Diophantine spectra

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Outline

1. Markoff and Lagrange spectra
2. Dirichlet spectrum
3. Minkowski spectrum

Introduction: Hurwitz theorem

A classical theorem by A.Hurwitz states that

Theorem

For any irrational number α the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$$

has infinitely many integer solutions for $(p, q) = 1$.

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$\sqrt{5}$ cannot be replaced by a greater constant. Indeed, denote $\varphi = \frac{\sqrt{5}+1}{2}$. Then $\forall \varepsilon > 0$ the inequality

$$\left|\varphi - \frac{p}{q}\right| < \frac{1}{(\sqrt{5}+\varepsilon)q^2}$$

has only finitely many solutions.

The Lagrange spectrum - Diophantine approach

Consider an arbitrary irrational number $\alpha = [0; a_1, \ldots, a_n, \ldots]$, such that the partial quotients a_i are uniformly bounded from above i.e. $\forall i \;\; a_i < C$ for some C. Then there exists a constant $L(\alpha)$ s.t. $\forall \varepsilon > 0$

$$\begin{aligned} \left|\alpha - \frac{p}{q}\right| &< \frac{1}{(L(\alpha)-\varepsilon)q^2} \qquad \textit{has } \infty \textit{ solutions.} \\ \left|\alpha - \frac{p}{q}\right| &< \frac{1}{(L(\alpha)+\varepsilon)q^2} \quad \textit{has } <\infty \textit{ solutions.} \end{aligned} \tag{1}$$

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$L(\alpha)$ is called *the Lagrange constant* of α, particularly $L(\varphi) = \sqrt{5}$. In other words,

$$L(\alpha) = \left(\liminf_{p,q\in\mathbb{Z}} q^2 \left|\alpha - \frac{p}{q}\right|\right)^{-1}.$$

The Lagrange spectrum $\mathbb{L}$ is by definition the set of all values taken by Lagrange constants. $\mathbb{L} = \{L(\alpha) | \alpha \in \mathbb{R} \setminus \mathbb{Q}\}$.

The Markoff and Lagrange spectra

Denote $\frac{p_n}{q_n} = [0; a_1, \dots, a_n]$. A classical lemma states that

$$\left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{\lambda_{n+1}(\alpha) q_n^2},$$

where

$$\lambda_i(\alpha) = [a_i; a_{i-1}, a_{i-2}, \dots, a_1] + [0; a_{i+1}, a_{i+2}, \dots]. \qquad (2)$$

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Hence $L(\alpha) = \limsup_{i\to\infty} \lambda_i(\alpha)$. It is convenient to use the "symmetric" form of (2). Consider

$$A = (\ldots a_{-1}, a_0, a_1, \ldots), \quad a_i \geqslant 1$$

and

$$L(A) = \limsup_{i\to\infty} \lambda_i(A), \quad M(A) = \sup_{i\in\mathbb{Z}} \lambda_i(A).$$

The set of values taken by $L(A)$ and $M(A)$ is called respectively the Lagrange and Markoff spectrum, $\mathbb{L}$ and $\mathbb{M}$.

The discrete part of $\mathbb{M}$ and $\mathbb{L}$

Hurwitz theorem implies that the minimal element of the Lagrange spectrum is $\sqrt{5}$. It is easy to show that the same statement is true for the Markoff spectrum.

Theorem (Markoff (1879, 1880))

The set of elements less than 3 *in* $\mathbb{L}$ *and* $\mathbb{M}$ *is countable and discrete, with 3 as its only limit point. Every sequence* A *such that* $\lambda_i(A) < 3$ *for all* $i \in \mathbb{Z}$ *is periodic.*

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As $M(A) = L(A)$, for periodic A, the least statement implies that $\mathbb{L}$ and $\mathbb{M}$ coincide below 3.
More precisely, the elements of the discrete part of the spectra are exactly the numbers of the form $\sqrt{9 - \frac{4}{z^2}}$, where z is a Markoff number i.e. positive integer such that there exist $0 < x \leqslant y \leqslant z$ and $x^2 + y^2 + z^2 = 3xyz$.

10 smallest numbers in $\mathbb{M}$ and $\mathbb{L}$.

The bar over the number or the sequence of numbers denotes the period. $a_n = (\underbrace{a, a, \ldots, a}_{\text{n times}})$.

