Metric Diophantine approximation, trigonometric products, and continued fraction statistics

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Sudler products I

For $\alpha \notin \mathbb{Q}$, consider the product

$$P_N(\alpha) = \prod_{n=1}^{N} 2\left|\sin(\pi n \alpha)\right|.$$

How large is $P_N(\alpha)$ as $N \to \infty$?

Sudler products II

Let ϕ denote the Golden Mean.

Theorem (Mestel and Verschueren, 2016)

There is a constant C such that

$$P_{F_m}(\phi) \to C$$

as $m \to \infty$.

Alternatively: $P_{F_m - 1}(\phi) \to \tilde{C} F_m$ as $m \to \infty$.

Compare: $\prod_{n=1}^{M-1} 2|\sin(\pi n/M)| = M$.

Keyword: cotangent sums.

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Sudler products III

Theorem (Grepstad, Kaltenböck, Neumüller, 2020)

$$\liminf_{N \to \infty} P_N(\phi) > 0.$$

Sudler products IV

Assume $N = F_{100} + F_{98}$.

$$P_N(\phi) = \prod_{n=1}^{N} 2|\sin(\pi n \phi)|$$

$$= \prod_{n=1}^{F_{100}} 2|\sin(\pi n \phi)| \times \prod_{n=F_{100}+1}^{F_{100}+F_{98}} 2|\sin(\pi n \phi)|$$

$$= \prod_{n=1}^{F_{100}} 2|\sin(\pi n \phi)| \times \prod_{n=1}^{F_{98}} 2|\sin(\pi (F_{100}+n) \phi)|$$

$$= \prod_{n=1}^{F_{100}} 2|\sin(\pi n \phi)| \times \prod_{n=1}^{F_{98}} 2|\sin(\pi (n\phi + \underbrace{F_{100}\phi}_{\text{the ``shift''}}))|.$$

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$$\begin{aligned} P_N(\phi) &= \prod_{n=1}^{N} 2|\sin(\pi n \phi)| \\ &= \prod_{n=1}^{F_{100}} 2|\sin(\pi n \phi)| \times \prod_{n=F_{100}+1}^{F_{100}+F_{98}} 2|\sin(\pi n \phi)| \\ &= \prod_{n=1}^{F_{100}} 2|\sin(\pi n \phi)| \times \prod_{n=1}^{F_{98}} 2|\sin(\pi (F_{100}+n) \phi)| \\ &= \prod_{n=1}^{F_{100}} 2|\sin(\pi n \phi)| \times \prod_{n=1}^{F_{98}} 2|\sin(\pi (n\phi + \underbrace{F_{100}\phi}_{\text{the "shift"}}))|. \end{aligned}$$

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Sudler products V

Generally:

$$P_N(\alpha, x) = \prod_{n=1}^{N} 2|\sin(\pi(n\alpha + x))|.$$

It turns out:

$$P_{q_k}(\alpha, (-1)^k x/q_k)$$

converges to a function.

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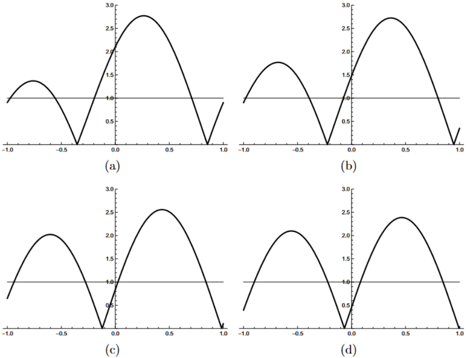
Sudler products VI



Sudler products VII

2 Shifted Sudler products

Let

$$P_N(\alpha, x) := \prod_{n=1}^{N} |2\sin(\pi(n\alpha + x))|, \qquad \alpha, x \in \mathbb{R}$$

denote a shifted form of the Sudler product. Given a non-negative integer with Ostrowski expansion $N = \sum_{k=0}^{K-1} b_k q_k$, let us also introduce the notation

$$\varepsilon_k(N) := q_k \sum_{\ell=k+1}^{K-1} (-1)^{k+\ell} b_\ell \|q_\ell \alpha\|. \tag{12}$$

It is then easy to see that

$$P_N(\alpha) = \prod_{k=0}^{K-1} \prod_{b=0}^{b_k-1} P_{q_k}(\alpha, (-1)^k (b q_k \|q_k \alpha\| + \varepsilon_k(N))/q_k), \tag{13}$$

which will serve as a fundamental tool in the proof of our results. This product form

Sudler products VIII

Theorem (A.–Technau–Zafeiropoulos, 2022)

Let $\alpha = [0; \bar{a}]$. *Then*

$$\liminf_{N \to \infty} P_N(\alpha) > 0 \qquad \text{and} \qquad \limsup_{N \to \infty} \frac{P_N(\alpha)}{N} < \infty$$

if and only if $a \in \{1, 2, 3, 4, 5\}$.

