Dirichlet spectrum and transference inequalities

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Theorem (Dirichlet, 1842)

Let m and n be two strictly positive integers. For $i = 1, \dots, n$ and $j = 1, \dots, m$, let $\boldsymbol{\theta} = (\theta_{i,j})_{i,j}$ be a mn-tuple of real numbers. Let $Q > 1$ be a real number. Then there exists integers $q_1, \dots, q_m, p_1, \dots, p_n$ such that

$$1 \leq \max_{1 \leq j \leq n} |q_j| \leq Q \quad and \quad \max_{1 \leq i \leq m} |\theta_{i,1} q_1 + \cdots + \theta_{i,n} q_n - p_i| \leq Q^{-\frac{n}{m}}$$

Definition

We define the exponent $\omega_{m,n}(\boldsymbol{\theta})$ as the supremum of ω such that the system

$$0 < \|\boldsymbol{\theta} \cdot \boldsymbol{p}\| \leq Q^{-\omega} \ , \ 1 \leq \max_{1 \leq i \leq n} |p_i| \leq Q$$

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and $\hat{\omega}_{m,n}(\boldsymbol{\theta})$ such that we have solutions for all large Q.

Theorem (Khintchine Transference Principle, 1926)

The inequalities

$$\frac{\omega_{1,n}(\boldsymbol{\theta})}{(n-1)\omega_{1,n}(\boldsymbol{\theta})+n} \leq \omega_{n,1}({}^t\boldsymbol{\theta}) \leq \frac{\omega_{1,n}(\boldsymbol{\theta})-n+1}{n}$$

holds for every point $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ *in* $\mathbb{R}^n$ *with* $1, \theta_1, \dots, \theta_n$ *linearly independant over* $\mathbb{Q}$.

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Theorem (Dyson, 1947)

Let $A \in M_{m\times n}(\mathbb{R})$. *Write* $\omega = \omega_{m,n}(A)$ *and* ${}^t\omega = \omega_{n,m}({}^tA)$. *Then,*

$${}^t\omega \geq \frac{n\omega+m-1}{(n-1)\omega+m}, \quad \text{and} \quad \omega \geq \frac{m\,{}^t\omega+n-1}{(m-1)\,{}^t\omega+n}.$$

Improving Jarník and Apfelbeck, German showed that for uniform exponents,

Theorem (German, 2011)

$$\hat{\omega}_{n,m}(\boldsymbol{\theta}) \geq \frac{n-1}{m-\hat{\omega}_{m,n}({}^t\boldsymbol{\theta})} \quad \textit{if } \hat{\omega}_{m,n}({}^t\boldsymbol{\theta}) \leq 1$$

$$\hat{\omega}_{n,m}(\boldsymbol{\theta}) \geq \frac{n-\hat{\omega}_{m,n}({}^t\boldsymbol{\theta})^{-1}}{m-1} \quad \textit{if } \hat{\omega}_{m,n}({}^t\boldsymbol{\theta}) \geq 1$$

Question A : Show optimality for $m, n > 1$ of Dyson and German transference inequalities and their respective splitting via intermediate exponents.
Idea : play with *systems* in the frame of parametric geometry of numbers.

For each $t \in \mathbb{R}$ and for each matrix A, let

$$g_t = \begin{bmatrix} e^{t/m}\mathrm{I}_m & \\ & e^{-t/n}\mathrm{I}_n \end{bmatrix}, \qquad u_A = \begin{bmatrix} \mathrm{I}_m & A \\ & \mathrm{I}_n \end{bmatrix},$$

where I_k denotes the k-dimensional identity matrix. Finally, let $d = m + n$, and for each $j = 1, \ldots, d$, let $\lambda_j(\Lambda)$ denote the jth successive minimum of a lattice $\Lambda \subset \mathbb{R}^d$ (with respect to some fixed norm on $\mathbb{R}^d$)

We consider the function

$$\begin{array}{cccc} \boldsymbol{\Lambda_A}: & [0,\infty) & \to & \mathbb{R}^{n+m} \\ & t & \mapsto & (\log(\lambda_1(g_t u_A \mathbb{Z}^n), \ldots, \log(\lambda_{n+m}(g_t u_A \mathbb{Z}^n)). \end{array}$$

Proposition

With the previous notation, we have

$$\liminf_{t\to\infty} \frac{-1}{t} \log \lambda_1(g_t u_A \mathbb{Z}^d) = \frac{1}{n} \frac{\omega_{m,n} - \frac{m}{n}}{\omega_{m,n} + 1}.$$

Definition (Das – Fishman – Simmons – Urbański, 2019)

An $m \times n$ *template* is a continuous piecewise linear map $\boldsymbol{P} : \mathbb{R}_+ \to \mathbb{R}^d$ with the following properties:

(I) $P_1 \leq \cdots \leq P_d$.

(II) $-\frac{1}{m} \leq P_i' \leq \frac{1}{n}$ for all i.

(III) For all $j = 1, \ldots, d$ and for every interval I such that $P_j < P_{j+1}$ on I, the function $P_j = \sum_{i \leq j} P_i$ is convex and piecewise linear on I with slopes in $Z(j)$.

$$Z(j) = \left\{ \frac{k}{m} - \frac{l}{n} : k + l = j, \quad 0 \leq k \leq m, \quad 0 \leq l \leq n \right\}, \tag{1}$$

We use the convention that $P_0 = -\infty$ and $P_{d+1} = +\infty$.

