Mutual behaviour of irrationality measure functions

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Irrationality measure function

α - real irrational, $||\cdot||$ - distance to the nearest integer

$$\psi_\alpha(t) = \min_{1\le q\le t,\, q\in\mathbb{Z}} ||q\alpha||, \quad t\ge 1$$

continued fractions (Lagrange Theorem):

$$\alpha = [a_0; a_1, a_2, ..., a_n, ...], \;\; \frac{p_n}{q_n} = [a_0; a_1, a_2, ..., a_n], \;\; \xi_n = |q_n\alpha - p_n|$$

$$\psi_\alpha(t) = \xi_n \;\text{ for }\; q_n \le t < q_{n+1}.$$

for any t we have

$$\psi_\alpha(t) < t^{-1}$$

Irrationality measure functions for two numbers

Theorem (I.D. Kan + N.M., Unif. Distr. Th. 5:2 (2010))
Suppose $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$ such that $\alpha \pm \beta \notin \mathbb{Z}$. Then the difference function

$$\psi_\alpha(t) - \psi_\beta(t)$$

changes its sign infinitely many times as $t \to +\infty$

- Informal question: Is there other objects that behave themselves in a similar way?
- variations on multi-dimensional irrationality measure functions
- variations on one-dimensional irrationality measure functions
- many irrationality measure functions

Multidimensional irrationality measure functions: definitions

$$\Theta = \begin{pmatrix} \theta_1^1 & \cdots & \theta_1^m \\ \cdots & \cdots & \cdots \\ \theta_n^1 & \cdots & \theta_n^m \end{pmatrix} - \text{ real matrix}, \quad d = m + n$$

$$\mathbf{x} = (x_1, ..., x_m) \in \mathbb{Z}^m \mapsto \begin{cases} L_j(\mathbf{x}) = \theta_j^1 x_1 + \cdots + \theta_j^m x_m, \\ \qquad\qquad 1 \le j \le n \\ M(x) = \max_{1 \le i \le m} |x_i| \end{cases}$$

$$\psi_\Theta(t) = \min_{\mathbf{x} \in \mathbb{Z}^m:\, 0 < M(\mathbf{x}) \le t} \ \max_{1 \le j \le n} ||L_j(\mathbf{x})|| \ - \text{ piecewise constant function}$$

- Minkowski: $\psi_\Theta(t) < t^{-\frac{m}{n}} \ \ \forall\ t > 1$
- badly approximability: $\exists C > 0\ \exists \Theta$ s.t. $\psi_\Theta(t) > Ct^{-\frac{m}{n}} \ \ \forall\ t > 1$
- irrationality $\psi_\Theta(t) > 0\ \forall t$
- it is possible to consider different norms

Singular matricies: Khintchine-Jarník's theory, $m + n \geq 3$

$$\Theta = \begin{pmatrix} \theta_1^1 & \cdots & \theta_1^m \\ \cdots & \cdots & \cdots \\ \theta_n^1 & \cdots & \theta_n^m \end{pmatrix} \quad \mapsto \quad \psi_\Theta(t)$$

Theorem (V. Jarník, 1959) *Suppose that $\psi(t)$ satisfies*

1. $n \geq 1$ *and* $m \geq 2$; $\quad \psi(t) \downarrow 0$ *as* $t \to +\infty$.

2. $n \geq 1$ *and* $m = 1$; $\quad \psi(t) \downarrow 0$ *as* $t \to +\infty$ *and* $\psi(t)t \uparrow +\infty$.

Then there exists a matrix Θ such that
– numbers θ_j^i are algebraically independent,
– one has $\psi_\Theta(t) \leq \psi(t) \ \ \forall\, t$.

Multidimensional irrationality measure functions: singularity

$$\Theta \quad \mapsto \quad \psi_\Theta(t)$$

General Proposition

Suppose that $m + n \geq 3$*. Then for any* Θ

$$\textit{with } \psi_\Theta(t) > 0 \ \forall\, t$$

there exists Θ_1 *with*

$$\psi_\Theta(t) > \psi_{\Theta_1}(t) \quad \textit{for all } t \quad \textit{large enough}$$

About badly approximability

badly approximability: $\exists C > 0 \ \exists \Theta$ s.t. $\psi_\Theta(t) > Ct^{-\frac{m}{n}} \ \ \forall \ t > 1$

Example $m = 1, n = 2$, $\Theta = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$. Dirichlet spectrum

$$\mathbb{D} = \{d : \ \exists(\alpha_1, \alpha_2) \text{ such that } \ d = \limsup_{t \to \infty} t^{\frac{1}{2}} \cdot \psi_\Theta(t)\}$$

$$\mathbb{D}^{bad} = \{d : \ \exists(\alpha_1, \alpha_2) \text{- badly approximable such that with } d = \limsup_{t \to \infty} t^{\frac{1}{2}} \cdot \psi_\Theta(t)\}$$

$$\psi_\Theta^e(t) = \min_{q \le t} \sqrt{||q\alpha_1||^2 + ||q\alpha_2||^2}$$

Theorem (Akhunzhanov+M. 2021)

