SYSTEMS OF FUNCTIONAL EQUATIONS AND INFINITE DIMENSIONAL GAUSSIAN LIMITING DISTRIBUTIONS IN COMBINATORIAL ENUMERATION

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Abstract. In this paper systems of functional equations in infinitely many variables arising in combinatorial enumeration problems are studied. We prove sufficient conditions under which the combinatorial random variables encoded in the generating functions of the system tend to an infinite dimensional Gaussian limiting distribution.

1. Introduction

Systems of functional equations for generating functions appear in many combinatorial enumeration problems, for example in tree enumeration problems or in the enumeration of planar graphs (and related problems), see [1, 8, 15]. Usually, these enumeration techniques can be extended to take several parameters into account: the number of vertices, the number of edges, the number of vertices of a given degree, et cetera.

One of the simplest examples is that of rooted plane trees, which are defined as rooted trees, where each node has an arbitrary number of successors with a natural left-to-right-order. Let y_n be the number of rooted plane trees with n vertices. By splitting up at the root one obtains a recursive description (see Figure 1) which translates into corresponding relations for the counting generating function $y(x) = \sum_{n\ge 1} y_n x^n$:

$$y(x) = x + xy(x) + xy(x)^2 + xy(x)^3 + \cdots = \frac{x}{1-y(x)}.$$

This leads to

$$y(x) = \frac{1-\sqrt{1-4x}}{2} \tag{1}$$

and to

$$y_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

Now let $\mathbf{k} = (k_0, k_1, k_2, \ldots)$ be a sequence of non-negative integers and $y_{n,\mathbf{k}}$ the number of rooted plane trees with n vertices such that k_j vertices have exactly j successors (that is, the out-degree equals j) for all $j \ge 0$. Then the formal generating function $y(x,\mathbf{u}) = \sum_{n,\mathbf{k}} y_{n,\mathbf{k}} x^n \mathbf{u}^{\mathbf{k}}$, where $\mathbf{u} = (u_0, u_1, u_2, \ldots)$ and $\mathbf{u}^{\mathbf{k}} = u_0^{k_0} u_1^{k_1} u_2^{k_2} \cdots$, satisfies the equation

$$y(x,\mathbf{u}) = xu_0 + xu_1 y(x,\mathbf{u}) + xu_2 y(x,\mathbf{u})^2 + xu_3 y(x,\mathbf{u})^3 + \cdots = F(x, y(x,\mathbf{u}), \mathbf{u}). \tag{2}$$

If $\|\mathbf{u}\|_\infty$ is bounded then this can be considered as an analytic equation for $y(x,\mathbf{u})$, and $y(x,\mathbf{u})$ encodes the distribution of the number of vertices of given out-degree. More precisely, suppose that all rooted plane trees of size n are equally likely. Then the number of vertices with out-degree j becomes a random variable $X_n^{(j)}$. If we now consider the infinite dimensional random vector $\mathbf{X}_n = (X_n^{(0)}, X_n^{(1)}, X_n^{(2)}, \ldots)$ then we have in this uniform random model

$$\mathbb{E}\,\mathbf{u}^{\mathbf{X}_n} = \frac{1}{y_n}[x^n]\, y(x,\mathbf{u}),$$

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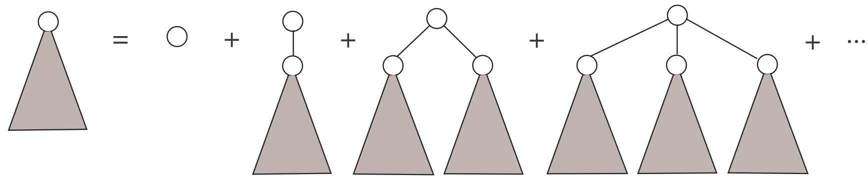

FIGURE 1. Recursive structure of a rooted plane tree

where $[x^n]\, y(x, \mathbf{u})$ denotes the coefficient of x^n in the series expansion of $y(x, \mathbf{u})$. Let ℓ be a linear functional of the form $\ell \cdot \mathbf{X}_n = \sum_{j \ge 0} s_j X_n^{(j)}$, then we also have

$$\mathbb{E}\, e^{it\ell \cdot \mathbf{X}_n} = \frac{1}{y_n} [x^n]\, y(x, e^{its_0}, e^{its_1}, \ldots).$$

This also means that the asymptotic behavior of the characteristic function of $\ell \cdot \mathbf{X}_n$, which determines the limiting distribution, can be derived from the asymptotic behavior of $[x^n]\, y(x, \mathbf{u})$. In this way one can prove by standard methods that $X_n^{(j)}$ and also all finite dimensional random vectors $(X_n^{(0)}, X_n^{(1)}, \ldots, X_n^{(K)})$ satisfy a (finite dimensional) central limit theorem. Nevertheless, it is not obvious that the infinite random vector $\mathbf{X}_n$ has Gaussian limiting distribution as well. (For a definition of infinite dimensional Gaussian distributions see Section 2.) In Theorem 3 we will give a sufficient condition for such a property when the generating function $y(x, \mathbf{u})$ satisfies a single functional equation $y(x, \mathbf{u}) = F(x, y(x, \mathbf{u}), \mathbf{u})$.

In more refined enumeration problems it will be necessary to replace the (single) equation for $y(x, \mathbf{u})$ by a finite or infinite system of equations $\mathbf{y} = \mathbf{F}(x, \mathbf{y}, \mathbf{u})$; see Section 4. More precisely, this means that we have to split up our enumeration problem into finitely or infinitely many subproblems that are interrelated. If y_i denotes the generating function of the i-th subproblem then this means that $y_i(x, \mathbf{u}) = F_i(x, \mathbf{y}(x, \mathbf{u}), \mathbf{u})$ for a certain function F_i. After having solved this system of equations the generating function $y(x, \mathbf{u})$ for the original problem can be computed with the help of the generating functions y_i, that is $y(x, \mathbf{u}) = G(x, \mathbf{y}(x, \mathbf{u}), \mathbf{u})$ for a properly chosen function G. (For example, we will show in Section 4 that such a procedure applies for the degree distribution $\mathbf{X}_n = (X_n^{(j)})_{j \ge 1}$ in series-parallel graphs.)

