THE MAXIMUM DEGREE OF SERIES PARALLEL GRAPHS

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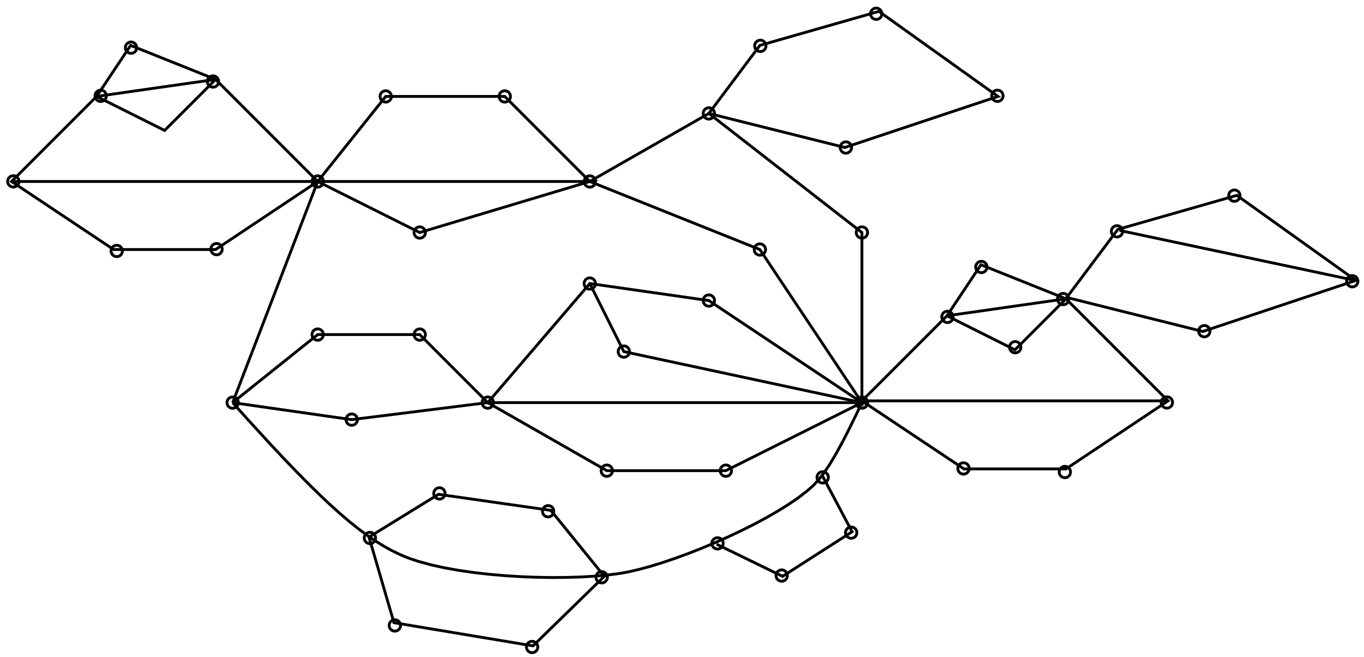
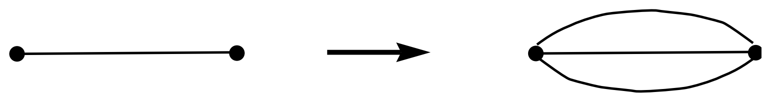
Series-Parallel Graphs

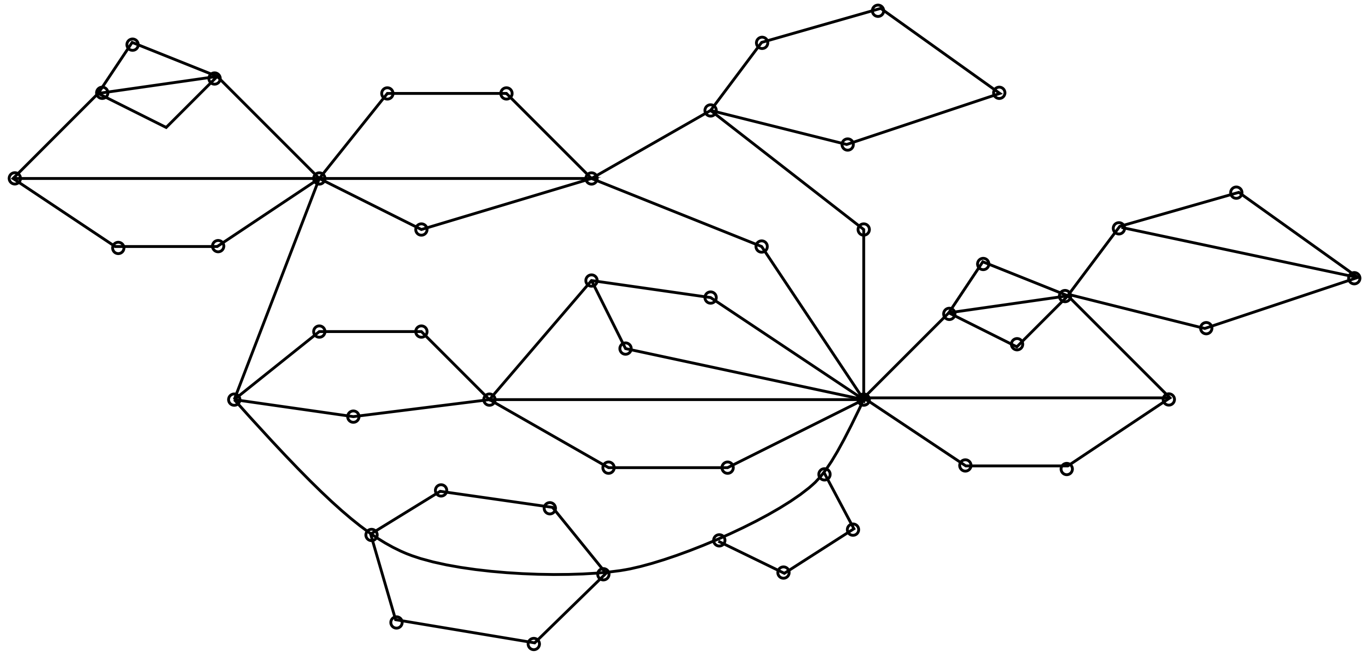


- Series-parallel extension of a tree or forest

Series-extension: 

Parallel-extension: 

Series-Parallel Graphs



- Ex(K_4) ... no K_4 as a minor
- Treewidth ≤ 2

Series-Parallel Graphs

Theorem 1 [D.+Giménez+Noy]

G_n ... random vertex labelled SP-graph with n vertices

Δ_n ... **maximum degree** of G_n

$$\Longrightarrow \quad \boxed{\frac{\Delta_n}{\log n} \to c \quad \text{in probability}} \quad \text{and} \quad \boxed{\mathbb{E}\,\Delta_n \sim c \log n}.$$

Series-Parallel Graphs

Remark 1. A corresponding result holds for 2-connected and connected SP-graphs:

$$c \approx 3.679771 \quad \text{for 2-connected SP-graphs,}$$
$$c \approx 3.482774 \quad \text{for connected and all SP-graphs.}$$

Remark 2. p_k ... (limiting) probability that a random vertex in a random SP-graph has degree k.

q^{-1} ... radius of convergence of $p(w) = \sum_{k \ge 1} p_k w^k$.

$$\Longrightarrow \quad \boxed{c = \frac{1}{\log(1/q)}}.$$

Series-Parallel Graphs

Heuristically: Δ_n concentrated around level k_0 which satisfies $\boxed{n\, p_{k_0} \approx 1}$.

• p_k has "geometric" behaviour: $\log p_k \sim k \log q$ (for $0 < q < 1$)

$$\Longrightarrow \quad \Delta_n \sim c \log n, \quad c = \frac{1}{\log(1/q)}$$

(E.g. plane trees)

• p_k has "Poisson" behaviour: $p_k \sim a^k e^{-a}/k!$

$$\Longrightarrow \quad \Delta_n \sim \frac{\log n}{\log \log n}$$

(E.g. labelled trees)

Historic Remarks

- Gao + Wormald: precise distribution of maximum degree in planar maps and triangulations.