For Euclidean norm $\mathbb{D} = \left[0, \sqrt{2/\sqrt{3}}\right]$ *is closure of* $\mathbb{D}^{bad}$

Remark. It is likely that a similar statement is true for any $m + n \geq 3$ and for any norm. (c.f. Schleischitz arXiv:2202.04951, 2022)

Multidimensional irrationality measure functions: badly approximability

badly approximability:

$$\exists C > 0 \ \exists \Theta = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \text{ such that } \psi_\Theta(t) > Ct^{-\frac{1}{2}} \ \ \forall \ t > 1$$

Proposition

Suppose that $m = 1, n = 2$. Then for any s badly approximable Θ there exists badly approximable Θ_1 with

$$\psi_\Theta(t) > \psi_{\Theta_1}(t) \quad \textit{for all } \ t \quad \textit{large enough}$$

One-dimensional settings: Minkowski function

Those denominators of convergent for which $\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{2q_n^2}$:

$$Q_1, Q_2, ..., Q_n, ...$$

$$\mu_\alpha(t) = \frac{Q_{n+1} - t}{Q_{n+1} - Q_n} \cdot ||Q_n\alpha|| + \frac{t - Q_n}{Q_{n+1} - Q_n} \cdot ||Q_{n+1}\alpha||, \quad Q_n \leq t \leq Q_{n+1}$$

Minkowski: $\mu_\alpha(t)$ is convex.

Example. $\mu_{\sqrt{2}}(t) < \mu_{\frac{1+\sqrt{5}}{2}}(t), \quad \forall t > 0$

One-dimensional settings: Second order approximation functions.

$$\psi_\alpha^{[2]}(t) = \min_{\substack{1 \le q \le t,\ q \in \mathbb{Z} \\ q \ne q_n \,\forall\, n = 1,2,3,\dots}} ||q\alpha||$$

$$\psi_\alpha^{[2]*}(t) = \min_{\substack{(q,p):\ q,p \in \mathbb{Z}, 1 \le q \le t, \\ p/q \ne p_n/q_n \,\forall\, n = 1,2,3,\dots}} |q\alpha - p|$$

$$\psi_\alpha(t) < \psi_\alpha^{[2]}(t) \le \psi_\alpha^{[2]*}(t)$$

Proposition *Differences*

$$\psi_\alpha^{[2]}(t) - \psi_\beta^{[2]}(t) \quad \textit{and} \quad \psi_\alpha^{[2]*}(t) - \psi_\beta^{[2]*}(t)$$

may not oscillate

Supplement to Gayfulin's lecture: $\mathbb{L}_2^*$ spectrum

$$\mathfrak{k}^*(\alpha) = \liminf_{t\to\infty} t \cdot \psi_\alpha^{[2]*}(t)$$

$$\mathbb{L}_2^* = \{\lambda : \ \exists\, \alpha \text{ such that } \lambda = \mathfrak{k}^*(\alpha)\}$$

Theorem.
1. $\mathbb{L}_2^*$ is $\sqrt{5}$ = maximal element of $\mathbb{L}_2^*$,
$\mathfrak{k}^*(\alpha) = \sqrt{5}$ iff $\alpha \sim \frac{1+\sqrt{5}}{2}$.
2. If $\alpha \not\sim \frac{1+\sqrt{5}}{2}$ then $\mathfrak{k}^*(\alpha) \le \frac{3}{2}$, so $\left(\frac{3}{2}, \sqrt{5}\right)$ is a gap in $\mathbb{L}_2^*$.
3. For $e = \sum_{k=0}^{\infty} \frac{1}{k!}$ one has $\mathfrak{k}^*(e) = \frac{3}{2}$.
4. $\frac{3}{2}$ is a limit point for $\mathbb{L}_2^*$.
5. $\mathbb{L}_2^*$ is $\frac{1}{2}$ is the minimal element of $\mathbb{L}_2^*$.
6. $\left[\frac{1}{2}, \frac{2}{3}\right] \subset \mathbb{L}_2^*$.