In this case we are faced with two different problems. First of all a system of equations is more difficult to solve than a single equation, in particular in the infinite dimensional case. However, this can be handled by assuming compactness of the Jacobian of the system, see Theorem 1. Furthermore, it turns out that the problem on the infinite dimensional Gaussian distribution is considerably more involved than in the single equation case. Nevertheless, we prove that all bounded functionals $\ell \cdot \mathbf{X}_n$ have a Gaussian limiting distribution and we give sufficient conditions under which the combinatorial random variables encoded in the generating functions tend to an infinite dimensional Gaussian limiting distribution. (For example, for series-parallel graphs we obtain an infinite dimensional Gaussian limiting distribution for $\mathbf{X}_n$.)

The structure of the paper is as follows. In Section 2 we collect some facts from functional analysis that are needed to formulate our main results that are stated in Section 3. Then we present some applications in Section 4. In Section 5 the proofs of our results will be found.

Finally we would like to mention that this paper is a continuation of the work of [7], [13], and [24].

2. PRELIMINARIES

Before we state the main result, we recall some definitions from the field of functional analysis in order to be able to specify the basic setting. Let B_1 and B_2 be Banach spaces. We denote by $L(B_1, B_2)$ the set of bounded linear operators from B_1 to B_2. If U is the open unit ball in B_1, then an operator $T : B_1 \to B_2$ is compact, if the closure of $T(U)$ is compact in B_2 (or, equivalently, if every bounded sequence $(x_n)_{n \ge 0}$ in B_1 contains a subsequence $(x_{n_i})_{i \ge 0}$ such that $(Tx_{n_i})_{i \ge 0}$

converges in B_2). If A is a bounded operator from B to B, then $r(A)$ denotes the spectral radius of A defined by $r(A) = \sup_{\lambda\in\sigma(A)} |\lambda|$, where $\sigma(A)$ is the spectrum of A.

A function $F : B_1 \to B_2$ is called Fréchet differentiable at x_0 if there exists a bounded linear operator $(\partial F/\partial x)(x_0) : B_1 \to B_2$ such that

$$F(x_0 + h) = F(x_0) + \frac{\partial F}{\partial x}(x_0)\, h + \omega(x_0, h) \quad \text{and} \quad \omega(x_0, h) = o(\|h\|), \quad (h \to 0). \tag{3}$$

The operator $\partial F/\partial x$ is called the Fréchet derivative of F. If the Banach spaces are complex vector spaces and (3) holds for all h, then F is said to be analytic in x_0. F is analytic in $D \subseteq B_1$, if it is analytic for all $x_0 \in D$. Analyticity is equivalent to the fact that for all $x_0 \in D$ there exist an $s > 0$ and continuous symmetric n-linear forms $A_n(x_0)$ such that $\sum_{n\ge1} \|A_n(x_0)\|\, s^n < \infty$ and

$$F(x_0 + h) = F(x_0) + \sum_{n\ge1} \frac{A_n(x_0)}{n!}(h^n)$$

in a neighborhood of x_0 (including the set $\{x_0 + h : \|h\| \le s\}$). (The "coefficients" A_n are equal to the (iteratively defined) *n-th* Fréchet derivatives of F). See for example [6, Section 7.7 and 15.1] and [30, Chapters 4 and 8] for analytic functions in Banach spaces.

Next, we want to recall some facts concerning probability theory on Banach spaces. For a detailed survey see [3] or [23]. Suppose that $\mathbf{X}$ is a random variable from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (here, Ω denotes a set with σ-algebra $\mathcal{F}$ and probability measure $\mathbb{P}$) to a separable Banach space B (equipped with the Borel σ-algebra). Let P be the law (the distribution) of $\mathbf{X}$ (that is, $P = \mathbb{P}\mathbf{X}^{-1}$). Since we assumed B to be separable, we have that the scalar valued random variables $\ell^*(\mathbf{X})$ for continuous functionals ℓ^* determine the distribution of $\mathbf{X}$ (see [23, Section 2.1]).

The random variables $\mathbf{X}_n$, $n \in \mathbb{N}$ (with possibly different probability spaces) are said to converge weakly to some B-valued random variable $\mathbf{X}$ (defined on some probability space and with law P) if the corresponding laws P_n converge weakly to P, i.e., if we have (as n goes to infinity)

$$\int_B f \,\mathrm{d}P_n \to \int_B f \,\mathrm{d}P$$

for every bounded continuous real function f. In what follows we denote this by $\mathbf{X}_n \xrightarrow{w} \mathbf{X}$. We call a set Π of probability measures tight if for each $\varepsilon > 0$ there exists a compact set $K = K_\varepsilon$ such that $P(K) > 1 - \varepsilon$ for every $P \in \Pi$. Let B^* be the dual space of B (the set of continuous functionals from B to $\mathbb{C}$). By Prohorov's theorem[1] we have that $\mathbf{X}_n$ weakly converges to $\mathbf{X}$ if and only if $\ell^*(\mathbf{X}_n)$ weakly converges to $\ell^*(\mathbf{X})$ for all $\ell^* \in B^*$ and the family of probability measures $\{P_n : n \in \mathbb{N}\}$ is tight. Since for scalar valued random variables the weak convergence is completely determined by the convergence of the corresponding characteristic functions, one has to check

(i) tightness of the set $\{P_n : n \in \mathbb{N}\}$

and

(ii) there exists an $\mathbf{X}$ such that $\mathbb{E}\left[e^{it\ell^*(\mathbf{X}_n)}\right] \to \mathbb{E}\left[e^{it\ell^*(\mathbf{X})}\right]$ for all $\ell^* \in B^*$,

in order to show $\mathbf{X}_n \xrightarrow{w} \mathbf{X}$. We call a random variable $\mathbf{X}$ Gaussian if $\ell^*(\mathbf{X})$ is a Gaussian variable (in the extended sense that $X \equiv 0$ is also normally distributed) for all $\ell^* \in B^*$. If it exists, we denote by $\mathbb{E}\mathbf{X}$ the (unique) element $\mathbf{y} \in B$ such that

$$\ell^*(\mathbf{y}) = \mathbb{E}(\ell^*(\mathbf{X}))$$

for all $\ell^* \in B^*$. Gaussian variables are called centered, if $\mathbb{E}\mathbf{X} = \mathbf{0}$.