- McDiarmid + Reed: $c \log n < \Delta_n < C \log n$ whp for random planar graphs.

- Bernasconi + Panagiotou + Steger: concentration results for degree distribution (uniform up to $k \le C \log n$)
  + **conjecture** for max-degree of SP-graphs.

Maximum Degree

Relation to number of vertices of given degree

$X_n^{(k)}$ ... **number of vertices of degree** k in G_n.

$X_n^{(>k)} = X_n^{(k+1)} + X_n^{(k+2)} + \cdots$ ... **number of vertices of degree** $> k$.

Δ_n ... maximum degree:

$$\boxed{\boxed{\Delta_n > k \iff X_n^{(>k)} > 0}}$$

$$\boxed{\mathbb{P}\{\Delta_n > k\} = \mathbb{P}\{X_n^{(>k)} > 0\}}$$

Maximum Degree

First moment method

Y ... a discrete random variable on non-negative integers.

$$\Longrightarrow \quad \boxed{\mathbb{P}\{Y > 0\} \le \min\{1, \mathbb{E}\, Y\}}$$

Second moment method

Y is a non-negative random variable with finite second moment.

$$\Longrightarrow \quad \boxed{\mathbb{P}\{Y > 0\} \ge \frac{(\mathbb{E}\, Y)^2}{\mathbb{E}\,(Y^2)}}$$

Maximum Degree

First and second moment method

$$\boxed{\frac{\left(\mathbb{E}\, X_n^{(>k)}\right)^2}{\mathbb{E}\,(X_n^{(>k)})^2} \le \mathbb{P}\{\Delta_n > k\} \le \min\{1, \mathbb{E}\, X_n^{(>k)}\}}$$

$X_n^{(>k)}$ ... **number of vertices of degree** $> k$.

Maximum Degree

First moments

$p_{n,k}$ … **probability that a random vertex** in G_n **has degree** k

$$\boxed{\mathbb{E}\, X_n^{(k)} = n\, p_{n,k}}$$

$$\Longrightarrow \quad \mathbb{E}\, X_n^{(>k)} = \mathbb{E}\left(\sum_{\ell>k} X_n^{(\ell)}\right) = n \sum_{\ell>k} p_{n,\ell}.$$

Precise asymptotics for $p_{n,k}$ are needed that are **uniform in** n **and** k.

Maximum Degree

Second moments

$p_{n,k,\ell}$ ... **probability that two different randomly selected vertices** in G_n **have degrees** k **and** ℓ**.**

$$\boxed{\mathbb{E}\left(X_n^{(k)} X_n^{(\ell)}\right) = n(n-1)\, p_{n,k,\ell}} \qquad (k \neq \ell)$$

$$\Longrightarrow \quad \mathbb{E}\,(X_n^{(>k)})^2 = \mathbb{E}\left(\sum_{j>k} X_n^{(j)}\right)^2 = n \sum_{\ell>k} p_{n,\ell} + n(n-1) \sum_{\ell_1,\ell_2>k} p_{n,\ell_1,\ell_1}.$$

Precise asymptotics for $p_{n,k,\ell}$ are needed that are **uniform in** n**,** k**, and** ℓ.

Maximum Degree

Bounds for the distribution of Δ_n

$$\frac{n^2\left(\sum_{\ell>k} p_{n,\ell}\right)^2}{n\sum_{\ell>k} p_{n,\ell} + n(n-1)\sum_{\ell_1,\ell_2>k} p_{n,\ell_1,\ell_1}} \le \mathbb{P}\{\Delta_n > k\} \le \min\left\{1, n\sum_{\ell>k} p_{n,\ell}\right\}.$$

"Master Theorem" Suppose that

$$p_{n,k} \sim c\, k^\alpha q^k$$
$$p_{n,k,\ell} \sim p_{n,k}p_{n,\ell} \sim c^2\,(k\ell)^\alpha q^{k+\ell}$$

$$\Longrightarrow \quad \boxed{\frac{\Delta_n}{\log n} \to \frac{1}{\log(1/q)} \qquad \text{in probability}}$$

Maximum Degree

Remark 1 More precisely we need

$$p_{n,k} \sim c\, k^{\alpha} q^{k} \quad \text{uniformly for } k \le C \log n$$

and

$$p_{n,k} = O(\overline{q}^{k}) \quad \text{uniformly for all } n, k \ge 0$$

for some q and $\overline{q}$ with $0 < q \le \overline{q} < 1$
(and similar conditions for $p_{n,k,\ell}$).

Remark 2 (Thanks to Kosta Panagiotou)
The relations for $p_{n,k,\ell}$ can be replaced by proper estimates for the covariance of $X_n^{(k)} X_n^{(\ell)}$. For example, if G_n has **many small blocks** whp then the **degrees** of two independently chosen vertices will be *almost* **independent** since they will be in different blocks whp.

Series-Parallel Graphs

Generating functions

$b_{n,m}$ ... number of **2-connected labelled series-parallel** graphs with n vertices and m edges, $b_n = \sum_m b_{n,m}$

$$B(x,y) = \sum_{n,m} b_{n,m} \frac{x^n}{n!} y^m$$

$c_{n,m}$ ... number of **connected labelled series-parallel** graphs with n vertices and m edges, $c_n = \sum_m c_{n,m}$

$$C(x,y) = \sum_{n,m} c_{n,m} \frac{x^n}{n!} y^m$$

$g_{n,m}$ ... number of **labelled series-parallel** graphs with n vertices and m edges, $g_n = \sum_m g_{n,m}$

$$G(x,y) = \sum_{n,m} g_{n,m} \frac{x^n}{n!} y^m$$

Series-Parallel Graphs

Generating functions

$$G(x,y) = e^{C(x,y)}$$

$$\frac{\partial C(x,y)}{\partial x} = \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right),$$

$$\frac{\partial B(x,y)}{\partial y} = \frac{x^2}{2}\,\frac{1+D(x,y)}{1+y} = \frac{x^2}{2}\,e^{S(x,y)}$$

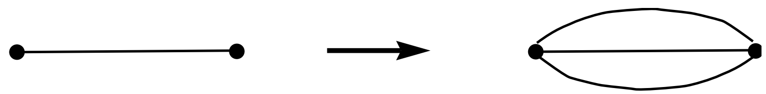
$$D(x,y) = (1+y)e^{S(x,y)} - 1,$$

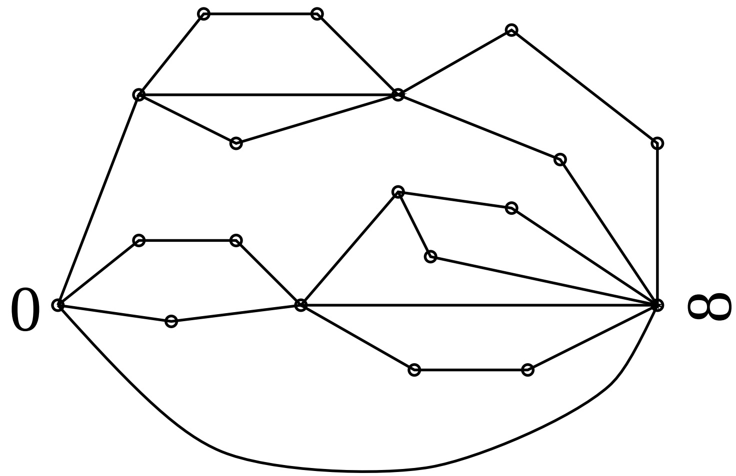
$$S(x,y) = (D(x,y) - S(x,y))xD(x,y).$$

Series-Parallel Graphs

Series-parallel networks: series-parallel extension of an edge

Series-extension: 

Parallel-extension: 