A paper by Dubickas JNT 177 (2017)

Theorem 1 *Under the condition* $\alpha \pm \beta \notin \mathbb{Z}$ *one has*

$$\limsup_{t\to+\infty} \left| \frac{1}{\psi_\alpha(t)} - \frac{1}{\psi_\beta(t)} \right| = +\infty.$$

Theorem 2 *There exist irrational* α, β *with* $\alpha \pm \beta \notin \mathbb{Z}$ *and*

$$\psi_\alpha(n+1) < \psi_\beta(n) \ \ \forall n \geq 2.$$

After Dubickas 1

Theorem (N.M., Arch. Math. 2019)
Under the condition $\alpha \pm \beta \notin \mathbb{Z}$ $\quad \exists t_\nu \to \infty$ *such that*

$$|\psi_\alpha(t_\nu) - \psi_\beta(t_\nu)| \geq K \cdot \min(\psi_\alpha(t_\nu), \psi_\beta(t_\nu)), \;\; K = \sqrt{\frac{\sqrt{5}+1}{2}} - 1.$$

Moreover the constant K here is optimal.

As $\psi_\alpha(t) < 1/t$ we have

Corollary *Under the conditions of Th. 1 one has*

$\left|\frac{1}{\psi_\alpha(t_\nu)} - \frac{1}{\psi_\beta(t_\nu)}\right| > K \cdot t_\nu$ *for certain* $t_\nu \to \infty$.

After Dubickas 2

Corollary *Under the condition* $\alpha \pm \beta \notin \mathbb{Z}$ *one has*

$$\left|\frac{1}{\psi_\alpha(t_\nu)} - \frac{1}{\psi_\beta(t_\nu)}\right| > K \cdot t_\nu \quad \textit{for certain} \quad t_\nu \to \infty$$

with $K = \sqrt{\frac{\sqrt{5}+1}{2}} - 1 = 0.2720^+$

Theorem (N. Shulga, MJCNT 2022)
Under the condition specified one has

$$\left|\frac{1}{\psi_\alpha(t_\nu)} - \frac{1}{\psi_\beta(t_\nu)}\right| > K_1 \cdot t_\nu \quad \textit{for certain} \quad t_\nu \to \infty$$

with $K_1 = \sqrt{5}\left(1 - \sqrt{\frac{\sqrt{5}-1}{2}}\right) = 0.47818^+$

Moreover the constant K_1 *here is optimal*

Simultaneous approximation revisited

$$\Theta = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad \psi^{e}_{\Theta}(t) = \min_{q \le t} \sqrt{||q\alpha_1||^2 + ||q\alpha_2||^2}$$

Theorem (N.M., 202?)
$\omega(t) \uparrow, \omega(0) = 0 \quad \exists\ \Theta = (\alpha_1, \alpha_2)$ and $\Theta_1 = (\beta_1, \beta_2)$ such that:
1. $1, \alpha_1, \alpha_2, \beta_1, \beta_2$ are aigebraic independent.
2. for every t one has

$$|\psi_{\Theta}(t) - \psi_{\Theta_1}(t)| \le \psi_{\Theta}(t)\omega(\psi_{\Theta}(t)).$$

$\mathfrak{k}$- index: definition

$\alpha, \beta \in \mathbb{R}$ *incommensurable* if $\psi_\alpha(t) \neq \psi_\beta(t)$ for all large enough t.
For n-tuple $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$ of pairwise incommensurable numbers consider permutation

$$\boldsymbol{\sigma}(t) : \{1, 2, 3, ..., n\} \mapsto \{\sigma_1, \sigma_2, \sigma_3, ..., \sigma_n\}$$

with $$\psi_{\alpha_{\sigma_1}}(t) > \psi_{\alpha_{\sigma_2}}(t) > \psi_{\alpha_{\sigma_3}}(t) > ... > \psi_{\alpha_{\sigma_n}}(t)$$

$\mathfrak{k}$-index $\mathfrak{k}(\boldsymbol{\alpha}) = \mathfrak{k}(\alpha_1, ..., \alpha_n)$ is defined by

$$\mathfrak{k}(\boldsymbol{\alpha}) = \max\{k : \text{there exist different pemutations } \boldsymbol{\sigma}_1, ..., \boldsymbol{\sigma}_k$$

$$\text{with the following property} : \forall j \ \forall t_0 > 0 \ \exists \ t > t_0 \text{ such that } \boldsymbol{\sigma}(t) = \boldsymbol{\sigma}_j\}$$

Kan-Moshchevitin Theorem states that $\mathfrak{k}(\alpha_1, \alpha_2) = 2$ if $\alpha_1 \pm \alpha_2 \notin \mathbb{Z}$

From Dubickas' approach it follows that there exists a incommensurable n-tuple $(\alpha_1, ..., \alpha_n)$ with $\mathfrak{k}(\alpha_1, ..., \alpha_n) = n$

$\mathfrak{k}$- index results: Manturov-M. arXiv:2108.08778 (2021)