In what follows, we mainly deal with the Banach space $\ell^p = \ell^p(\mathbb{N})$ $(1 \le p < \infty)$ of all complex valued sequences $(t_n)_{n\in\mathbb{N}}$ satisfying $\|(t_n)\|_p^p := \sum_{n=1}^\infty |t_n|^p < \infty$. (The space $\ell^\infty = \ell^\infty(\mathbb{N})$ is the space of all bounded complex sequences (z_n) with norm $\|(z_n)\|_\infty = \sup_{n\ge1} |z_n| < \infty$.) In this case, the Fréchet derivative is also called Jacobian operator (in analogy to the finite dimensional

[1] Prohorov's theorem says that in a separable and complete metric space a set of probability measures is tight if and only if it is relatively compact; see [3, Chapter I, Section 5].

case). We call a function $\mathbb{F} : \mathbb{C} \times \ell^p \to \ell^p$ positive (in $U \times V$), if there exist nonnegative real numbers $a_{i\mathbf{j},k}$ such that for all $k \geq 1$ and for all $(x, \mathbf{y}) \in U \times V$,

$$F_k(x, \mathbf{y}) = \sum_{i,\mathbf{j}} a_{i\mathbf{j},k} x^i y^{\mathbf{j}},$$

where $\mathbf{j} \in \mathbb{N}^{\mathbb{N}}$, only finitely many components are nonzero, and $y^{\mathbf{j}} = y_1^{j_1} y_2^{j_2} y_3^{j_3} \cdots$.

In our main theorem we have to assume that $\partial F/\partial \mathbf{y}$ is irreducible. In order to be able to define this property, we recall some basic notion from functional analysis on ℓ^p spaces. Any bounded linear operator on an ℓ^p space ($1 \leq p < \infty$) is uniquely determined by an infinite dimensional matrix $(a_{ij})_{1 \leq i,j < \infty}$ via the functional

$$(A\mathbf{x})_i = \sum_{k=1}^{\infty} a_{ik} x_k,$$

where $\mathbf{x} = (x_k)_{1 \leq k < \infty}$ is written with respect to the canonical standard bases in ℓ^p. We call the matrix $(a_{ij})_{1 \leq i,j < \infty}$ the matrix representation of A (and write $A = (a_{ij})_{1 \leq i,j < \infty}$ or just $A = (a_{ij})$). An operator A is called positive, if all entries of the matrix representation of A are nonnegative. A positive operator $A = (a_{ij})$ is said to be irreducible, if for every pair (i, j) there exists an integer $n = n(i, j) > 0$, such that $a_{ij}^{(n)} > 0$, where

$$A^n = \left(a_{ij}^{(n)} \right)_{1 \leq i,j < \infty}.$$

If u and v are real vectors or matrices, $u \geq v$ means that all entries of u are greater than or equal to the corresponding entries of v. Thus, an operator A is positive if $(a_{ij}) \geq 0$. Similarly, a vector x is called positive (or also nonnegative) if $x \geq 0$. We call x strictly positive, if all entries x_i of x satisfy $x_i > 0$. Moreover, if u is a vector with entries u_i, then $|u|$ denotes the vector with entries $|u_i|$ (a corresponding definition is used for matrices).

The dual space of ℓ^p, $1 < p < \infty$ is isomorphic to ℓ^q, where $1/p + 1/q = 1$. Note, that the dual space of ℓ^1 is ℓ^∞. If p is fixed, we use throughout this work the letter q for the real number which satisfies $1/p + 1/q = 1$ if $p > 1$ and $q = \infty$ if $p = 1$. If $x \in \ell^p$ and $\boldsymbol{\ell} \in \ell^q \cong (\ell^p)'$, we denote by $\boldsymbol{\ell}(x)$ the functional $\boldsymbol{\ell}$ evaluated at x. Analogous to the finite dimensional case, we also use the notation $\boldsymbol{\ell} \cdot x$ and $\boldsymbol{\ell}^T x$ instead of $\boldsymbol{\ell}(x)$.

If $1 < p < \infty$, the adjoint operator of an operator A (denoted by A^*) is acting on $\ell^{p'} \cong \ell^q$. The operator A^* can be associated with the matrix $(a_{ji})_{1 \leq i,j < \infty}$ acting on ℓ^q (which we do in the sequel without explicitly saying so). If x is an eigenvector of A we also call it right eigenvector of A and if y is an eigenvector of A^* we call it left eigenvector of A.

The study of operators (or matrices) in ℓ^∞ is different. In fact, the space ℓ^∞ is not separable and there is no one-to-one correspondence between operators and matrices. (Actually, there exist nontrivial compact operators, such that the corresponding "matrix representation" is the zero matrix.) Nevertheless, if we have a matrix $(a_{ij})_{1 \leq i,j < \infty}$, we define an operator A on ℓ^∞ via

$$(Ax)_i = \sum_{k=1}^{\infty} a_{ik} x_k,$$

if the summation is well-defined for all $i \geq 1$ and for all $x \in \ell^\infty$. In the case that $A = (a_{ij})_{1 \leq i,j < \infty}$ is an operator from ℓ^1 to ℓ^1, we get that the dual operator from ℓ^∞ to ℓ^∞ is given by $(a_{ji})_{1 \leq i,j < \infty}$ (as in the ℓ^p-case for $p > 1$).

Throughout, we denote by $\mathrm{I_p}$ the identity on ℓ^p (with matrix representation $(\delta_{ij})_{1 \leq i,j < \infty}$, where δ_{ij} denotes Kronecker's delta function).