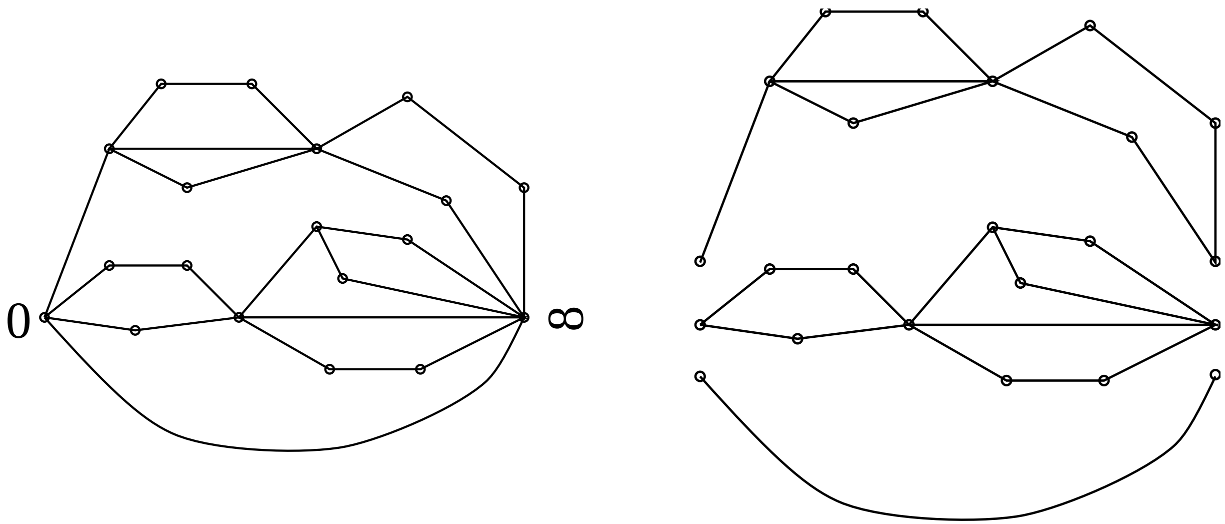


There are always two **poles** $(0, \infty)$ coming from the original two vertices.

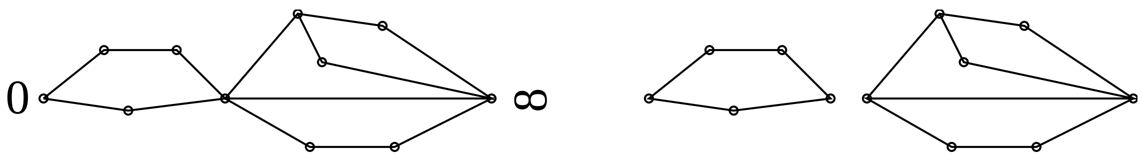
Series-Parallel Graphs

Series-parallel networks

Parallel decomposition of a Series-parallel network:



Series decomposition of a series-parallel network



Series-Parallel Graphs

Series-parallel networks

$d_{n,m}$ ... number of SP-networks with $n+2$ vertices and m edges

$s_{n,m}$ ... number of **series** SP-networks $n+2$ vertices and m edges

$$D(x,y) = \sum_{n,m} d_{n,m} \frac{x^n}{n!} y^m, \qquad S(x,y) = \sum_{n,m} s_{n,m} \frac{x^n}{n!} y^m,$$

$$\begin{aligned} D(x,y) &= e^{S(x,y)} - 1 + y e^{S(x,y)} \\ &= (1+y)e^{S(x,y)} - 1, \end{aligned}$$

$$S(x,y) = (D(x,y) - S(x,y)) x D(x,y)$$

Series-Parallel Graphs

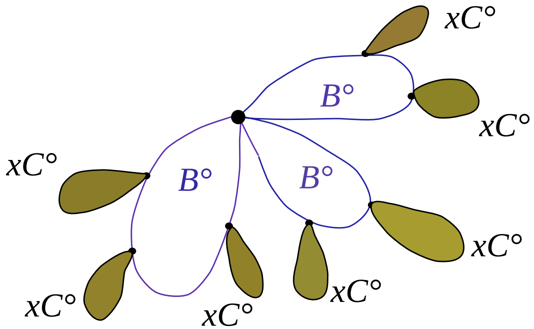
2-connected SP-graphs

A SP-network network with non-adjacent poles (which is counted by $e^{S(x,y)}$) is obtained by distinguishing, orienting and then deleting any edge of an arbitrary 2-connected series-parallel graph:

$$\frac{\partial B(x,y)}{\partial y} = \frac{x^2}{2} e^{S(x,y)} = \frac{x^2}{2} \frac{1 + D(x,y)}{1+y}$$

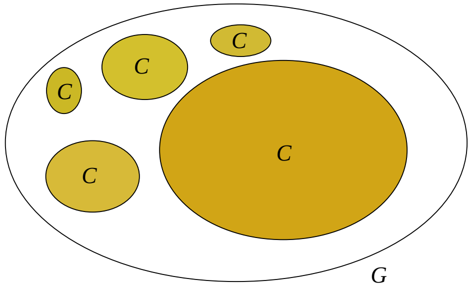
Series-Parallel Graphs

Connected SP-graphs



$$\frac{\partial C(x,y)}{\partial x} = \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right)$$

All SP-graphs



$$G(x,y) = e^{C(x,y)}$$

Series-Parallel Graphs

Asymptotic enumeration

$$b_n = b \cdot \rho_1^{-n} n^{-\frac{5}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right),$$
$$c_n = c \cdot \rho_2^{-n} n^{-\frac{5}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right),$$
$$g_n = g \cdot \rho_2^{-n} n^{-\frac{5}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right),$$

$$\rho_1 = 0.1280038...,$$
$$\rho_2 = 0.11021...,$$
$$b = 0.0010131...,$$
$$c = 0.0067912...,$$
$$g = 0.0076388...$$

Series-Parallel Graphs

Asymptotic enumeration

$$D(x,y) = (1+y)\,\mathrm{exp}\left(\frac{xD(x,y)^2}{1+xD(x,y)}\right) - 1 = \Phi(x,y,D(x,y))$$

$$\Longrightarrow \quad D(x,y) = g(x,y) - h(x,y)\sqrt{1 - \frac{x}{\rho(y)}},$$

with $\rho(1) = \rho_1 = 0.12800\ldots.$

Series-Parallel Graphs

Asymptotic enumeration

$$\Longrightarrow \quad \frac{\partial B(x,y)}{\partial y} = \frac{x^2}{2}\frac{1+D(x,y)}{1+y}$$

$$= g_2(x,y) - h_2(x,y)\sqrt{1-\frac{x}{\rho(y)}}$$

!!!! $$\Longrightarrow \quad B(x,y) = g_3(x,y) + h_3(x,y)\left(1-\frac{x}{\rho(y)}\right)^{\frac{3}{2}}$$

$$\Longrightarrow \quad \boxed{b_n \sim b\cdot \rho(1)^{-n} n^{-\frac{5}{2}} n!}$$

Series-Parallel Graphs

Asymptotic enumeration $(C' := \frac{\partial}{\partial x} C)$

$$C'(x,y) = e^{B'(xC'(x,y),y)},\ v(x,y) = xC'(x,y),\ \Phi(x,y,v) = xe^{B'(v,y)}$$

$$\Longrightarrow \quad \boxed{v(x,y) = \Phi(x,y,v(x,y))}$$

$$\Longrightarrow \quad v(x,y) = xC'(x,y) = g_4(x,y) - h_4(x,y)\sqrt{1 - \frac{x}{\rho_2(y)}}$$

with $\rho_2(1) = 0.11021....$ (**Note that** $v(\rho) = 0.1279695... < \rho_1$!!!)