Theorem 1 *For an n-tuple* $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$ *of pairwise incommensurable numbers one has*

$$\mathfrak{k}(\boldsymbol{\alpha}) \geq \sqrt{\frac{n}{2}}$$

Theorem 2 *Let* $k \geq 3$ *and* $n = \frac{k(k+1)}{2}$. *Then there exists a pairwise incommensurable n-tuple* $\boldsymbol{\alpha}$ *with*

$$\mathfrak{k}(\boldsymbol{\alpha}) = k$$

$\mathfrak{k}$- index results: V. Rudykh, arXiv:2204.05769 (2022)

Theorem 2 *Let $k \geq 3$ and $n = \frac{k(k+1)}{2}$. Then there exists a pairwise incommensurable n-tuple $\boldsymbol{\alpha}$ with*

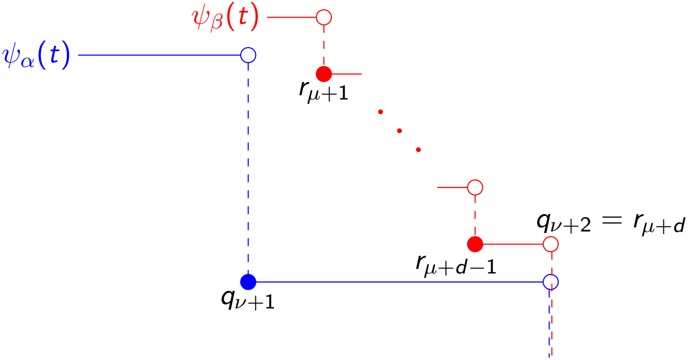
$$\mathfrak{k}(\boldsymbol{\alpha}) = k$$

Theorem 1' (Viktoria Rudykh, April 2022)
The size of an n-tuple $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$ of pairwise independent numbers with $\mathfrak{k}(\boldsymbol{\alpha}) = k$ is

$$n \leq \frac{k(k+1)}{2}$$

Main lemma

this never happens infinitely often for $\alpha \pm \beta \in \mathbb{Z}$:



Main lemma

$$\alpha = [a_0; a_1, a_2, \ldots], \quad \frac{p_\nu}{q_\nu} = [a_0; a_1, a_2, \ldots, a_\nu]$$

$$\alpha^*_\nu = [0; a_\nu, a_{\nu-1}, \ldots, a_1], \quad \xi_\nu = |q_\nu \alpha - p_\nu|$$

$$\beta = [b_0; b_1, b_2, \ldots], \quad \frac{s_\mu}{r_\mu} = [b_0; b_1, b_2, \ldots, b_\nu]$$

$$\beta^*_\mu = [0; b_\mu, b_{\mu-1}, \ldots, b_1], \quad \eta_\mu = |r_\nu \beta - s_\mu|$$

Lemma *Let $d \geq 1$. Suppose that $q_{\nu+2} = r_{\mu+d}$ and*

$$\xi_\nu \leq \eta_\mu,$$

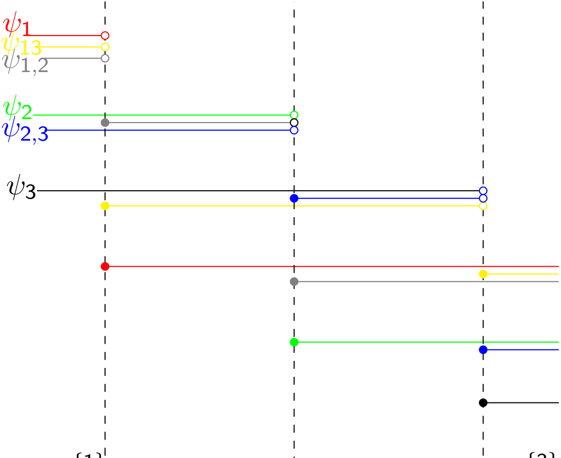
$$\xi_{\nu+1} \leq \eta_{\mu+d-1},$$

$$q_{\nu+1} \leq r_{\mu+1}$$

Then everywhere we have equalities and

$$d = 2 \quad \textit{und} \quad \alpha^*_{\nu+2} = \beta^*_{\mu+2}$$

Example $k = 3, n = 6$



$$t_{3\nu} = q_{\nu}^{\{1\}} = = q_{2\nu}^{\{1,2\}} = q_{2\nu}^{\{1,3\}}$$

$$t_{3\nu+1} = q_{\nu}^{\{2\}} = = q_{2\nu}^{\{2,3\}} = q_{2\nu+1}^{\{1,2\}}$$

$$t_{3\nu+2} = q_{\nu}^{\{3\}} = = q_{2\nu+1}^{\{1,3\}} = q_{2\nu+1}^{\{2,3\}}$$

Thank you for your attention!