3. Main Theorems

In Section 3.1 we state some theorems for infinite systems of functional equations of the form $\mathbf{y} = \mathbf{F}(x, \mathbf{y}, \mathbf{v})$ where the function $\mathbf{F}$ is defined on some subset of $\mathbb{C} \times \ell^p \times \ell^r$ and the range of $\mathbf{F}$ is a subset of ℓ^p, where $1 \leq p < \infty$ and $1 \leq r \leq \infty$. The same result (with obvious modifications of the proof) holds true if one replaces one (or both) of the spaces ℓ^p and ℓ^r by finite dimensional spaces $\mathbb{R}^m$ and $\mathbb{R}^n$. In the case that both spaces are replaced by finite dimensional ones, the statement was proven in [7]; compare also with Lalley [21] and Woods [29]. In Section 3.2 we state sufficient conditions under which the combinatorial random variables encoded in the generating functions of the system tend to an infinite dimensional Gaussian limiting distribution.

3.1. **Systems of functional equations.** Our first result is a generalization of a result of [24], where only one counting variable was considered. It determines the kind of singularity of the solution of a positive irreducible and infinite system of equations. Note that it is more convenient to write $\mathbf{u}$ in the form $\mathbf{u} = e^{\mathbf{v}}$, that is, $u_j = e^{v_j}$. The reason is that in the functional analytic context of our results it is natural to work in a neighborhood of $\mathbf{v} = \mathbf{0}$ instead of a neighborhood of $\mathbf{u} = \mathbf{1}$. Anyway, in the applications we will use again $\mathbf{u}$ since this is more natural for counting problems.

Theorem 1. *Let $1 \le p < \infty$, $1 \le r \le \infty$ and $\mathbf{F} : \mathbb{C} \times \ell^p \times \ell^r \to \ell^p$, $(x, \mathbf{y}, \mathbf{v}) \mapsto \mathbf{F}(x, \mathbf{y}, \mathbf{v})$ be a function satisfying:*

1. *there exist open balls $B \in \mathbb{C}$, $U \in \ell^p$ and $V \in \ell^r$ such that $(0, \mathbf{0}, \mathbf{0}) \in B \times U \times V$ and $\mathbf{F}$ is analytic in $B \times U \times V$,*
2. *the function $(x, \mathbf{y}) \mapsto \mathbf{F}(x, \mathbf{y}, \mathbf{0})$ is a positive function,*
3. *$\mathbf{F}(0, \mathbf{y}, \mathbf{v}) = \mathbf{0}$ for all $\mathbf{y} \in U$ and $\mathbf{v} \in V$,*
4. *$\mathbf{F}(x, \mathbf{0}, \mathbf{v}) \not\equiv \mathbf{0}$ in B for all $\mathbf{v} \in V$,*
5. *$\frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x, \mathbf{y}, \mathbf{0}) = A(x, \mathbf{y}) + \alpha(x, \mathbf{y})\, \mathrm{I_p}$ for all $(x, \mathbf{y}) \in B \times U$, where α is an analytic function and there exists an integer n such that A^n is compact,*
6. *$A(x, \mathbf{y})$ is irreducible for strictly positive $(x, \mathbf{y})$ and $\alpha(x, \mathbf{y})$ has nonnegative Taylor coefficients.*

Let $\mathbf{y} = \mathbf{y}(x, \mathbf{v})$ be the unique solution of the functional equation

$$\mathbf{y} = \mathbf{F}(x, \mathbf{y}, \mathbf{v}) \tag{4}$$

with $\mathbf{y}(0, \mathbf{v}) = \mathbf{0}$. Assume that for $\mathbf{v} = \mathbf{0}$ the solution has a finite radius of convergence $x_0 > 0$ such that $\mathbf{y}_0 := \mathbf{y}(x_0, \mathbf{0})$ exists and $(x_0, \mathbf{y}_0) \in B \times U$.

Then there exists $\varepsilon > 0$ such that $\mathbf{y}(x, \mathbf{v})$ admits a representation of the form

$$\mathbf{y}(x, \mathbf{v}) = \mathbf{g}(x, \mathbf{v}) - \mathbf{h}(x, \mathbf{v}) \sqrt{1 - \frac{x}{x_0(\mathbf{v})}} \tag{5}$$

for $\mathbf{v}$ in a neighborhood of $\mathbf{0}$, $|x - x_0(\mathbf{v})| < \varepsilon$ and $\arg(x - x_0(\mathbf{v})) \ne 0$, where $\mathbf{g}(x, \mathbf{v})$, $\mathbf{h}(x, \mathbf{v})$ and $x_0(\mathbf{v})$ are analytic functions with $\mathbf{h}_i(x_0(\mathbf{0}), \mathbf{0}) > 0$ for all $i \ge 1$.

Moreover, if there exist two integers n_1 and n_2 that are relatively prime such that $[x^{n_1}]\mathbf{y}_1(x, \mathbf{0}) > 0$ and $[x^{n_2}]\mathbf{y}_1(x, \mathbf{0}) > 0$, then $x_0(\mathbf{v})$ is the only singularity of $\mathbf{y}(x, \mathbf{v})$ on the circle $|x| = x_0(\mathbf{v})$ and there exist constants $0 < \delta < \pi/2$ and $\eta > 0$ such that $\mathbf{y}(x, \mathbf{v})$ is analytic in a region of the form

$$\Delta := \{x : |x| < x_0(\mathbf{0}) + \eta, |\arg(x/x_0(\mathbf{v}) - 1)| > \delta\}.$$

Remark 1. As we will show in the proof of Theorem 1, the point $(x_0, \mathbf{y}_0)$ satisfies the equations

$$\mathbf{y}_0 = \mathbf{F}(x_0, \mathbf{y}_0, \mathbf{0}),$$
$$r\left(\frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x_0, \mathbf{y}_0, \mathbf{0})\right) = 1.$$

The main reason for this property is the fact that we have assumed that $(x_0, \mathbf{y}_0)$ lies in the domain of analyticity of $\mathbf{F}$. For a detailed study in the finite dimensional case of such so called critical points see [2]. Note furthermore, that the existence of a point $(x_0, \mathbf{y}_0)$ satisfying the above equations implies that $\mathbf{F}$ is a nonlinear function in $\mathbf{y}$.