Sudler products - the quantum connection I

Let

$$J_{4_1,0}(q) = \sum_{n=0}^{\infty} \left|(1-q)(1-q^2)\dots(1-q^n)\right|^2 = \sum_{n=0}^{\infty} |(q;q)|^2.$$

If $q = e^{2\pi i a/Q}$, then

$$J_{4_1,0}(q) = \mathbf{J}_{4_1}(a/Q) = \sum_{N=1}^{Q} P_N^2(a/Q).$$

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Note that trivially $P_N(x) = P_N(x+1)$, and so $\mathbf{J}_{4_1}(x) = \mathbf{J}_{4_1}(x+1)$.

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$$h(x) = \log \frac{\mathbf{J}_{4_1}(x)}{\mathbf{J}_{4_1}(1/x)}.$$

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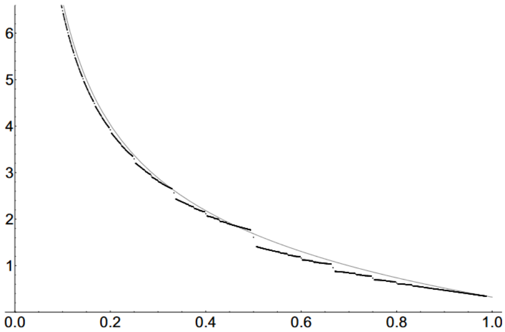

Figure 1: The function $h(x)$, evaluated at all rationals in $(0,1)$ with denominator at most 80 (black graph with jumps). For comparison, the plot also shows the function $\frac{\mathrm{Vol}(4_1)}{2\pi x} - \frac{3}{2}\log x$ (thin gray solid line), which is suggested as a continuous approximation to $h(x)$ by Formula (4).

Sudler products - the quantum connection III

Theorem (A.–Borda, 2021)

The function $h(x)$ is continuous at all irrationals x that have unbounded partial quotients.

Connection with value distribution of quantum modular forms (Bettin–Drappeau).

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The Duffin–Schaeffer conjecture I

Theorem (Khintchine, 1924)

Assume that ψ is decreasing. The equation

$$\left| \alpha - \frac{p}{q} \right| \leq \frac{\psi(q)}{q}$$

has finitely resp. infinitely many solutions for almost all α, if the series

$$\sum_{q=1}^{\infty} \psi(q)$$

converges resp. diverges.

The conclusion fails without the monotonicity assumption (Duffin–Schaeffer, 1942).

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The Duffin–Schaeffer conjecture II

Theorem (Koukoulopoulos–Maynard, 2020)

~~*Assume that ψ is decreasing.*~~ *The equation*

$$\left| \alpha - \frac{a}{q} \right| \leq \frac{\psi(q)}{q}$$

has finitely resp. infinitely many solutions with a, q co-prime for almost all α, if the series

$$\sum_{q=1}^{\infty} \frac{\psi(q)\varphi(q)}{q}.$$

converges resp. diverges.

The Duffin–Schaeffer conjecture III

Let

$$\mathcal{E}_q := \bigcup_{\substack{1 \leq a \leq q, \\ (a,q)=1}} \left(\frac{a}{q} - \frac{\psi(q)}{q}, \frac{a}{q} + \frac{\psi(q)}{q} \right).$$

Pollington–Vaughan:

$$\lambda(\mathcal{E}_q \cap \mathcal{E}_r) \ll \lambda(\mathcal{E}_q)\lambda(\mathcal{E}_r) \prod_{\substack{p \mid \frac{qr}{(q,r)^2}, \\ (q,r) \text{ ``large''}}} \left(1 + \frac{1}{p}\right).$$

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The Duffin–Schaeffer conjecture IV

Theorem (Koukoulopoulos–Maynard, 2020)

Let $\mathcal{Q}$ be a set of integers. If

$$\sum_{q \in \mathcal{Q}} \lambda(\mathcal{E}_q) \geq 1,$$

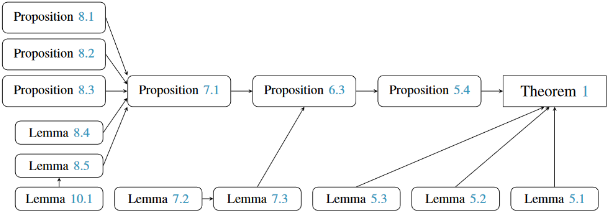
then

$$\frac{\sum_{q,r \in \mathcal{Q}} \lambda(\mathcal{E}_q \cap \mathcal{E}_r)}{\left(\sum_{q \in \mathcal{Q}} \lambda(\mathcal{E}_q)\right)^2} \ll 1.$$

The Duffin–Schaeffer conjecture V

half of the paper to demonstrate the key statements of Propositions 8.1-8.3.

