Transference inequalities

Austrian–Russian two-days workshop

Oleg N. German

Moscow State University

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Diophantine exponents

$$\Theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix} \in \mathbb{R}^n, \quad \ell = \mathbb{R}\begin{pmatrix} 1 \\ \Theta \end{pmatrix} \qquad \Theta^* = \begin{pmatrix} \theta_1 & \cdots & \theta_n \end{pmatrix} \in (\mathbb{R}^n)^*, \quad \ell^\perp$$

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$$(*) \quad \begin{cases} |x_0| \leqslant t \\ \max\limits_{1\leqslant i\leqslant n} |x_0\theta_i - x_i| \leqslant t^{-\gamma} \end{cases} \qquad (**) \quad \begin{cases} |x_0 + \theta_1 x_1 + \ldots + \theta_n x_n| \leqslant t^{-\gamma} \\ \max\limits_{1\leqslant i\leqslant n} |x_i| \leqslant t \end{cases}$$

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Regular exponents

$$\lambda = \lambda(\Theta) = \sup\left\{\gamma \in \mathbb{R} \,\middle|\, \exists t \text{ however large: } (*) \text{ has a solution } (x_0,\ldots,x_n) \in \mathbb{Z}^{n+1}\backslash\{\mathbf{0}\}\right\}$$

$$\omega = \omega(\Theta) = \sup\left\{\gamma \in \mathbb{R} \,\middle|\, \exists t \text{ however large: } (**) \text{ has a solution } (x_0,\ldots,x_n) \in \mathbb{Z}^{n+1}\backslash\{\mathbf{0}\}\right\}$$

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Dirichlet's theorem for simultaneous approximation, 1842

For every $t \geqslant 1$ $(*)$ with $\gamma = 1/n$ admits a nonzero solution $(x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}$

Dirichlet's theorem for a linear form, 1842

For every $t \geqslant 1$ $(**)$ with $\gamma = n$ admits a nonzero solution $(x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}$

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Corollary

$$\lambda \geqslant \hat{\lambda} \geqslant \frac{1}{n} \qquad \omega \geqslant \hat{\omega} \geqslant n$$

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"Trivial" bounds

$$\lambda \geqslant \hat{\lambda} \geqslant \frac{1}{n} \qquad \omega \geqslant \hat{\omega} \geqslant n \qquad \hat{\lambda} \leqslant 1$$

Transference theorems

A. Khintchine, 1926

$$\frac{1+\omega}{1+\lambda} \geqslant n, \qquad \frac{1+\omega^{-1}}{1+\lambda^{-1}} \geqslant \frac{1}{n}$$

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$$\frac{1+\omega}{1+\lambda} \geqslant \frac{n-1}{1-\hat{\lambda}}, \qquad \frac{1+\omega^{-1}}{1+\lambda^{-1}} \geqslant \frac{1-\hat{\omega}^{-1}}{n-1}$$

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$$\hat{\omega}^{-1}+\hat{\lambda}=1$$

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A. Marnat, N. Moshchevitin, 2020

$$\frac{\omega}{\hat{\omega}} \geqslant G_{\text{lin}}(\hat{\omega}), \qquad \frac{\lambda}{\hat{\lambda}} \geqslant G_{\text{sim}}(\hat{\lambda})$$

$G_{\text{lin}}(\hat{\omega})$ is the largest root of $f(x) = \hat{\omega}^{-1}x^n - x + (1-\hat{\omega}^{-1})$

$G_{\text{sim}}(\hat{\lambda})$ is the largest root of $g(x) = (1-\hat{\lambda})x^n - x^{n-1} + \hat{\lambda}$

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Question

Is it true that these inequalities describe the whole spectrum of the quadruples $(\lambda, \hat\lambda, \omega, \hat\omega)$?

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Schmidt–Summerer implies the weakest of Marnat–Moshchevitin

Schmidt–Summerer: $\dfrac{\omega}{\hat{\omega}} \geqslant \dfrac{1+\lambda}{1+\omega^{-1}}\,, \qquad \dfrac{\lambda}{\hat{\lambda}} \geqslant \dfrac{1+\lambda}{1+\omega^{-1}}$

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A strange corollary to Schmidt–Summerer

$$\frac{1+\lambda}{1+\omega^{-1}} \geqslant \frac{1-\hat{\omega}^{-1}}{1-\hat{\lambda}}$$

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Quantities in between

We have either

$$G_{\mathrm{lin}}(\hat\omega) \leqslant (\hat\omega\hat\lambda)^{\frac{1}{n-1}} \leqslant \frac{1-\hat\omega^{-1}}{1-\hat\lambda} \leqslant G_{\mathrm{sim}}(\hat\lambda),$$

or

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or $G_{\mathrm{sim}}(\hat{\lambda}) \leqslant \frac{1-\hat{\omega}^{-1}}{1-\hat{\lambda}} \leqslant (\hat{\omega}\hat{\lambda})^{\frac{1}{n-1}} \leqslant G_{\mathrm{lin}}(\hat{\omega})$

Question

What does all this mean geometrically?

Intermediate Diophantine exponents

$$\Theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, \qquad \mathbf{L} = \begin{pmatrix} 1 \\ \Theta \end{pmatrix}, \qquad \ell = \mathbb{R}\mathbf{L} \qquad\qquad (*) \quad \begin{cases} 0 < |\mathbf{X}| \leqslant t \\ |\mathbf{L} \wedge \mathbf{X}| \leqslant t^{-\gamma} \end{cases}$$

k^{th} intermediate Diophantine exponent of Θ

$$\omega_k(\Theta) = \sup\left\{\gamma \in \mathbb{R} \,\middle|\, \exists t \text{ however large: } (*) \text{ has a decomposable solution } \mathbf{X} \in \textstyle\bigwedge^k(\mathbb{Z}^{n+1})\right\}$$

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$$\omega_1 = \lambda, \quad \hat{\omega}_1 = \hat{\lambda}, \qquad\qquad \omega_n = \omega, \quad \hat{\omega}_n = \hat{\omega}$$

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$$\frac{1 + \omega_{k+1}}{1 + \omega_k} \geqslant \frac{n + 1 - k}{n - k}, \qquad k = 1, \ldots, n - 1$$

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Bugeaud–Laurent splits Khintchine

$$\frac{1+\omega_{k+1}}{1+\omega_k} \geqslant \frac{n+1-k}{n-k} \qquad \Longrightarrow \qquad \frac{1+\omega}{1+\lambda} \geqslant n$$

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O.G., 2012

$$1 + \hat{\omega}_2 \geqslant \frac{1}{1 - \hat{\omega}_1} + \frac{1}{n-1}, \qquad \frac{1 + \hat{\omega}_{k+1}}{1 + \hat{\omega}_k} \geqslant \frac{n+1-k}{n-k}, \qquad k = 2, \ldots, n-1$$

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Uniform transference also gets split

$$1 + \hat{\omega}_2 \geqslant \frac{1}{1 - \hat{\omega}_1} + \frac{1}{n-1}, \qquad \frac{1 + \hat{\omega}_{k+1}}{1 + \hat{\omega}_k} \geqslant \frac{n+1-k}{n-k} \qquad \Longrightarrow \qquad \hat{\omega} \geqslant \frac{n-1}{1 - \hat{\lambda}}$$

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Question: How can Schmidt–Summerer be split?

$$??? \qquad \Longrightarrow \qquad \hat{\omega} \leqslant \frac{1+\omega}{1+\lambda}, \quad \hat{\lambda} \leqslant \frac{1+\omega^{-1}}{1+\lambda^{-1}}$$

Thank you!