Remark 2. Condition (3) of Theorem 1, that is $\mathbf{F}(0, \mathbf{y}, \mathbf{v}) = \mathbf{0}$,, is not really necessary. It is sufficient to assume that $\mathbf{F}(0, \mathbf{0}, \mathbf{v}) = \mathbf{0}$ and that $r\left(\frac{\partial \mathbf{F}}{\partial \mathbf{y}}(0, \mathbf{0}, \mathbf{0})\right) < 1$. In both cases the implicit function theorem implies that the equation (4) has an analytic solution $\mathbf{y}(x, \mathbf{v})$ with $\mathbf{y}(0, \mathbf{v}) = \mathbf{0}$ (if $\mathbf{v}$ is sufficiently close to $\mathbf{0}$).

As mentioned in the introduction, we are often faced with a slightly different situation: Indeed, many combinatorial enumeration problems are described by a generating function of the form

$$y(x,\mathbf{v}) = G(x,\mathbf{y}(x,\mathbf{v}),\mathbf{v}) = \sum_{n=0}^{\infty}\sum_{\mathbf{m}\in\ell^p} c_{n,\mathbf{m}} e^{\mathbf{m}\cdot\mathbf{v}} x^n = \sum_{n=0}^{\infty}\sum_{\mathbf{m}\in\ell^p} c_{n,\mathbf{m}} \mathbf{u}^{\mathbf{m}} x^n, \tag{6}$$

where $c_{n,\mathbf{m}}$ denotes the number of objects of size n and characteristic $\mathbf{m}$ and the auxiliary function $\mathbf{y}(x,\mathbf{v})$ is the solution of a finite or infinite system of equations $\mathbf{y} = \mathbf{F}(x,\mathbf{y},\mathbf{v})$.

Corollary 1. *Let $\mathbf{y} = \mathbf{y}(x,\mathbf{v})$ be the unique solution of the functional equation* (4) *and assume that all assumptions of Theorem 1 are satisfied. Suppose that $G : (\mathbb{C},\ell^p,\ell^r) \to \mathbb{C}$ is an analytic function such that $(x_0(\mathbf{0}),\mathbf{y}(x_0(\mathbf{0}),\mathbf{0}),\mathbf{0})$ is contained in the interior of the region of convergence and that*

$$\frac{\partial G}{\partial \mathbf{y}}(x_0(\mathbf{0}),\mathbf{y}(x_0(\mathbf{0}),\mathbf{0}),\mathbf{0}) \neq \mathbf{0}.$$

Then there exists $\delta,\varepsilon > 0$ such that $G(x,\mathbf{y}(x,\mathbf{v}),\mathbf{v})$ has a representation of the form

$$G(x,\mathbf{y}(x,\mathbf{v}),\mathbf{v}) = \bar{g}(x,\mathbf{v}) - \bar{h}(x,\mathbf{v})\sqrt{1-\frac{x}{x_0(\mathbf{v})}} \tag{7}$$

for $|x-x_0(\mathbf{v})| \le \varepsilon$ and $\arg(x-x_0(\mathbf{0})) \neq 0$ and for $\mathbf{v}$ in a neighborhood of $\mathbf{0}$. The functions $\bar{g}(x,\mathbf{v})$, $\bar{h}(x,\mathbf{v})$ and $x_0(\mathbf{v})$ are analytic in this domain and $\bar{h}(x_0(\mathbf{0}),\mathbf{0}) \neq 0$. Moreover, $G(x,\mathbf{y}(x,\mathbf{v}),\mathbf{v})$ is analytic for $\mathbf{v}$ in a neighborhood of $\mathbf{0}$ and $|x-x_0(\mathbf{v})| \ge \varepsilon$ but $|x| \le |x_0(\mathbf{v})| + \eta$ and we have

$$[x^n]G(x,\mathbf{y}(x,\mathbf{v}),\mathbf{v}) = \frac{\bar{h}(x_0(\mathbf{v}),\mathbf{v})}{2\sqrt{\pi}} x_0(\mathbf{v})^{-n} n^{-3/2}\left(1+O\left(\frac{1}{n}\right)\right)$$

uniformly for $\mathbf{v}$ in a neighborhood of $\mathbf{0}$.

3.2. **Central limit theorems.** Consider a set of combinatorial objects with a given size associated to them and assume a uniform distribution on the set of objects of size n. Moreover, let χ denote an ℓ^p-valued characteristic of these objects which induces an ℓ^p-valued random variable $\mathbf{X}_n$ defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ $(1 \le p < \infty)$. Assume furthermore that the generating function associated to the numbers $c_{n,\mathbf{m}}$ of combinatorial objects of size n having characteristic $\mathbf{m}$ is of the form (6). In this setting we have

$$\mathbb{P}\{\mathbf{X}_n = \mathbf{m}\} = \frac{c_{n,\mathbf{m}}}{\sum_{\mathbf{k}\in\ell^p} c_{n,\mathbf{k}}}$$

and if we write

$$y(x,\mathbf{v}) = G(x,\mathbf{y}(x,\mathbf{v}),\mathbf{v}) = \sum_{n=0}^{\infty} c_n(\mathbf{v}) x^n,$$

then

$$\mathbb{E}\left[e^{it\boldsymbol{\ell}\cdot\mathbf{X}_n}\right] = \frac{c_n(it\boldsymbol{\ell})}{c_n(\mathbf{0})} \tag{8}$$

for all $\boldsymbol{\ell} \in \ell^q$.

Our second result shows that in the case where $\mathbf{y}(x,\mathbf{v})$ satisfies a functional equation $\mathbf{y} = \mathbf{F}(x,\mathbf{y},\mathbf{v})$ with $\mathbf{y}(0,\mathbf{v}) = \mathbf{0}$ and such that the assumptions of Theorem 1 are fulfilled all bounded functionals of $\mathbf{X}_n$ satisfy a central limit theorem.

