The Sarnak conjecture and related questions on the Fibonacci-Thue-Morse sequence

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Russian-Austrian Joint Meeting, July 11-12, 2022

Summary

1. Thue-Morse Sequence
2. Fibonacci-Thue-Morse Sequence
3. Sarnak conjecture
4. Results on Sarnak conjecture
5. Proof Methods
6. Open Problems

Digital Expansions

- ***q*-adic expansion**

$$\boxed{n = \sum_{j\geqslant 0} \varepsilon_{q,j}(n) q^j}, \quad \varepsilon_{q,j}(n) \in \{0, 1, \ldots, q-1\}$$

Sum-of-digits function

$$s_q(n) = \sum_{j\geqslant 0} \varepsilon_{q,j}(n)$$

- **Zeckendorf expansion**

($F_0 = 0, F_1 = 1, F_{k+1} = F_k + F_{k-1}$ Fibonacci numbers)

$$\boxed{n = \sum_{j\geqslant 2} \varepsilon_{Z,j}(n) F_j}, \quad \varepsilon_{Z,j}(n) \in \{0, 1\}, \ \varepsilon_{Z,j}(n)\varepsilon_{Z,j+1}(n) = 0.$$

Zeckendorf Sum-of-digits function

$$s_Z(n) = \sum_{j\geqslant 2} \varepsilon_{Z,j}(n)$$

★ Thue-Morse sequence

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0

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Thue-Morse sequence $(t(n))_{n \geqslant 0}$:

01

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Thue-Morse sequence $(t(n))_{n\geqslant 0}$:

0110

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01101001

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0110100110010110

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01101001100101101001011001101001

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$$t_0 = 0, \qquad t_{2^n+k} = 1 - t_k \quad (0 \leqslant k < 2^n)$$

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$$\boxed{t(n) = s_2(n) \bmod 2}$$

Multiplicative version:

$$\boxed{T(n) = (-1)^{s_2(n)} = (-1)^{t(n)}}$$

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$$t_0 = 0, \qquad t_{2^n+k} = 1-t_k \quad (0 \leqslant k < 2^n) \qquad \text{or} \qquad t_{2k} = t_k,\ t_{2k+1} = 1-t_k$$

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Multiplicative version:

$$\boxed{T(n) = (-1)^{s_2(n)} = (-1)^{t(n)}}$$

★ Thue-Morse sequence

- TM sequence is **not periodic** and **cubeless**.
- TM sequence is **almost periodic**:
  Every appearing consecutive block appears infinitely many times with bounded gaps.
- **Subword complexity is linear**: $p_k \leqslant \frac{10}{3} k$
  p_k ... subword complexity (*number of different consecutive blocks of length k that appear in the TM sequence*).
- **Zero topological entropy** of the corresponding dynamical system:
  $$h = \lim_{k\to\infty} \frac{1}{k} \log p_k = 0$$
- **Linear subsequences** $(t_{an+b})_{n\geqslant 0}$ have the same properties.
- The TM sequence and its linear subsequences are **automatic sequences**.

★ Thue-Morse sequence

Substitution that generates the Thue-Morse sequence

Consider the **substitution**

$$\sigma : \quad a \mapsto ab, \quad b \mapsto ba$$

and the projection

$$\pi : \quad a \mapsto 0, \quad b \mapsto 1,$$

Then the Thue-Morse-sequence $t(n) = s_2(n) \bmod 2$ is just the fixed point of σ that starts with a $(= abbaabbabaab\ldots)$ followed by the projection π.

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Remark. Every sequence that can be generated in such a way by a constant length substitution (followed by a projection) is a so-called **automatic sequence**.

★ Fibonacci-Thue-Morse Sequence

Substitution that generates the Fibonacci-Thue-Morse sequence

Consider the **substitution**

$$\sigma : \quad a \mapsto ab, \quad b \mapsto c, \quad c \mapsto cd, \quad d \mapsto a$$

and the projection

$$\pi : \quad a \mapsto 1, \quad b \mapsto -1, \quad c \mapsto -1, \quad d \mapsto 1.$$

Then the sequence $\boxed{T_F(n) = (-1)^{s_Z(n)}}$ is just the fixed point of σ (that starts with a) followed by the projection π. $T_F(n)$ is a **morphic sequence** (but not an automatic one).

