(D)!} = \frac{[w^\ell]D(\rho_D, 1, w)}{D(\rho_D, 1)},$$

where $[x^n w^\ell]D(x, 1, w)$ is the number of networks with n labeled vertices and root-degree ℓ. By following the representation (3.12) of $D(x, 1, w)$ and by setting $x = x_0$ in (3.13) we obtain a singular representation of the form

$$D(x_0, 1, w) = a(w) + b(w)\left(1 - \frac{w}{w_0}\right)^{3/2}$$

for some functions $a(w), b(w)$ that are non-zero and analytic at w_0. It is also easy to see that $D(x_0, 1, w)$ has an analytic continuation to a proper Δ-domain in w. We just have to modify the arguments at the end of Section 3.2. Hence we can apply the Transfer Lemma of Flajolet and Odlyzko and obtain

$$\Pr[rd(\gamma) = \ell] \sim c_1 \ell^{-5/2} w_0^{-\ell}$$

for some constant $c_1 > 0$. By adding these values up for $\ell \geq k$ we obtain the statement of the lemma. □

4.2 Proof of the Lower Bound in Theorem 1.3

Let D_n denote a random network from $\mathcal{D}_n$, and recall the definition of w_0 in (3.1). Let $\varepsilon = \varepsilon(n)$ be a function that tends slowly to 0, for example $\varepsilon(n) = c' \log\log n / \log_{w_0} n$, for some $c' > 0$ to be chosen later, is sufficient for our purpose. By applying Proposition 4.1 we infer that if

$$\Pr\left[\Delta(D_{n-2}) \leq (1-\varepsilon)\log_{w_0} n\right] = o(1),$$

then it follows also that $\Pr[\Delta(B_n) \leq (1-\varepsilon)\log_{w_0} n] = o(1)$. We thus proceed with the estimation of the above probability, where we write n instead of $n-2$ for brevity.

First of all, by applying Theorem 4.2 we obtain that

$$p = \Pr\left[\Delta(D_n) \leq (1-\varepsilon)\log_{w_0} n\right] = \Pr\left[\Delta(D_n) \leq (1-\varepsilon)\log_{w_0} n \text{ and } C(D_n) > n/2\right] + o(1),$$

where $C(D)$ denotes the size of the largest core in a network D. Let us write $\gamma = \Gamma D(\rho_N, 1)$, where ΓD is the Boltzmann sampler for the class of networks described in Section 2.7. By using the first equality in (2.2) we infer that

$$p = \Pr\left[\Delta(\gamma) \leq (1-\varepsilon)\log_{w_0} n \text{ and } C(\gamma) > n/2 \mid \gamma \in \mathcal{D}_n\right] + o(1),$$

By Lemma 2.3 we obtain that $\Pr[\gamma \in \mathcal{D}_n] = \Theta(n^{-5/2})$. So,

$$p = O(n^{5/2}) \Pr\left[\Delta(\gamma) \leq (1-\varepsilon)\log_{w_0} n \text{ and } C(\gamma) > n/2 \text{ and } \gamma \in \mathcal{D}_n\right] + o(1). \tag{4.1}$$

For the remainder of this section let us fix $x = \rho_D = \rho_B$ and $y = 1$. In the subsequent analysis we will make the following modification of the Boltzmann sampler $\Gamma D(x, y)$. Let $L = (L_1, L_2, \dots)$ be an infinite list, where for all $i \geq 1$ we have $L_i \in \mathcal{D}$. Recall the definition of the sampler $\Gamma H(x, y)$ that generates core networks. $\Gamma H(x, y)$ first samples a network from $\bar{\mathcal{T}}$, and then replaces independently every edge by a network that is drawn from the Boltzmann distribution with parameters x and y for $\mathcal{D}$. Instead of doing this, we modify $\Gamma H(x, y)$ so that it uses graphs from L instead, provided that the network sampled from $\bar{\mathcal{T}}$ is large. In particular, the sampler $\widetilde{\Gamma} H(x, y;\, n, L)$ works as follows.