Theorem 2. *Let $1 \le p < \infty$ and suppose that $\mathbf{X}_n$ is a sequence of ℓ^p-valued random variables defined by* (8)*. Furthermore, let $\boldsymbol{\ell} \in \ell^q$. Then we have $\boldsymbol{\ell}\cdot\mathbb{E}\mathbf{X}_n = \mu_{\boldsymbol{\ell}} n + O(1)$ with $\mu_{\boldsymbol{\ell}} = -\frac{\partial x_0}{\partial \mathbf{v}}(\mathbf{0})\cdot\boldsymbol{\ell}/x_0$ and*

$$\boldsymbol{\ell}\cdot\left(\frac{\mathbf{X}_n - \mathbb{E}\mathbf{X}_n}{\sqrt{n}}\right)$$

weakly converges for n to infinity to a centered real Gaussian variable with variance $\sigma^2_{\boldsymbol{\ell}} = \boldsymbol{\ell}^T B\boldsymbol{\ell}$, where $B \in L(\ell^q,\ell^p)$ is given by the matrix

$$\frac{1}{x_0^2}\left(\frac{\partial x_0}{\partial v_i}(\mathbf{0})\cdot\frac{\partial x_0}{\partial v_j}(\mathbf{0})^T\right)_{1\le i,j<\infty} - \frac{1}{x_0}\left(\frac{\partial^2 x_0}{\partial v_i \partial v_j}(\mathbf{0})\right)_{1\le i,j<\infty}.$$

Corollary 2. *Let $1 \le p < \infty$ and suppose that $\mathbf{X}_n$ is a sequence of ℓ^p-valued random variables defined by (8) such that the set of laws of $(\mathbf{X}_n - \mathbb{E}\mathbf{X}_n)/\sqrt{n}$, $n \ge 1$ is tight. Then there exists a centered Gaussian random variable $\mathbf{X}$ such that*

$$\frac{\mathbf{X}_n - \mathbb{E}\mathbf{X}_n}{\sqrt{n}} \xrightarrow{w} \mathbf{X},$$

where $\mathbf{X}$ is uniquely determined by the operator $B \in L(\ell^q, \ell^p)$ stated in Theorem 2.

Remark 3. There are some natural situation where our results cannot be applied. For example if we have

$$y(x, \mathbf{v}) = xe^{\sum_{j\ge 0} v_j} + \frac{xy(x, \mathbf{v})}{1 - y(x, \mathbf{v})}$$

then all random variables $X_n^{(j)}$ $(j \ge 0)$ count the number of leaves in rooted plane trees and the sequence $(\mathbf{X}_n - \mathbb{E}\mathbf{X}_n)/\sqrt{n}$ is tight in ℓ^∞ but not contained in some ℓ^p with $p < \infty$.

Tightness is usually a rather difficult matter. Grenander [17, Theorem 6.2.3] states a sufficient condition for tightness in ℓ^2 which is sometimes easy to check: the corresponding laws of $(\mathbf{X}_n - \mathbb{E}\mathbf{X}_n)/\sqrt{n}$ are tight if

$$\lim_{N\to\infty} \sup_{n\ge 1} \mathbb{E}\left[\sum_{j>N} \frac{(X_n^{(j)} - \mathbb{E}X_n^{(j)})^2}{n}\right] = 0, \tag{9}$$

where $\mathbf{X}_n = (X_n^{(j)})_{j\ge 0}$. Based on this fact, we provide a sufficient condition in terms of the functional equation itself. However, even in the case of a single equation we have to check several non-trivial assumptions. It is far from being obvious how these properties might generalize to the general case.

Theorem 3. *Suppose that $y(x, \mathbf{v})$ is the unique solution of a single functional equation $y = F(x, y, \mathbf{v})$, where $F : B \times U \times V \to \mathbb{C}$ is a positive analytic function on $B \times U \times V \subseteq \mathbb{C}^2 \times \ell^2$ such that there exist positive real $(x_0, y_0) \in B \times U$ with $y_0 = F(x_0, y_0, \mathbf{0})$ and $1 = F_y(x_0, y_0, \mathbf{0})$ such that $F_x(x_0, y_0, \mathbf{0}) \ne 0$ and $F_{yy}(x_0, y_0, \mathbf{0}) \ne 0$. Furthermore assume that the corresponding random variables $X_n^{(j)}$ have the property that $X_n^{(j)} = 0$ if $j > cn$ for some constant $c > 0$ and that the following conditions are satisfied:*

$$\sum_{j\ge 0} F_{v_j} < \infty, \quad \sum_{j\ge 0} F_{yv_j}^2 < \infty, \quad \sum_{j\ge 0} F_{v_j v_j} < \infty,$$

$$F_{xv_j} = o(1), \quad F_{xv_jv_j} = o(1), \quad F_{yyv_j} = o(1), \quad F_{yyv_jv_j} = o(1),$$

$$F_{xxv_j} = O(1), \quad F_{xyv_j} = O(1), \quad F_{xyyv_j} = O(1), \quad F_{yyyv_j} = O(1), \quad (j \to \infty)$$

where all derivatives are evaluated at $(x_0, y_0, \mathbf{0})$.

Then the the set of laws of $(\mathbf{X}_n - \mathbb{E}\mathbf{X}_n)/\sqrt{n}$, $n \ge 1$ is tight and has a Gaussian limit.

Remark 4. The drawback of Theorem 3 is that it only applies for a single equation. It is certainly possible to formulate proper tightness conditions for finite systems, however, it seems that there is no *simple statement* that refers just to the derivatives of $\mathbf{F}$; in the infinite case there is probably no direct approach. Nevertheless, what one really has to check is condition (9), that is, one has to obtain proper asymptotic information on the variances. (In the case of one equation the conditions stated in Theorem 3 are sufficient to check (9).) Actually there are usually different approaches to obtain asymptotic information on the variances. In Section 4.3 we will present an example, where the defining system of equations is infinite and where it is possible to check tightness directly with the help (9).

Finally, we mention the (simpler) case when the function $\mathbf{F}$ is linear in $\mathbf{y}$ (as noted in Remark 1, we considered until now only the nonlinear case). We just state and prove the following result from which one can deduce corresponding asymptotic expansions of the coefficients and limit theorems.