$$\Longrightarrow \quad C(x,y) = g_5(x,y) + h_5(x,y)\left(1 - \frac{x}{\rho_2(y)}\right)^{\frac{3}{2}}.$$

$$\Longrightarrow \quad \boxed{c_n \sim c\,\rho_2^{-n} n^{-\frac{5}{2}} n!}$$

Series-Parallel Graphs

Asymptotic enumeration

$$C(x,y) = g_5(x,y) + h_5(x,y)\left(1 - \frac{x}{\rho(y)}\right)^{\frac{3}{2}}$$

$$\Longrightarrow \quad G(x,y) = e^{C(x,y)} = g_6(x,y) + h_6(x,y)\left(1 - \frac{x}{\rho_2(y)}\right)^{\frac{3}{2}}.$$

$$\Longrightarrow \quad \boxed{g_n \sim g \cdot \rho_2^{-n} n^{-\frac{5}{2}} n!}$$

Root Degree

Random vertex *versus* **root vertex**

G_n ... random vertex labelled SP-graph with n vertices

$G_n^\bullet$ ... random vertex labelled SP-graph with n vertices, where one vertex is distinguished (= **root**)

$$p_{n,k} = \text{probability that a random vertex in } G_n \text{ has degree } k$$

$$= \textbf{probability that the root in } G_n^\bullet \textbf{ has degree } k$$

Root Degree

Generating functions

$b^{\bullet}_{n,k}$ ... number of **rooted 2-connected labelled series-parallel** graphs with n vertices and root-degree k.

$$B^{\bullet}(x,w) = \sum_{n,k} b^{\bullet}_{n,k} \frac{x^n}{n!} w^k$$

$c^{\bullet}_{n,k}$ ... number of **rooted connected labelled series-parallel** graphs with n vertices and root-degree k.

$$C^{\bullet}(x,w) = \sum_{n,k} c^{\bullet}_{n,k} \frac{x^n}{n!} w^k$$

$g^{\bullet}_{n,k}$ ... number of **rooted labelled series-parallel** graphs with n vertices and root-degree k.

$$G^{\bullet}(x,w) = \sum_{n,k} g_{n,k} \frac{x^n}{n!} w^k$$

Root Degree

Computation of $p_{n,k}$

$$\boxed{p_{n,k} = \frac{g^{\bullet}_{n,k}}{n g_n} = \frac{[x^n w^k] G^{\bullet}(x,w)}{[x^n] G^{\bullet}(x,1)}}$$

Root Degree

Generating functions

$$G^\bullet(x,w) = C^\bullet(x,w)e^{C(x)},$$

$$C^\bullet(x,w) = e^{B^\bullet(xC'(x),w)},$$

$$w\frac{\partial}{\partial w}B^\bullet(x,w) = \sum_{k\geq 1} kB_k(x)w^k = xwe^{S^\bullet(x,w)},$$

$$D^\bullet(x,w) = (1+w)e^{S^\bullet(x,w)} - 1,$$

$$S^\bullet(x,w) = (D^\bullet(x,w) - S^\bullet(x,w))xD(x,1).$$

Root Degree

Series-parallel networks

$d^\bullet_{n,k}$ ... number of SP-networks with $n+2$ vertices, where the first pole has degree k

$s^\bullet_{n,m}$ ... number of **series** SP-networks $n+2$ vertices, where the first pole has degree k

$$D^\bullet(x,y) = \sum_{n,k} d^\bullet_{n,k} \frac{x^n}{n!} w^k, \qquad S^\bullet(x,y) = \sum_{n,k} s^\bullet_{n,k} \frac{x^n}{n!} w^k,$$

$$D^\bullet(x,w) = (1+w)e^{S^\bullet(x,w)} - 1,$$

$$S^\bullet(x,w) = (D^\bullet(x,w) - S^\bullet(x,w))xD(x,1)$$

Root Degree

2-connected SP-graphs

A SP-network network with non-adjacent poles (which is counted by $e^{S^\bullet(x,w)}$) is obtained by distinguishing, orienting and then deleting any edge of an arbitrary 2-connected series-parallel graph:

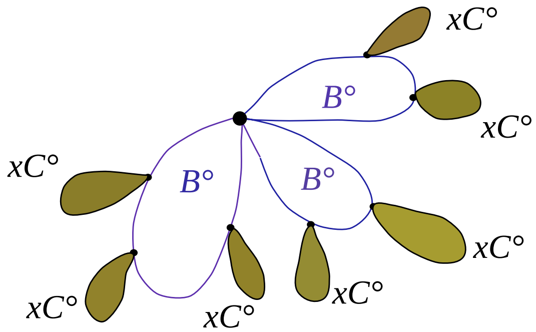
$$w\frac{\partial}{\partial w}B^\bullet(x,w) = \sum_{k\ge 1} kB_k(x)w^k = xe^{S^\bullet(x,w)},$$

$$= \frac{1 + D^\bullet(x,w)}{1+w}$$

Root Degree

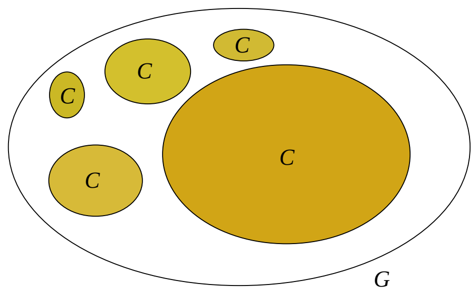
Connected SP-graphs

$$C^{\bullet}(x,w) = e^{B^{\bullet}(xC'(x),w)}$$



All SP-graphs

$$G^{\bullet}(x,w) = C^{\bullet}(x,w)e^{C(x)}$$



Degree Distribution

Theorem 2 [D.+Giménez+Noy]

Let $p_{n,k}$ be the probability that a random vertex in a random 2-connected, connected or unrestricted series-parallel graph with n vertices has degree k. Then the limit

$$\boxed{p_k := \lim_{n\to\infty} p_{n,k}}$$

exists. The probability generating function

$$\boxed{p(w) = \sum_{k\geq 1} p_k w^k}$$

can be computed explicitly and we have asymptotically

$$p_k \sim c\, q^k\, k^{-\frac{3}{2}}.$$

Degree Distribution

For **2-connected series-parallel graphs** the series $p(w) = \sum_{k \geq 1} p_k w^k$ is given by:

$$\boxed{p(w) = \frac{B_1(1, w)}{B_1(1, 1)}},$$

where $B_1(y, w)$ is given by the following procedure ...

Degree Distribution

$$\begin{aligned}
\frac{E_0(y)^3}{E_0(y)-1} &= \left(\log\frac{1+E_0(y)}{1+R(y)} - E_0(y)\right)^2,\\
R(y) &= \frac{\sqrt{1-1/E_0(y)}-1}{E_0(y)},\\
E_1(y) &= -\left(\frac{2R(y)E_0(y)^2(1+R(y)E_0(y))^2}{(2R(y)E_0(y)+R(y)^2E_0(y)^2)^2+2R(y)(1+R(y)E_0(y))}\right)^{\frac{1}{2}},\\
D_0(y,w) &= (1+yw)e^{\frac{R(y)E_0(y)}{1+R(y)E_0(y)}D_0(y,w)}-1,\\
D_1(y,w) &= \frac{(1+D_0(y,w))^{\frac{R(y)E_1(y)D_0(y,w)}{1+R(y)E_0(y)}}}{1-(1+D_0(y,w))^{\frac{R(y)E_0(y)D_0(y,w)}{1+R(y)E_0(y)}}},\\
B_0(y,w) &= \frac{R(y)D_0(y,w)}{1+R(y)E_0(y)} - \frac{R(y)^2E_0(y)D_0(y,w)^2}{2(1+R(y)E_0(y))},\\
B_1(y,w) &= \frac{R(y)D_1(y,w)}{1+R(y)E_0(y)} - \frac{R(y)^2E_0(y)D_0(y,w)D_1(y,w)}{1+R(y)E_0(y)}\\
&\quad - \frac{R(y)^2E_1(y)D_0(y,w)(1+D_0(y,w)/2)}{(1+R(y)E_0(y))^2}.
\end{aligned}$$

Degree Distribution

Remark 3 [D.+Giménez+Noy] $X_n^{(k)}$ satisfies a **central limit theorem** with

$$\mathbb{E}\, X_n^{(k)} \sim \mu_k n \quad \text{and} \quad \mathbb{V}\, X_n^{(k)} \sim \sigma_k^2 n.$$

Remark. $\mu_k = p_k$.