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Remark. Every sequence that can be generated by a substitution followed by a projection is a **morphhic sequences**. Automatic sequences are special morphic sequences.

★ Sarnak Conjecture

A bounded complex valued sequence $u(n)$, $n \in \mathbb{N}$, is said to be **deterministic** if for every $\epsilon > 0$ the set $\{(u(n+1),\ldots,u(n+m)) : n \in \mathbb{N}\}$ can be covered by $O(\exp(o(m)))$ balls of radius ϵ (as $m \to \infty$).

For example, if $\boxed{u(n) := f(T^n x)}$ for a minimal topological dynamical system (X, T) with **zero topological entropy** (and a continuous function f) then $u(n)$ is deterministic.

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In particular **automatic sequences** and **morphic sequences** are deterministic.

★ Sarnak Conjecture

Conjecture (Sarnak conjecture)

Every deterministic bounded complex valued sequence $u(n)$, $n \in \mathbb{N}$ is orthogonal to the Möbius function:

$$\sum_{n \le N} \mu(n) u(n) = o(N) \qquad (N \to \infty).$$

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Problem (Prime Number Theorem)

For which bounded complex sequences $u(n)$ do we have a relation of the form

$$\sum_{n\le N} \Lambda(n)u(n) \sim cN \qquad (N \to \infty)?$$

($\Lambda(n)$ denotes the Von-Mangoldt-function given by $\Lambda(p^k) = \log p$ and 0 else.)

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Theorem (Müllner 2017)

Every automatic sequences $a(n)$ satisfies the Sarnak conjecture:

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Partial results by Dartyge+Tenenbaum, Mauduit+Rivat, Tao (Thue-Morse sequence, Rudin-Shapiro sequence).
The methods introduded by Mauduit+Rivat were a breakthrough (solution of the Gelfond problem for primes).

★ Sarnak Conjecture

The Thue-Morse-sequence $T(n)$ satisfies $T(a2^k + b) = T(a)T(b)$ (for $0 \leqslant b < 2^k$). Such sequences are called **strongly** 2**-multiplicative.** Similarly strongly q-multipliciative functions are defined.

Martin+Mauduit+Rivat proved a PNT and the Sarnak conjecture for general **strongly** q**-multipliciative functions**, e.g. for $f(n) = \mathrm{e}(\alpha s_q(n))$.

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Theorem (D.+Mauduit+Rivat 2020)

If f is a strongly q_1-multiplicative function and g a strongly q_2-multiplicative function such that $\gcd(q_1, q_2) = 1$, then we have uniformly for $\vartheta \in \mathbb{R}$

$$\left| \sum_{n \le N} \mu(n) f(n) g(n) \mathrm{e}(\vartheta n) \right| = o(N).$$

Furthermore, a PNT holds if f or g is non-periodic.

★ Sarnak Conjecture

Theorem (D.+Mauduit+Rivat+Spiegelhofer 2022)

Let $t(n)$ denote the Thue–Morse sequence. Then we have, as $N \to \infty$,

$$\sum_{n<N} \mu(n) t(n^2) = o(N).$$

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The sequence $t(n^2)$ is **not deterministic.** It is actually a **normal sequence** (by a result of D+Mauduit+Rivat).

Recently Andrei Shubin extended this result to the subsequences $t(\lfloor n^c \rfloor)$ for $1 < c < 2$.

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Theorem (D.+Müllner+Spiegelhofer 2018)

We have for non-integers α

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Theorem (D.+Müllner+Spiegelhofer 2021+)

We have for non-integers α

$$\sum_{n\leqslant N} \Lambda(n) \mathrm{e}\big(\alpha\, s_Z(n)\big) \ll (\log N)^5 N^{1-c\|\alpha\|^2}$$

for some $c > 0$.

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This implies in particular for $\alpha = \frac{1}{2}$: $\mathrm{e}\big(\frac{1}{2}\, s_Z(n)\big) = (-1)^{s_Z(n)} = T_F(n)$.