Theorem 4. *Let $1 \le p < \infty$, $1 \le r \le \infty$ and $\mathbf{F} : \mathbb{C} \times \ell^p \times \ell^r \to \ell^p$, $(x, \mathbf{y}, \mathbf{v}) \mapsto \mathbf{F}(x, \mathbf{y}, \mathbf{v})$ be an affine function in $\mathbf{y}$, i.e. $\mathbf{F}(x, \mathbf{y}, \mathbf{v}) = \mathbf{L}(x, \mathbf{v})\mathbf{y} + \mathbf{b}(x, \mathbf{v})$, satisfying the assumptions (1)–(6) of Theorem 1. Let $\mathbf{y} = \mathbf{y}(x, \mathbf{v})$ be the solution of the functional equation*

$$\mathbf{y} = \mathbf{L}(x, \mathbf{v})\mathbf{y} + \mathbf{b}(x, \mathbf{v})$$

with $\mathbf{y}(0, \mathbf{v}) = \mathbf{0}$. Assume that there exists a positive number $x_0 > 0$ in the domain of analyticity of $\mathbf{L}(x, \mathbf{v})$ such that

$$r\Big(\mathbf{L}(x_0, \mathbf{0})\Big) = 1.$$

Then there exists $\varepsilon > 0$ such that $\mathbf{y}(x, \mathbf{v})$ admits a representation of the form

$$\mathbf{y}(x, \mathbf{v}) = \frac{1}{1 - \frac{x}{x_0(\mathbf{v})}} \mathbf{f}(x, \mathbf{v}) \tag{10}$$

for $\mathbf{v}$ in a neighborhood of $\mathbf{0}$, $|x - x_0(\mathbf{v})| < \varepsilon$ and $\arg(x - x_0(\mathbf{v})) \neq 0$, where $\mathbf{f}(x, \mathbf{v})$ and $x_0(\mathbf{v})$ are analytic functions with $f_i(x_0(\mathbf{0}), \mathbf{0}) \neq 0$ for all $i \ge 1$.

4. Applications

In this section we present some applications. The first few examples lead to a single functional equation for a generating function in an infinite number of variables. At the end of this section we present two examples where we obtain an infinite system of equations where the unknown generating functions have again infinitely many variables.

4.1. **Rooted Trees.**

4.1.1. *Rooted Plane Trees.* As in the Introduction we consider rooted plane trees, where we count the size as well as the numbers of vertices with out-degree $j \ge 0$ simultaneously for all j. The problem of the degree distribution in trees was already studies by Robinson and Schwenk [27] (one particular j) and later extended in [12] (finitely many j's), [16, 25] (non-constant j) and [5] (more general patterns). Compare also with the discussion in [8, Ch. 3].

The problem we consider here was studied by Pittel [26] who showed, using a different approach, the convergence of the finite-dimensional projections, but without tightness.

Proposition 1. *Fix some $1 \le p < \infty$ and let $X_n^{(j)}$ denote the number of vertices of degree j in a random rooted tree of size n. Then for every functional $\boldsymbol{\ell} \in \ell^q$ the random variable $\boldsymbol{\ell} \cdot \mathbf{X}_n$ is asymptotically normally distributed with asymptotic mean*

$$\mu_{\boldsymbol{\ell}} = \sum_{i \ge 0} 2^{-i-1} \ell_i \tag{11}$$

and asymptotic variance $\sigma_{\boldsymbol{\ell}} = \boldsymbol{\ell}^T B \boldsymbol{\ell}$ with $B = (B_{ij})_{i,j \ge 0}$ where

$$B_{ij} = \frac{1}{2^{i+j+2}} - \frac{(i-1)(j-1)}{2^{i+j+3}} + \frac{\delta_{ij}}{2^{i+1}}. \tag{12}$$

Furthermore, the sequence $(\mathbf{X}_n - \mathbb{E}\mathbf{X}_n)/\sqrt{n}$ (considered as a sequence in ℓ^2) is tight and, thus, there exists a centered Gaussian random variable $\mathbf{X}$ with

$$\frac{\mathbf{X}_n - \mathbb{E}\mathbf{X}_n}{\sqrt{n}} \xrightarrow{w} \mathbf{X}.$$

Proof. We rewrite (2) as

$$y(x, \mathbf{v}) = x \sum_{i \ge 0} e^{v_i} y(x, \mathbf{v})^i =: F(x, y(x, \mathbf{v}), \mathbf{v}).$$

Obviously, $F(x, y, \mathbf{v})$ is analytic at $(0, 0, \mathbf{0})$, it has only nonnegative coefficients. Moreover, we observe easily that $F(0, y, \mathbf{v}) \equiv \mathbf{0}$ and $F(x, 0, \mathbf{v}) = \mathbf{x} \not\equiv \mathbf{0}$. Furthermore, we have $\frac{\partial F}{\partial y}(x, y, \mathbf{0}) = x \sum_{i \ge 1} i y^{i-1}$. Thus the Jacobian is a one by one matrix with a nonzero entry and hence trivially compact and irreducible. Solving the system $F(x, y, \mathbf{0}) = y$, $F_y(x, y, \mathbf{0}) = 1$ gives $x_0 = 1/4$ and $y_0 = 1/2$, that is $x_0 = 1/4$ is the radius of convergence and $y_0 = y(x_0, \mathbf{0}) = 1/2$ is finite. Hence, all the assumptions of Theorem 1 are satisfied.

In order to complete the proof we have to show tightness. By differentiating the equation $F(x_0(\mathbf{v}), y(x_0(\mathbf{v}), \mathbf{v}), \mathbf{v}) = y(x_0(\mathbf{v}), \mathbf{v})$ with respect to v_i we obtain $\frac{\partial x_0}{\partial v_i}(\mathbf{0}) = -2^{-i-3}$ yielding (11). The variance can be expressed in terms of partial derivatives of $F(x, y, \mathbf{v})$. This was already done in [12] and yields (12).