Asymptotic Analysis

We know

$$p_k = \lim_{n\to\infty} p_{n,k} \sim c\, q^k\, k^{-\frac{3}{2}}$$

We need (uniformly for $k \le C \log n$)

$$p_{n,k} \sim c\, q^k\, k^{-\frac{3}{2}}.$$

The **goal** is to extend Theorem 2 to a bivariate asymptotics.

Asymptotic Analysis

Series-parallel networks

$$D(x,1) = 2\exp\left(\frac{xD(x,1)^2}{1+xD(x,1)}\right) - 1 = \Phi(x, D(x,1))$$

$$\Longrightarrow \quad D(x,1) = g_1(x) - h_1(x)\sqrt{1-\frac{x}{\rho_1}},$$

with $\rho_1 = 0.12800...$.

[Repetition of the previous case with $y = 1$].

Asymptotic Analysis

Series-parallel networks

$$D^\bullet(x,w) = 2\exp\left(\frac{xD(x,1)D^\bullet(x,w)}{1+xD^\bullet(x,w)}\right) - 1 = \Phi(x,w,D(x,1),D^\bullet(x,w))$$

$$\Longrightarrow \quad D^\bullet(x,w) = g_2(x,w,D(x,1)) - h_2(x,w,D(x,1))\sqrt{1-\frac{w}{\rho(x,D(x,1))}},$$

with

$$\rho(x,D(x,1)) = \overline{g}(x) - \overline{h}(x)\sqrt{1-\frac{x}{\rho_1}}$$

Asymptotic Analysis

2-connected SP-graphs

$$\Longrightarrow \quad \frac{\partial B^\bullet(x,w)}{\partial w} = \frac{1+D^\bullet(x,w)}{1+w} D^\bullet(x,w)$$

$$= g_3(x,w,D(x,1)) - h_3(x,w,D(x,1))\sqrt{1-\frac{w}{\rho(x,D(x,1))}}$$

$$\Longrightarrow \quad B^\bullet(x,w) = g_4(x,w,D(x,1)) + h_4(x,w,D(x,1))\left(1-\frac{w}{\rho(x,D(x,1))}\right)^{\frac{3}{2}}$$

$$= \boxed{G(x,w) + H(x,w)\,(1-y(x)w)^{\frac{3}{2}}}$$

with

$$y(x) = \rho(x,D(x,1))^{-1} = g(x) - h(x)\sqrt{1-x/\rho_1},$$

$$G(x,w) = g_4(x,w,D(x,1)) = G_1(x,w) - G_2(x,w)\sqrt{1-x/\rho_1},$$

$$H(x,w) = h_4(x,w,D(x,1)) = H_1(x,w) - H_2(x,w)\sqrt{1-x/\rho_1}.$$

Asymptotic Analysis

Connected SP-graphs

$$\Longrightarrow \quad C^\bullet(x,w) = e^{B^\bullet(xC'(x),w)}$$

$$= \boxed{\overline{G}(x,w) + \overline{H}(x,w)\,(1-\overline{y}(x)w)^{\frac{3}{2}}}$$

with

$$\overline{y}(x) = y(xC'(x)) = \overline{g}(x) - \overline{h}(x)\sqrt{1-x/\rho_2},$$
$$\overline{G}(x,w) = \overline{G}_1(x,w) - \overline{G}_2(x,w)\sqrt{1-x/\rho_2},$$
$$\overline{H}(x,w) = \overline{H}_1(x,w) - \overline{H}_2(x,w)\sqrt{1-x/\rho_2}.$$

Asymptotic Analysis

Lemma 1

$$f(x,w) = \sum_{n,k\ge 0} f_{n,k} x^n w^k$$
$$= \boxed{G(x,w) + H(x,w)\,(1-y(x)w)^{\frac{3}{2}}},$$

where

$$y(x) = g(x) - h(x)\sqrt{1 - x/x_0},$$
$$G(x,w) = G_1(x,w) - G_2(x,w)\sqrt{1 - x/x_0},$$
$$H(x,w) = H_1(x,w) - H_2(x,w)\sqrt{1 - x/x_0}.$$

with analytic functions g, h, G_1, G_2, H_1, H_2
(+ some technical conditions)

$$\Longrightarrow \quad \boxed{f_{n,k} = \frac{3h(x_0)H(x_0,0,1/g(x_0))}{8\pi} g(x_0)^{k-1} x_0^{-n} k^{-\frac{3}{2}} n^{-\frac{3}{2}} \left(1 + O\left(\frac{1}{k}\right)\right)}$$

uniformly for $k \le C \log n$ (for any constant $C > 0$) and

$$f_{n,k} = O\left((g(x_0)+\varepsilon)^k \rho^{-n} n^{-\frac{3}{2}}\right).$$

Asymptotic Analysis

Application

$$B^\bullet(x,w) = G(x,w) + H(x,w)\,(1 - y(x)w)^{\frac{3}{2}}\,,$$

$$\Longrightarrow \quad \frac{b^\bullet_{n,k}}{n!} \sim c_1\, q^k\, x_0^{-n}\, k^{-\frac{3}{2}}\, n^{-\frac{3}{2}}.$$

with $q = g(x_0) < 1$.

$$\frac{b_n}{n!} \sim b x_0^{-n} n^{-\frac{5}{2}} \quad \text{(from above)}$$

$$\Longrightarrow \quad \boxed{p_{n,k} = \frac{b^\bullet_{n,k}}{n b_n} \sim c\, q^k\, k^{-\frac{3}{2}}}$$

Double Rooting

Generating Functions

$$
\begin{aligned}
G^{\bullet\bullet}(x,w,t) &= e^{C(x)}G^{\bullet}(x,w)G^{\bullet}(x,t) + e^{C(x)}C^{\bullet\bullet}(x,w,t),\\
C^{\bullet\bullet}(x,w,t) &= \frac{x}{(xC'(x))'}\frac{\partial}{\partial x}C^{\bullet}(x,w)\frac{\partial}{\partial x}C^{\bullet}(x,t)\\
&\quad + B^{\bullet\bullet}(xC'(x),w,t)C^{\bullet}(x,w)C^{\bullet}(x,t),\\
w\frac{\partial}{\partial w}B^{\bullet\bullet}(x,w,t) &= wte^{S_1(x,w,t})+we^{S(x,w)}S_2(x,w,t),\\
D_1(x,w,t) &= (1+wt)e^{S_1(x,w,t)}-1,\\
S_1(x,w,t) &= x(D^{\bullet}(x,w)-S^{\bullet}(x,w))D^{\bullet}(x,t),\\
D_2(x,w,t) &= (1+wt)e^{S_2(x,w,t)},\\
S_2(x,w,t) &= x(D_2(x,w,t)-S_2(x,w,t))D(x,1)\\
&\quad + x(D_1(x,w,t)-S_1(x,w,t))D^{\bullet}(x,t)\\
&\quad + x(D^{\bullet}(x,w)-S^{\bullet}(x,w))D_2(x,1,t).
\end{aligned}
$$

Asymptotic Analysis

$$B^{\bullet\bullet}(x,w,t) = \frac{G(x,w,t) + H(x,w,t)W + I(x,w,t)T + J(x,w,t)WT}{\sqrt{1 - x/\rho_1}}$$

with the abbeviations

$$W = \sqrt{1 - y(x)w} \quad \text{and} \quad T = \sqrt{1 - y(x)t}$$

and with

$$\begin{aligned} & y(x)g(x) - h(x)\sqrt{1 - x/\rho_1}, \\ G(x,w,t) &= G_1(x,w,t) - G_2(x,w,t)\sqrt{1 - x/\rho_1}, \\ H(x,w,t) &= H_1(x,w,t) - H_2(x,w,t)\sqrt{1 - x/\rho_1}, \\ I(x,w,t) &= I_1(x,w,t) - I_2(x,w,t)\sqrt{1 - x/\rho_1}, \\ J(x,w,t) &= J_1(x,w,t) - J_2(x,w,t)\sqrt{1 - x/\rho_1}. \end{aligned}$$

The analytic behaviour of $C^{\bullet\bullet}(x,w,t)$ is of the same kind.