Sequence	Markoff Number	Element of $\mathbb{M}$ and $\mathbb{L}$
$\overline{1}$	1	$\sqrt{5} \approx 2.23607$
$\overline{2}$	2	$\sqrt{8} \approx 2.82843$
$\overline{2,2,1,1}$	5	$\frac{\sqrt{221}}{5} \approx 2.973214$
$\overline{2,2,1,1,1,1}$	13	$\frac{\sqrt{1517}}{13} \approx 2.996053$
$\overline{2,2,2,2,1,1}$	29	$\frac{\sqrt{7565}}{29} \approx 2.999207$
$\overline{2,2,1_6}$	34	$\frac{\sqrt{2600}}{17} \approx 2.999423$
$\overline{2,2,1_8}$	89	$\frac{\sqrt{71585}}{89} \approx 2.999916$
$\overline{2_6,1,1}$	169	$\frac{\sqrt{257045}}{169} \approx 2.999977$
$\overline{2,2,1,1,2,2,1_4}$	194	$\frac{\sqrt{84680}}{97} \approx 2.999982$
$\overline{2,2,1_{10}}$	233	$\frac{\sqrt{488597}}{233} \approx 2.999988$

Hall's ray

Theorem (M. Hall, 1947; Vinogradov, Delone, Fuchs, 1958)

The Lagrange spectrum, and so the Markoff spectrum, contains every number greater than $5+\sqrt{2}$.

Denote by μ_1 the minimal real number such that $[\mu_1,+\infty)\subset\mathbb{M}$.

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Improvements

μ_1 estimation	Author	Year
$\mu_1 \leqslant 5.102939$	Freiman, Judin	1966
$\mu_1 \leqslant 5.09406$	Bumby	1973
$\mu_1 \leqslant \sqrt{21} \approx 4.5825757$	Freiman, Schecker	1973,1977
$\mu_1 = 4.527829566$	Freiman	1975

The Freiman's theorem also states that $[\mu_1, +\infty) \subset \mathbb{L}$.

Some properties of $\mathbb{M} \setminus \mathbb{L}$.

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Theorem (Freiman, 1968)

$\sigma = M(\overline{2_2, 1, 2_2, 1_2, 2_2}, 1_2, \overline{2_2, 1_2, 2_2, 1, 2_2}) \approx 3.11812 \in \mathbb{M} \setminus \mathbb{L}$

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Several examples were constructed since 1968. It is known that $0.537 < HD(\mathbb{M} \setminus \mathbb{L}) < 0.796$.

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Several examples were constructed since 1968. It is known that $0.537 < HD(\mathbb{M} \setminus \mathbb{L}) < 0.796$.
Open problem: find minimal and maximal element in $\mathbb{M} \setminus \mathbb{L}$. It was shown (Moreira, Santos) that there are elements over $\sqrt{12}$ in $\mathbb{M} \setminus \mathbb{L}$.

Conjecture

There exists a decreasing sequence m_n s.t. $\lim_{n \to \infty} m_n = 3$ and $\forall n \in \mathbb{N}$ one has $m_n \in \mathbb{M} \setminus \mathbb{L}$.

Attainable numbers

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one can easily see that α is attainable if and only if $\lambda_n(\alpha) \geqslant L(\alpha)$ for infinitely many indices n.

Admissible numbers

We call the Lagrange spectrum element λ *admissible* if there exists an attainable irrational number α such that $L(\alpha) = \lambda$.

The problem

Are all elements of the Lagrange spectrum admissible?

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The answer is

Theorem (D.G., 2016)

The quadratic irrationality
$\lambda_0 = [3; 3, 3, 2, 1, \overline{1, 2}] + [0; 2, 1, \overline{1, 2}] = \frac{62976 - 1498\sqrt{3}}{16357} \approx 3.6914708$
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Note that $\lambda_0 = M(\overline{2, 1}, 1, 2, 3, 3, 3, 2, 1, \overline{1, 2})$.

Gaps in the Lagrange spectrum

The Lagrange spectrum (as well as $\mathbb{M}$) is closed set. The complement of $\mathbb{L}$ is a countable union of intervals called *maximal gaps* of the spectrum.

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Theorem (D.G., 2016, 2017)

(i) If $\lambda \in \mathbb{L}$ is not a left endpoint of some maximal gap in the Lagrange spectrum then λ is an admissible number.
(ii) A left endpoint of maximal gap in the Lagrange spectrum a is admissible if and only if there exists a quadratic irrationality α such that $Ł(\alpha) = a$.

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The following theorem gives the description of left endpoints of maximal gaps in $\mathbb{L}$.

Theorem (D.G., 2016)

If (a, b) is a maximal gap in $\mathbb{L}$ then a can be represented by a sum of two quadratic irrationalities.

Pictures

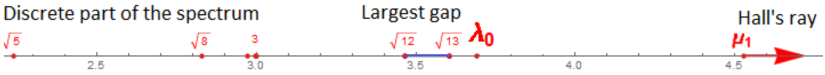
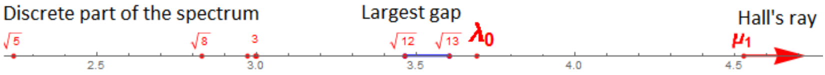


Figure: Lagrange specrtum



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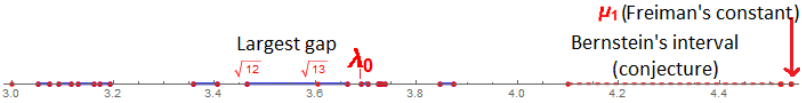

Figure: Lagrange specrtum: gaps

Some other unsolved questions

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Does $\mathbb{M}$ contain the interval $[4.1, 4.52]$?