★ Local Result

Theorem (D.+Müllner+Spiegelhofer 2021+)

For each $\varepsilon > 0$*, we have*

$$\#\{p \leqslant x : s_Z(p) = k\} = \frac{\pi(x)}{\sqrt{2\pi\sigma^2 \log_\gamma x}} \left(e^{-\frac{(k-\mu \log_\gamma x)^2}{2\sigma^2 \log_\gamma x}} + O\left((\log x)^{-\frac{1}{2}+\varepsilon}\right) \right)$$

uniformly for all integers $k \geqslant 0$*, where*

$$\mu = \frac{1}{\gamma^2 + 1} \quad \text{and} \quad \sigma^2 = \frac{\gamma^3}{(\gamma^2+1)^3},$$

$\pi(x)$ *denotes the number of primes* $\leqslant x$*, and* $\log_\gamma x = \log x / \log \gamma$*.*

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In particular, for every sufficiently large integer k there exists a prime number p with

$$s_Z(p) = k.$$

★ Proof Methods

- Daboussi-Kátai-criterion
- Vaughan's method
- Discrete Fourier techniques
- Level of distribution and Gowers norm estimates
- Exponential sum estimates
- Diophantine approximation (Baker's theorem, p-adic subspace theorem)
- Discrepancy estimates
- Digit detection techniques

★ Daboussi-Kátai-criterion

Lemma

Let $f(n)$ be a bounded sequence such that

$$\sum_{n \leqslant N} f(pn)\overline{f(qn)} = o(N)$$

for all distinct prime numbers p, q. Then

$$\sum_{n \leqslant N} \mu(n) f(n) = o(N).$$

There is a similar criterion by Bourgain-Sarnak-Ziegler.

★ Vaughan's method

Lemma

Let $f : \mathbb{N} \to \mathbb{C}$ such that $|f(n)| \leqslant 1$ for all $n \ge 1$. For all $N, U, V \geqslant 2$ such that $UV \le N$ we have

$$\sum_{n\le N} f(n)\Lambda(n) \ll U + (\log N) \sum_{t\le UV} \max_{w} \left| \sum_{w\le r\le N/t} f(rt) \right|$$

$$+ \sqrt{N}(\log N)^3 \max_{\substack{U\le M\le N/V \\ V\le q_2\le N/M}} \left(\sum_{V<q_1\le N/M} \left| \sum_{\substack{M<m\le 2M \\ m\le \min(N/q_1, N/q_2)}} f(mq_1)\overline{f(mq_2)} \right| \right)^{1/2},$$

with an absolute implied constant.

★ Level of distribution and Gowers norm

Level of distribution of the Thue-Morse sequence: For all $\varepsilon > 0$ we have

$$\sum_{1 \le d \le x^{1-\varepsilon}} \max_{\substack{y,z \ge 0 \\ z-y \le x}} \max_{0 \le a < d} \left| \sum_{\substack{y \le n < z \\ n \equiv a \bmod d}} (-1)^{s_2(n)} \right| = \mathcal{O}(x^{1-\eta})$$

for some $\eta > 0$ depending on ε; the level of distribution equals **1.**

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Gowers norm of the Thue-Morse sequence:

$$\sum_{0\le n,r_1,\dots,r_m<2^\rho} \mathrm{e}\left(\alpha \sum_{\varepsilon\in\{0,1\}^m} s_\rho(n+\varepsilon\cdot r)\right) \ll 2^{(m+1)\rho\cdot(1-c\|\alpha\|^2)}$$

for all $\rho \ge 0$, where $\varepsilon\cdot r = \sum_{1\le i\le m}\varepsilon_i r_i$ and s_ρ is the *truncated sum-of-digits function* in base 2.

★ Open Problems

- Sarnak conjecture for Tribonacci-sum-of-digits function
- PNT for Tribonacci-sum-of-digits function
- Extension to β-sum-of-digits function for Pisot numbers β
- **Sarnak conjecture for general morphic sequences**
- Distribution of $(-1)^{s_Z(n^2)}$
- Möbius orthogonality for $(-1)^{s_Z(n^2)}$
- ...

Thank you very much for your attention!

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Reference:

Michael Drmota, Clemens Müllner, Lukas Spiegelhofer

Primes as sums of Fibonacci numbers

https://arxiv.org/abs/2109.04068