$\widetilde{\Gamma} H(x, y;\, n, L)$: $T \leftarrow \Gamma\bar{T}(x, D(x, y))$ (∗)
if T has more than $n/2$ vertices
  $i \leftarrow 1$
  foreach edge e of T
    $\gamma_e \leftarrow L_i$
    $i \leftarrow i + 1$
else
  foreach edge e of T
    $\gamma_e \leftarrow \Gamma D(x, y)$
replace every e in T by γ_e
return T, relabeling randomly its non-pole vertices

Note that if we choose the L_i's independently from the Boltzmann distribution with parameters x and y for $\mathcal{D}$, then for any $D \in \mathcal{D}$ we have for all values of n that that

$$\Pr[\Gamma H(x,y) = D] = \Pr[\widetilde{\Gamma} H(x,y;\, n, L) = D].$$

In other words, we can work with $\widetilde{\Gamma} H$ instead of ΓH. In particular, we shall assume that ΓD, ΓS and ΓP use $\widetilde{\Gamma} H$ instead of ΓH, where the L_i's are independent samples from the Boltzmann distribution with parameters x and y for $\mathcal{D}$.

With these assumptions in mind, let us proceed with the estimation of the probability on the right-hand side of (4.1). First of all, the event "$C(\gamma) > n/2$" implies that at some point in time in the construction of $\gamma = \Gamma D(x,y)$ the sampler $\widetilde{\Gamma} H(x,y;\, n, L)$ is used, and the graph T (generated in the line marked with $(*)$) has $> n/2$ vertices. Since T is a 3-connected planar graph minus an edge, it has $\geq n/2$ edges. Thus, in the construction of γ certainly the first $\lfloor n/2 \rfloor$ graphs from L are used. Recall that every edge $e = \{u, v\}$ of T is subsequently replaced by some distinct network L_i from L, so that the degree of, say, u is at least $rd(L_i)$. In other words, the event "$\Delta(\gamma) \leq (1-\varepsilon)\log_{1/q} n$ and $C(\gamma) > n/2$ and $\gamma \in \mathcal{D}_n$" implies that the first $\lfloor n/2 \rfloor$ graphs in L have the property that the root degree of their left pole is $\leq (1-\varepsilon)\log_{w_0} n$. Hence, by using (4.1), the desired probability is at most

$$p = O(n^{5/2}) \Pr\left[\forall 1 \leq i \leq \lfloor n/2 \rfloor \;:\; rd(L_i) \leq (1-\varepsilon)\log_{w_0} n\right] + o(1).$$

Recall that the L_i's are independent samples from the Boltzmann distribution with parameters ρ_D and 1 for $\mathcal{D}$. By applying Lemma 4.3 we obtain for sufficiently large n that

$$\Pr\left[rd(L_i) \leq (1-\varepsilon)\log_{w_0} n\right] \leq 1 - (\log n)^{-3} w_0^{-(1-\varepsilon)\log_{w_0} n} = 1 - (\log n)^{-3} n^{-(1-\varepsilon)}.$$

So, since $\varepsilon = c' \log\log n / \log_{w_0} n$, by choosing, say, $c' = 10/\log(w_0)$

$$p \leq O(n^{5/2}) \left(1 - (\log n)^{-3} n^{-(1-\varepsilon)}\right)^{\lfloor n/2 \rfloor} + o(1) = o(1),$$

and the proof is completed.

4.3 Proof of the Lower Bound in Theorem 1.1

The proof of the lower bound in Theorem 1.1 follows directly from the lower bound in Theorem 1.3. More precisely, in [20, 15] it was shown that a random planar graph contains with probability $1 - o(1)$ a very large 2-connected subgraph.

Theorem 4.4. *There is a constant $c > 1/2$ such that the following is true. Let $\varepsilon > 0$ and denote by $b(C_n)$ the size of the largest 2-connected subgraph in a random graph C_n from $\mathcal{C}_n$. Then, with probability* $1 - o(1)$,

$$|b(C_n) - cn| \leq \varepsilon n.$$

Conditional on any specific value of $b(C_n)$ that is within the bounds given in the above theorem, note that any 2-connected planar graph with $b(C_n)$ vertices is equally likely to be the largest 2-connected subgraph of C_n. The lower bound for the maximum degree in C_n then follows immediately from Theorem 1.3.

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