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Conjecture 3 (Berstein)

Does $\mathbb{M}$ contain the interval $[4.1, 4.52]$?

Conjecture 4

Find the maximal number μ_2 such that $\lambda\Big((-\infty, \mu_2) \cap \mathbb{M}\Big) = 0.$

The best known lower estimate is $\mu_2 \geqslant 3.3343$ (Bumby, 1982).

Dirichlet theorem

A classical theorem by P.Dirichlet states that

Theorem

For any irrational number α for all $t > 1$ there exists a pair $(p, q) \in \mathbb{Z} \times \mathbb{Z}_+$ such that $q \leqslant t$ and

$$|q\alpha - p| < \frac{1}{t}.$$

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It is well-known fact that 1 in the numerator cannot be replaced by any smaller constant. However, for some α it is possible.

Denote by $\psi_x(t)$ the irrationality measure function of $x = [0; a_1, a_2, \ldots, a_n, \ldots] \in \mathbb{R} \setminus \mathbb{Q}$ i.e.

$$\psi_x(t) := \min_{0<q\leqslant t} \min_{p\in\mathbb{Z}} |qx - p|.$$

Dirichlet spectrum

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An equivalent (Perron-type) definition: denote

$$d_n(x) = [a_{n+1}; a_{n+2}, \dots] \cdot [a_n; a_{n-1}, \dots, a_1]$$

and $D(x) = \limsup\limits_{n\to\infty} d_n(x)$.

It was shown by Davenport and Schmidt that

$$D(x) = \frac{c(x)}{1 - c(x)}.$$

The set $\mathbb{D}$ of all values taken by $D(x)$ is called the Dirichlet spectrum.

Some properties of $\mathbb{D}$

Once again, the smallest element of $\mathbb{D}$ corresponds to the golden section and equals $\frac{3+\sqrt{5}}{2}$. The discrete part of D is exactly the sequence $D(x_n)$, where $x_1 = [1; \overline{1}]$ and $x_n = [\overline{2; 1_{2n-1}}]$. The limit point of $D(x_n)$ is $2+\sqrt{5}$.

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There exists an analogue of Hall's ray in $\mathbb{D}$, but the exact value of the origin μ_1 of the ray is an open question. The best known estimate is $\frac{5+3\sqrt{5}}{2} < \mu_1 < 10 + 6\sqrt{2}$.

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The Dirichlet spectrum also has a complicated fractal structure with countable number of gaps. But very few facts are known about their structure. Another open question – find the supremum of μ such that $(0, \mu) \cap \mathbb{D}$ has Lebesgue measure zero.

Minkowski spectrum: Legendre theorem

A classical theorem by A.Legendre states that

Theorem

If for any irrational number α there exist coprime integers p, q such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}, \tag{3}$$

then p/q is a convergent fraction of the continued fraction expansion of α.

The converse statement is not true. However, if we consider two consecutive convergent fractions, for at least one of them the inequality (3) holds.

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The converse statement is not true. However, if we consider two consecutive convergent fractions, for at least one of them the inequality (3) holds. Let us consider the infinite sequence of denominators $Q_1, Q_2, \ldots, Q_n, \ldots$ that satisfy (3).

Minkowski spectrum: definition

For all $t \geqslant Q_1$ define the function

$$\mu_\alpha(t) = \frac{Q_{n+1} - t}{Q_{n+1} - Q_n}\|Q_n\alpha\| + \frac{t - Q_n}{Q_{n+1} - Q_n}\|Q_{n+1}\alpha\|, \quad Q_n \leqslant t \leqslant Q_{n+1}.$$

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One can show that the plot of $\mu_\alpha(t)$ is the convex hull of the points $(q_n, \|q_n\alpha\|)$, here q_n is the set of **all** convergent fractions to x.

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One can show that the plot of $\mu_\alpha(t)$ is the convex hull of the points $(q_n, \|q_n\alpha\|)$, here q_n is the set of **all** convergent fractions to x.
Similarly to $c(x)$ define

$$\mathcal{M}(\alpha) = \limsup_{t\to\infty} t\mu_\alpha(t)$$

and we call the Minkowski spectrum $\mathbb{M}$ the set of all values taken by $\mathcal{M}(\alpha)$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

One can easily see that $\mathbb{M} \subset [0, \frac{1}{2}]$. The following statement was proved by N. Moshchevitin:

Theorem

$$\min \mathbb{M} = \frac{1}{4}, \quad \max \mathbb{M} = \frac{1}{2}.$$

Open questions – everything about the structure: gaps, Hall's ray, isolated points etc.

Thank you for your attention!