Asymptotic Analysis

Lemma 2

$$f(x,w,t) = \sum_{n,k,\ell} f_{n,k,\ell} x^n w^k t^\ell$$

$$= \boxed{\frac{G(x,w,t) + H(x,w,t)W + I(x,w,t)T + J(x,w,t)WT}{\sqrt{1 - x/x_0}}},$$

with the abbeviations $W = \sqrt{1 - y(x)w}$ and $T = \sqrt{1 - y(x)t}$, wher

$$y(x)g(x) - h(x)\sqrt{1 - x/x_0},$$

$$G(x,w,t) = G_1(x,w,t) - G_2(x,w,t)\sqrt{1 - x/x_0},$$

$$H(x,w,t) = H_1(x,w,t) - H_2(x,w,t)\sqrt{1 - x/x_0},$$

$$I(x,w,t) = I_1(x,w,t) - I_2(x,w,t)\sqrt{1 - x/x_0},$$

$$J(x,w,t) = J_1(x,w,t) - J_2(x,w,t)\sqrt{1 - x/x_0}$$

with analytic functions $g, h, G_1, G_2, H_1, H_2, I_1, I_2, J_1, J_2$
(+ some technical conditions)

Asymptotic Analysis

Lemma 2 (cont.)

$$\Longrightarrow \quad \boxed{f_{n,k,\ell} \sim \frac{J\left(x_0, 0, \frac{1}{g(x_0)}, \frac{1}{g(x_0)}\right)}{4\pi^{3/2}} g(x_0)^{k+\ell} x_0^{-n} (k\ell)^{-\frac{3}{2}} n^{-\frac{1}{2}}}$$

uniformly for $k, \ell \leq C \log n$ (for any constant $C > 0$) and

$$f_{n,k,\ell} = O\left((g(x_0) + \varepsilon)^{k+\ell} x_0^{-n} n^{-\frac{1}{2}}\right).$$

uniformly for all $n, k, \ell \geq 0$ for every $\varepsilon > 0$.

Remark This proves $p_{n,k,\ell} \sim c^2 q^{k+\ell} (k\ell)^{-\frac{3}{2}}$.

Proof of Lemma 1

1. Singularity Analysis
(following Flajolet-Odlyzko)

Suppose that

$$\boxed{y(x) = (1 - x/x_0)^{-\alpha}}.$$

Then

$$\boxed{y_n = [x^n]y(x) = (-1)^n \binom{-\alpha}{n} x_0^n = \frac{n^{\alpha-1}}{\Gamma(\alpha)} x_0^n + \mathcal{O}\left(n^{\alpha-2} x_0^n\right).}$$

Proof of Lemma 1

1. Singularity Analysis

Cauchy's formula:

$$(-1)^n \binom{-\alpha}{n} x_0^n = \frac{1}{2\pi i} \int_\gamma (1 - x/x_0)^{-\alpha} \, x^{-n-1} \, dx.$$

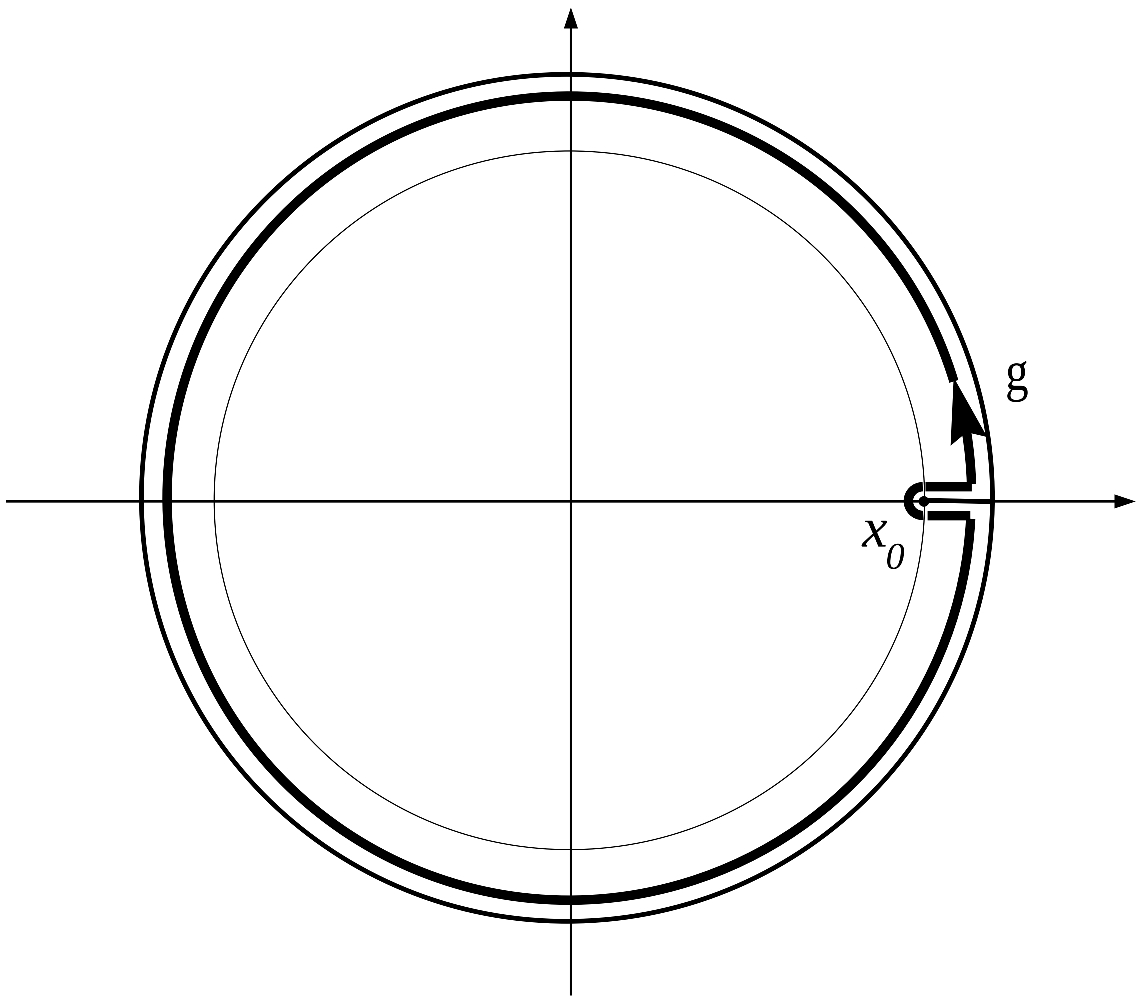
$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$:

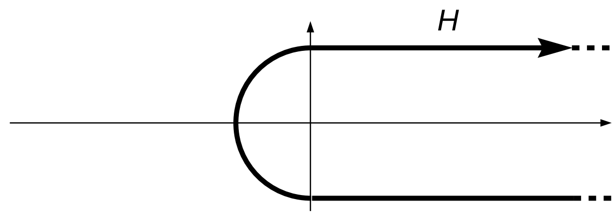
$$\begin{aligned}
\gamma_1 &= \left\{ x = x_0 \left(1 - \frac{i + (\log n)^2 - t}{n} \right) : 0 \le t \le (\log n)^2 \right\}, \\
\gamma_2 &= \left\{ x = x_0 \left(1 - \frac{1}{n} e^{-i\phi} \right) : -\frac{\pi}{2} \le \phi \le \frac{\pi}{2} \right\}, \\
\gamma_3 &= \left\{ x = x_0 \left(1 + \frac{i + t}{n} \right) : 0 \le t \le (\log n)^2 \right\},
\end{aligned}$$

and γ_4 is a circular arc centred at the origin and making γ a closed curve.