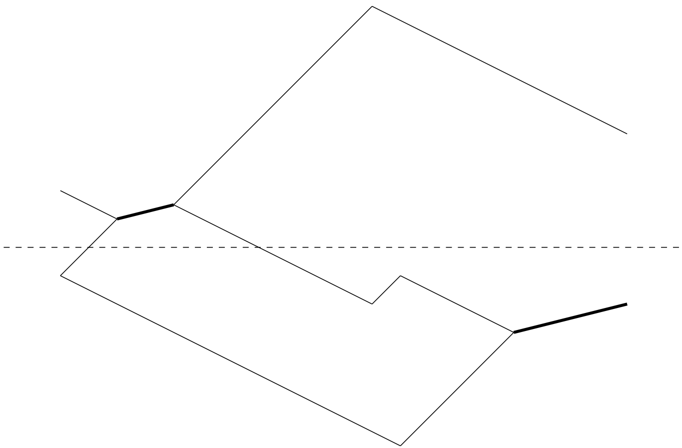

Figure: The joint graph of a 1×2 partial template $f = (f_1, f_2, f_3)$, where the *joint graph* of a template is the union of the graphs of its component functions

For a template $\boldsymbol{P}$ on $[T_0, \infty)$, we define the *local contraction rate* $\delta(\boldsymbol{P}, t)$ by

$\delta(\boldsymbol{P}, t) = \#\{indices(k, l), \quad k < l, \quad P_k \text{ goes up and } P_l \text{ goes down}$

We then consider the *average contraction rate* defined by

$$\Delta(\boldsymbol{P}, T) = \frac{1}{T - T_0} \int_{T_0}^{T} \delta(\boldsymbol{P}, t) dt,$$

Consider the lower average contraction rates $\underline{\delta}(\boldsymbol{P})$ defined by

$$\underline{\delta}(\boldsymbol{P}) \quad = \quad \liminf_{T \to \infty} \Delta(\boldsymbol{P}, T),$$

Theorem (Variational principle, Das – Fishman – Simmons – Urbański, 2019)

Be $\mathcal{P}$ a set of templates on $[T_0, \infty)$ closed under finite perturbation. Let

$$\mathcal{M}(\mathcal{P}) = \{A \in \mathbb{R}^{m\times n} \mid \exists \boldsymbol{P} \in \mathcal{P}, C \in \mathbb{R}, \|\Lambda_A - \boldsymbol{P}\| \leq C\}.$$

Then,

$$\dim_H(\mathcal{M}(\mathcal{P})) \quad = \sup_{\boldsymbol{P}\in\mathcal{P}} \underline{\delta}(\boldsymbol{P}),$$

Question B : determine the spectrum of the 4 exponents $\omega_{n,1}, \omega_{1,n}, \hat{\omega}_{n,1}, \hat{\omega}_{1,n}$.

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Theorem (Bugeaud, Laurent, 2009)

For all n-tuple $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ with $1, \theta_1, \ldots, \theta_n$ $\mathbb{Q}$-linearly independent, we have the following transference inequalities.

$$\frac{(\hat{\omega}_{n,1}(\boldsymbol{\theta}) - 1)\omega_{n,1}(\boldsymbol{\theta})}{((n-2)\hat{\omega}_{n,1}(\boldsymbol{\theta}) + 1)\omega_{n,1}(\boldsymbol{\theta}) + (n-1)\hat{\omega}_{n,1}(\boldsymbol{\theta})} \leq \omega_{1,n}(\boldsymbol{\theta})$$
$$\leq \frac{(1 - \hat{\omega}_{1,n}(\boldsymbol{\theta}))\omega_{n,1}(\boldsymbol{\theta}) - n(2 - \hat{\omega}_{1,n}(\boldsymbol{\theta}))}{n-1}.$$

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Idea: Start with $n = 3$. For general n, Schleischitz showed that BL is reached when the ratio $\omega_{n,1}/\hat{\omega}_{n,1}$ is large. One could try to decompose the spectrum in parts depending on this ratio. See recent work with Moshchevitin.

For a given norm consider the distance to a nearest integer $\|.\|$. Given $\theta \in \mathbb{R}^n$, define

$$\psi_\theta(Q) := \min_{1 \leq q \leq Q} \|q\theta\|$$

and consider the associated Dirichlet constant

$$d(\theta) := \limsup_{Q \to \infty} Q^{1/n} \psi_\theta(Q).$$

The Dirichlet spectrum is the set of all values that the Dirichlet constant takes.

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In recent work, Schleischitz shows that for $n \geq 2$ and the sup norm, the Dirichlet spectrum is $[0,1]$ for both simultaneous approximation and approximation by one linear form.
Question C : can this construction be adapted for other norms ? Can it be extended to n systems in m linear forms ?

For simultaneous approximation, Schleischitz constructs Liouville numbers. For approximation by one linear form, he explains that it does not need to be the case. This may be refined.
Question D : For $d \in [0,1]$ and θ such that $d(\theta) = d$, how large can be $d^* := \liminf_{Q\to\infty} Q^{1/n}\psi_\theta(Q)$? Can we have badly approximable numbers ?
Question E : What do we know about Hausdorff dimension of point with given d (and d^*) ? This relates to the (difficult) open problem of the Hausdorff dimension of ϵ- Dirichlet improvable numbers $DI_n(\epsilon) := \{\theta \mid \psi_\theta(Q) \leq \epsilon Q^{1/n}\}$.