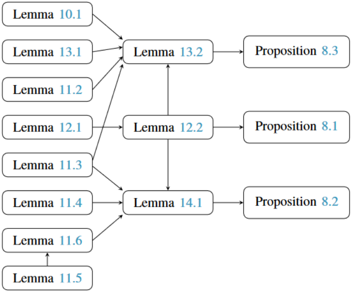
The dependency diagram for the first half of the paper is as follows:



The second half of the paper consists of Sections 11-14, and it is devoted to proving each of

The Duffin–Schaeffer conjecture VI

follows quickly from Lemma 14.1, which in turn relies on Lemmas 11.3-11.6. The dependency diagram for the second half of the paper is as follows:



(We have not included the essentially trivial statement of Lemma 11.1 or Lemma 6.7 which are

The Duffin–Schaeffer conjecture VII

Theorem (A.–Borda–Hauke, 2022)

If

$$\sum_{q \in \mathcal{Q}} \lambda(\mathcal{E}_q) = \Psi(\mathcal{Q}),$$

then

$$\frac{\sum_{q,r \in \mathcal{Q}} \lambda(\mathcal{E}_q \cap \mathcal{E}_r)}{\left(\sum_{q \in \mathcal{Q}} \lambda(\mathcal{E}_q)\right)^2} \leq 1 + O\left(\frac{1}{(\log \Psi(\mathcal{Q}))^C}\right).$$

The Duffin–Schaeffer conjecture VIII

Theorem (A.–Borda–Hauke, 2022)

Let

$$\sum_{q=1}^{Q} \lambda(\mathcal{E}_q) = \Psi(Q).$$

Then for almost all α the number of co-prime solutions [of the inequality] is of order

$$\Psi(Q) + O\left(\frac{\Psi(Q)}{(\log \Psi(Q))^C}\right).$$

Normal numbers I

A number x is normal in base b if the sequence $(b^n x)_{n \geq 1}$ is u.d. mod 1.

If x is normal in base b, then $x + p/q$ and $p/q \cdot x$ are also normal in base b.

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Normal numbers II

There is a characterization of those y for which

$$x \text{ normal in base } b \Rightarrow x + y \text{ normal in base } b$$

(Rauzy 1976; keyword: entropy).

Q1: This set depends on b. Are there numbers y which preserve normality under addition *in all bases*?

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Q2: Is there an irrational y such that

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Q3: Same question in all bases.

Normal numbers III

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Q3: Same question in all bases.

Normal numbers IV

Q4: What is the minimal order of the discrepancy of $(b^n x)$? (Korobov's problem; Levin 1999).

Q5: What is the minimal order of the discrepancy of $(b^n x)$ in all bases? (A.–Becher–Scheerer–Slaman, 2017).

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Q5: What is the minimal order of the discrepancy of $(b^n x)$ in all bases? (A.–Becher–Scheerer–Slaman, 2017).

Continued fractions I

Let $S(a/N)$ denote the sum of partial quotients of a/N.
Let $M(a/N)$ denote the maximal partial quotient.

Theorem (Zaremba, 1974)

For all N there is a reduced fraction a/N with $M(a/N) \ll \log N$.

Theorem (Larcher, 1986)

For all N there is a reduced fraction a/N with

$$S(a/N) \ll \log N \log\log N \frac{N}{\varphi(N)}.$$

Theorem (Rukavishnikova, 2011)

Let $g(N) \to \infty$. For all N there is a reduced fraction a/N with

$$S(a/N) \ll \log N \log\log N \; g(N).$$

Continued fractions II

Theorem (Bettin and Drappeau, 2020)

There is a limit distribution of

$$\frac{S(a/N) - \frac{12}{\pi^2} \log\log Q}{\log Q}$$

w.r.t. the normalized counting measure on the Farey fractions of order Q.

Similar results for the distribution of the length (Baladi–Vallee), the maximal partial quotient (Hensley), etc.

Continued fractions III

Q: In which way are different partial quotients of reduced fractions a/N "independent"?