1. Singularity Analysis

Path of integration





1. Singularity Analysis

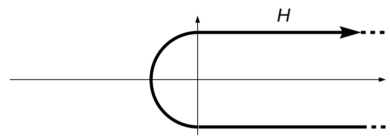
Substitution for $x \in \gamma_1 \cup \gamma_2 \cup \gamma_3$:

$$x/x_0 = 1 + \frac{t}{n} \Longrightarrow x^{-n-1} = e^{-t}\left(1 + \mathcal{O}\left(\frac{t^2}{n}\right)\right)$$

With Hankel's integral representation for $1/\Gamma(\alpha)$

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\gamma_1\cup\gamma_2\cup\gamma_3} (1 - x/x_0)^{-\alpha} x^{-n-1}\, dx &= \frac{n^{\alpha-1}x_0^n}{2\pi i}\int_H (-t)^{-\alpha} e^{-t}\, dt \\
&+ \frac{n^{\alpha-2}x_0^n}{2\pi i}\int_H (-t)^{-\alpha} e^{-t} \cdot \mathcal{O}\left(t^2\right)\, dt \\
&= n^{\alpha-1}\frac{1}{\Gamma(\alpha)} x_0^n + \mathcal{O}\left(n^{\alpha-2x_0^n}\right).
\end{aligned}$$

$H = \{t \,|\, |t| = 1, \Re t \le 0\} \cup \{t \,|\, 0 < \Re t \le \log^2 n, \Im t = \pm 1\}$:



1. Singularity Analysis

Remark

$$x \in \gamma_1 \cup \gamma_2 \cup \gamma_3 \quad \Longrightarrow \quad \boxed{\frac{1}{n} \le \left|1 - \frac{x}{x_0}\right| \le \frac{(\log n)^2}{n}}$$

Asymptotic Analysis

Lemma 1 (the same as before)

$$f(x,w) = \sum_{n,k\ge 0} f_{n,k} x^n w^k$$

$$= \boxed{G(x,w) + H(x,w)\,(1 - y(x)w)^{\frac{3}{2}}},$$

where

$$y(x) = g(x) - h(x)\sqrt{1 - x/x_0},$$

$$G(x,w) = G_1(x,w) - G_2(x,w)\sqrt{1 - x/x_0},$$

$$H(x,w) = H_1(x,w) - H_2(x,w)\sqrt{1 - x/x_0}.$$

with analytic functions g, h, G_1, G_2, H_1, H_2
(+ some technical conditions)

$$\Longrightarrow \quad \boxed{f_{n,k} = \frac{3h(x_0)H(x_0,0,1/g(x_0))}{8\pi} g(x_0)^{k-1} x_0^{-n} k^{-\frac{3}{2}} n^{-\frac{3}{2}} \left(1 + O\left(\frac{1}{k}\right)\right)}$$

uniformly for $k \le C \log n$ (for any constant $C > 0$) and

$$f_{n,k} = O\left((g(x_0) + \varepsilon)^k \rho^{-n} n^{-\frac{3}{2}} \right).$$

Proof of Lemma 1

2. Cauchy's formula

$$f_{n,k} = \frac{1}{(2\pi i)^2} \int_\gamma \int_\Gamma \frac{f(x,w)}{x^{n+1} w^{k+1}} \, dx \, dw$$

Integration with respect to x: $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where

$$\begin{aligned} \gamma_1 &= \left\{ x = x_0 \left(1 - \frac{i + (\log n)^2 - t}{n} \right) : 0 \le t \le (\log n)^2 \right\}, \\ \gamma_2 &= \left\{ x = x_0 \left(1 - \frac{1}{n} e^{-i\phi} \right) : -\frac{\pi}{2} \le \phi \le \frac{\pi}{2} \right\}, \\ \gamma_3 &= \left\{ x = x_0 \left(1 + \frac{i + t}{n} \right) : 0 \le t \le (\log n)^2 \right\}, \end{aligned}$$

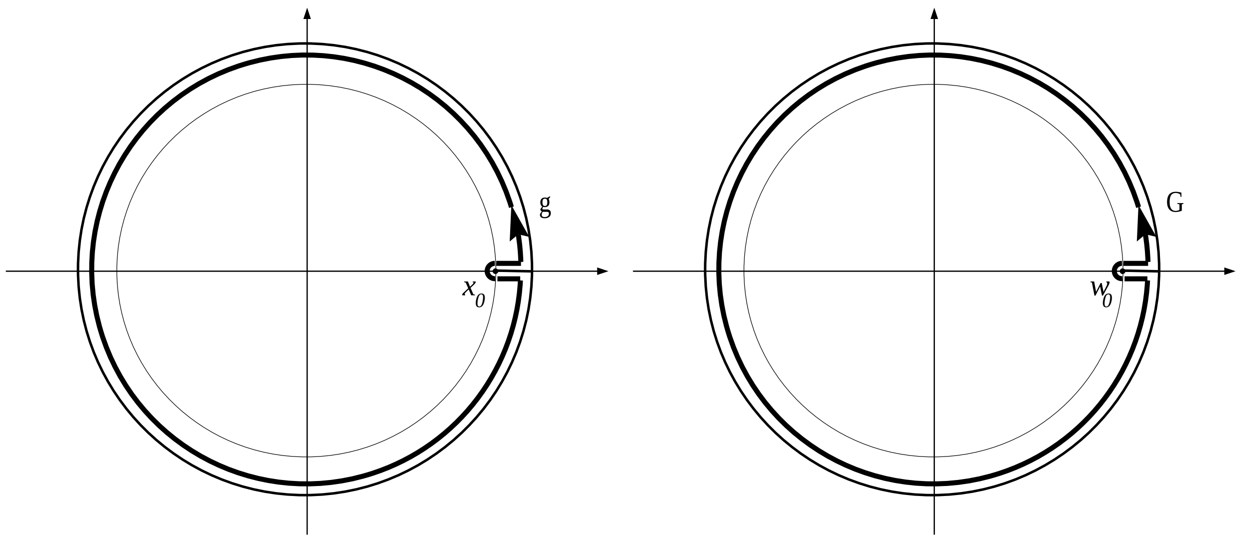
and γ_4 is a circular arc centred at the origin and making γ a closed curve.

2. Cauchy's formula

Integration with respect to w: $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where

$$\Gamma_1 = \left\{ w = w_0 \left(1 - \frac{i + (\log k)^2 - r}{k} \right) : 0 \le s \le (\log k)^2 \right\},$$
$$\Gamma_2 = \left\{ w = w_0 \left(1 - \frac{1}{k} e^{-i\psi} \right) : -\frac{\pi}{2} \le \psi \le \frac{\pi}{2} \right\},$$
$$\Gamma_3 = \left\{ w = w_0 \left(1 + \frac{i + s}{w} \right) : 0 \le s \le (\log k)^2 \right\},$$

and Γ_4 is a circular arc centred at the origin and making Γ a closed curve.
$(w_0 = 1/g(x_0))$



2. Cauchy's formula

Remark

$x \in \gamma_1 \cup \gamma_2 \cup \gamma_3$ and $w \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$:

$$\frac{1}{n} \leq \left|1 - \frac{x}{x_0}\right| \leq \frac{(\log n)^2}{n} \quad \text{and} \quad \frac{1}{k} \leq \left|1 - \frac{w}{w_0}\right| \leq \frac{(\log k)^2}{k}$$

For $k \leq C \log n$ we thus have

$$\boxed{X = \sqrt{1 - \frac{x}{x_0}} \quad \textbf{is much smaller than} \quad W = 1 - \frac{w}{w_0}}$$

Proof of Lemma 1

3. Local expansion around the singularity

$$
\begin{aligned}
y(x) &= g(x) - h(x)\sqrt{1 - x/x_0} \\
&= g(x_0) - h(x_0)X + O(X^2)
\end{aligned}
$$

$$
w = w_0 + w - w_0 = w_0(1 - W)
$$

$$
1 - y(x)w = W + h(x_0)w_0X + O(X^2)
$$

$$
\begin{aligned}
\boxed{(1 - y(x)w)^{\frac{3}{2}}} &= \left(W + h(x_0)w_0X + O(X^2)\right)^{3/2} \\
&= W^{3/2}\left(1 + \frac{(3/2)h(x_0)w_0X}{W} + O\left(\frac{X^2}{W}\right)\right) \\
&= W^{3/2} + \frac{3}{2}h(x_0)w_0\boxed{X\,W^{1/2}} + O\left(X^2W^{1/2}\right)
\end{aligned}
$$

3. Local expansion around the singularity

$$X\,W^{1/2} = \left(1 - \frac{x}{x_0}\right)^{\frac{1}{2}} \left(1 - \frac{w}{w_0}\right)^{\frac{1}{2}}$$

... Cauchy integration provides the asymptotic leading term

$$\boxed{\frac{1}{4\pi} x_0^{-n}\, w_0^{-k}\, n^{-\frac{3}{2}}\, k^{-\frac{3}{2}}}$$

Random Planar Graphs

Conjecture for maximum degree Δ_n

$$\frac{\Delta_n}{\log n} \to \frac{1}{\log(1/q)} \qquad \text{in probability}$$

and

$$\mathbb{E}\,\Delta_n \sim \frac{\log n}{\log(1/q)}$$

where $q = 0.6734506...$ appear in the asymptotics of $p_k \sim c\,k^{-\frac{1}{2}} q^k$; $1/\log(1/q) = 2.529464248...$

Random Planar Graphs

Degree Distribution

Theorem [D.+Giménez+Noy]

Let $p_{n,k}$ be the probability that a random vertex in a random planar graph $\mathcal{R}_n$ has degree k. Then the limit

$$\boxed{p_k := \lim_{n\to\infty} p_{n,k}}$$

exists. The probability generating function

$$\boxed{p(w) = \sum_{k\geq 1} p_k w^k}$$

can be explicitly computed; $\boxed{p_k \sim c\, k^{-\frac{1}{2}} q^k}$ for some $c > 0$ and $0 < q < 1$.

p_1	p_2	p_3	p_4	p_5	p_6
0.0367284	0.1625794	0.2354360	0.1867737	0.1295023	0.0861805

Random Planar Graphs

Counting Generating Functions

$$
\begin{aligned}
G(x,y) &= \exp\left(C(x,y)\right),\\
\frac{\partial C(x,y)}{\partial x} &= \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right),\\
\frac{\partial B(x,y)}{\partial y} &= \frac{x^2}{2}\,\frac{1+D(x,y)}{1+y},\\
\frac{M(x,D)}{2x^2 D} &= \log\left(\frac{1+D}{1+y}\right) - \frac{xD^2}{1+xD},\\
M(x,y) &= x^2y^2\left(\frac{1}{1+xy} + \frac{1}{1+y} - 1 - \frac{(1+U)^2(1+V)^2}{(1+U+V)^3}\right),\\
U &= xy(1+V)^2,\\
V &= y(1+U)^2.
\end{aligned}
$$

Random Planar Graphs

Asymptotic enumeration of planar graphs

$$b_n = b \cdot \rho_1^{-n} \, n^{-\frac{7}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right),$$
$$c_n = c \cdot \rho_2^{-n} \, n^{-\frac{7}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right),$$
$$g_n = g \cdot \rho_2^{-n} \, n^{-\frac{7}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right)$$

$$\rho_1 = 0.03819...,$$
$$\rho_2 = 0.03672841...,$$
$$b = 0.3704247487... \cdot 10^{-5},$$
$$c = 0.4104361100... \cdot 10^{-5},$$
$$g = 0.4260938569... \cdot 10^{-5}$$

Random Planar Graphs

Generating functions for the degree distribution of planar graphs

$C^\bullet = \frac{\partial C}{\partial x}$ ... GF, where one vertex is marked

w ... additional variable that *counts* the **degree of the marked vertex**

Generating functions:

$G^\bullet(x,y,w)$	**all rooted** planar graphs
$C^\bullet(x,y,w)$	**connected rooted** planar graphs
$B^\bullet(x,y,w)$	**2-connected rooted** planar graphs
$T^\bullet(x,y,w)$	**3-connected rooted** planar graphs

Random Planar Graphs

$$
\begin{aligned}
G^\bullet(x,y,w) &= \exp\left(C(x,y,1)\right) C^\bullet(x,y,w),\\
C^\bullet(x,y,w) &= \exp\left(B^\bullet\left(xC^\bullet(x,y,1),y,w\right)\right),\\
w\frac{\partial B^\bullet(x,y,w)}{\partial w} &= xyw\exp\left(S(x,y,w)+\frac{1}{x^2D(x,y,w)}T^\bullet\left(x,D(x,y,1),\frac{D(x,y,w)}{D(x,y,1)}\right)\right)\\
D(x,y,w) &= (1+yw)\exp\left(S(x,y,w)+\frac{1}{x^2D(x,y,w)}\times\right.\\
&\qquad \left.\times\, T^\bullet\left(x,D(x,y,1),\frac{D(x,y,w)}{D(x,y,1)}\right)\right)-1\\
S(x,y,w) &= xD(x,y,1)\left(D(x,y,w)-S(x,y,w)\right),\\
T^\bullet(x,y,w) &= \frac{x^2y^2w^2}{2}\left(\frac{1}{1+wy}+\frac{1}{1+xy}-1-\right.\\
&\qquad \left.-\frac{(u+1)^2\left(-w_1(u,v,w)+(u-w+1)\sqrt{w_2(u,v,w)}\right)}{2w(vw+u^2+2u+1)(1+u+v)^3}\right),\\
u(x,y) &= xy(1+v(x,y))^2,\quad v(x,y)=y(1+u(x,y))^2.
\end{aligned}
$$

Degree Distribution

with polynomials $w_1 = w_1(u, v, w)$ and $w_2 = w_2(u, v, w)$ given by

$$\begin{aligned} w_1 =& - uvw^2 + w(1 + 4v + 3uv^2 + 5v^2 + u^2 + 2u + 2v^3 + 3u^2v + 7uv) \\ &+ (u + 1)^2(u + 2v + 1 + v^2), \end{aligned}$$

$$\begin{aligned} w_2 =& u^2v^2w^2 - 2wuv(2u^2v + 6uv + 2v^3 + 3uv^2 + 5v^2 + u^2 + 2u + 4v + 1) \\ &+ (u + 1)^2(u + 2v + 1 + v^2)^2. \end{aligned}$$

Random Planar Graphs

Singular structure of $B^\bullet(x,1,w)$

$$\boxed{\frac{\partial B^\bullet(x,1,w)}{\partial w} = K(X,W) + \sqrt{L(X,W)}}$$

$$X = \sqrt{1-\frac{x}{x_0}}, \quad W = 1-\frac{w}{w_0}$$

$$L(X,W) = X^3 h_1(W) + X^2 W h_2(X,W) + 0 + W^3 h_4(W)$$

Random Planar Graphs

Lemma 1.2

$$f(x,w) = \sum_{n,k\ge 0} f_{n,k} x^n w^k$$
$$= \boxed{K(X,W) + \sqrt{L(X,W)}},$$

where $X = \sqrt{1 - x/x_0}$ and $W = 1 - w/w_0$ and

$$L(X,W) = X^3 h_1(W) + X^2 W h_2(X,W) + 0 + W^3 h_4(W)$$

with analytic functions K, h_1, h_2, h_4
(+ some technical conditions)

$$\Longrightarrow \quad \boxed{f_{n,k} = c\, x_0^{-n}\, w_0^{-k} k^{\frac{1}{2}} n^{-\frac{5}{2}} \left(1 + O\left(\frac{1}{k}\right)\right)}$$

Random Planar Graphs

Work in progress...

- Generating functions for double rooting

- Singular structure of generating functions

- Lemma 2.2

Thank You for Your Attention!