Finally, note that $F(x, y, \mathbf{0}) = \frac{x}{1-y}$ and that the partial derivatives needed in Theorem 3, evaluated at $(x, y, \mathbf{0})$ are therefore

$$\sum_{j\ge 0} F_{v_j} = \sum_{j\ge 0} F_{v_j v_j} = \frac{x}{1-y} = \frac{1}{2}, \quad \sum_{j\ge 0} F_{y v_j}^2 = \frac{x_0^2(1+y_0^2)}{(1-y_0^2)^3} = \frac{5}{27}$$

$$F_{x v_j} = F_{x v_j v_j} = 2^{-j} = o(1), \quad F_{yy v_j} = F_{yy v_j v_j} = j(j-1)2^{-j} = o(1),$$

and the terms

$$F_{xx v_j} = 0, \quad F_{xy v_j} = j2^{-j+1}, \quad F_{xyy v_j} = j(j-1)2^{-j+2}, \quad F_{yyy v_j} = j(j-1)(j-2)2^{-j+1},$$

as $j \to \infty$, are all bounded. Thus the assumptions of Theorem 3 are satisfied and consequently the random vector $\mathbf{X}_n = (X_n^{(j)})_{j\ge 0}$ satisfies a central limit theorem. □

4.1.2. *Simply Generated Trees.* Proposition 1 can be easily generalized to simply generated trees. These are trees with generating function given by the functional equation $y(x) = x\phi(y(x))$ where $\phi(t) = \sum_i \phi_i t^i$ is a power series with nonnegative coefficients such that $\phi_0 > 0$ and the unique positive solution y_0 of $\phi(t) = t\phi'(t)$ lies inside the radius of convergence of ϕ. Under aperiodicity conditions $y(x)$ has a unique singularity x_0 in the circle of convergence where (x_0, y_0) is the solution of the system

$$y = x\phi(y), \quad 1 = x\phi'(y).$$

If we keep track of the nodes of degree j, simultaneously for all $j \ge 0$, the we are faced with the generating function $y(x, \mathbf{v})$ given by the functional equation

$$y(x, \mathbf{v}) = F(x, y(x, \mathbf{v}), \mathbf{v}) := x \sum_{i\ge 0} \phi_i e^{v_i} y(x, \mathbf{v})^i. \tag{13}$$

Proposition 2. *Fix some $1 \le p < \infty$ and let $X_n^{(j)}$ denote the number of vertices of degree j in a random simply generated tree of size n. Then for any functional $\boldsymbol{\ell} \in \ell^q$ the random variable $\boldsymbol{\ell} \cdot \mathbf{X}_n$ is asymptotically normally distributed with asymptotic mean*

$$\mu_{\boldsymbol{\ell}} = \sum_{i\ge 0} \frac{\phi_i y_0^i}{\phi(y_0)} \ell_i$$

and asymptotic variance $\sigma_{\boldsymbol{\ell}} = \boldsymbol{\ell}^T B \boldsymbol{\ell}$ with $B = (B_{ij})_{i,j\ge 0}$ where

$$B_{ij} = -\frac{1}{x_0\phi(y_0)}(F_{x,j}\alpha_i + F_{x,i}\alpha_j + F_{xy}(\alpha_i\beta_j + \alpha_j\beta_i) + F_{y,i}\beta_j + F_{y,j}\beta_i + F_{yy}\beta_i\beta_j) + \delta_{ij} x_0 \phi_i y_0^i$$

where

$$F_{x,k} = \phi_k y_0^k, \quad F_{xy} = \phi'(y_0), \quad F_{y,k} = k\phi_k x_0 y_0^{k-1}, \quad F_{yy} = x_0\phi''(y_0),$$

and

$$\alpha_k = \frac{\partial x_0}{\partial v_k} = -\frac{x_0 \phi_k y_0^k}{\phi(y_0)}, \quad \beta_k = x_0\phi_k y_0^k + \frac{1}{\phi''(y_0)}\left(\frac{\phi'(y_0)\phi_k y_0^k}{\phi(y_0)} - k\phi_k y_0^{k-1}\right).$$

Furthermore, the sequence $(\mathbf{X}_n - \mathbb{E}\mathbf{X}_n)/\sqrt{n}$ (considered as a sequence in ℓ^2) is tight and, thus, there exists a centered Gaussian random variable $\mathbf{X}$ with

$$\frac{\mathbf{X}_n - \mathbb{E}\mathbf{X}_n}{\sqrt{n}} \xrightarrow{w} \mathbf{X}.$$

Proof. Starting from the functional equation (13) we observe that

$$\frac{\partial F}{\partial y}(x_0(\mathbf{v}), y(x_0(\mathbf{v}), \mathbf{v}), \mathbf{v}) = 1. \tag{14}$$

Similarly as in the proof of Proposition 1 we easily check that the conditions of Theorem 1 are satisfied and then partial derivation of (13) and (14) leads to the result. □

4.1.3. *Other trees.* Of course, other classes of trees can be treated in a similar way. For instance, if we consider unlabeled trees, then the functional equation for the generating function is

$$y(x,\mathbf{v}) = x + x\sum_{i\ge 1} e^{vi} Z_i(y(x,\mathbf{v}), y(x^2,\mathbf{v}^{(2)}),\ldots, y(x^i,\mathbf{v}^{(i)})$$

where $Z_i(x_1,x_2,\ldots,x_i)$ denotes the cycle index of the symmetric group on i elements and $\mathbf{v}^{(k)} = (v_0^k, v_1^k, v_2^k,\ldots)$. The calculations are then more involved but the assumptions of Theorem 1 can be verified as well and we obtain again a normal limit. The expressions we get then for the mean and the covariance matrix are quite lengthy and omitted here. They are essentially the infinite dimensional counterparts of those in [12, Th. 2.1], where the joint distribution of the numbers of nodes for finitely many degrees was computed.

4.2. **Bipartite Planar Maps.** Planar maps are connected graphs that are embedded on the sphere. Rooted (and also pointed) maps can be counted by several techniques (for example by the quadratic method et cetera). Recently, a bijection between rooted maps and so-called mobiles has been established that makes the situation much more transparent, see [4]. We restrict ourselves to the case of bipartite maps, that is, all faces have an even degree.

In particular let $R(x,z,\mathbf{u})$ denote the generating function that solves the equation

$$R = xz + x\sum_{j\ge1} u_j \binom{2j-1}{j} R^j.$$

Then the generating function $M(x,z,\mathbf{u})$ of bipartite maps, where x counts the number of edges, z the number of vertices, and u_j the number of faces of valency $2j$ for $j\ge 1$, satisfies $M_z = R$.

Here we can also apply Theorem 3 (in this case $x_0 = 1/8$ and $R_0 = 3/16$). Furthermore, since Eulerian maps are dual to bipartite maps we also get a central limit theorem for the degree distribution of Eulerian maps. For the sake of shortness we omit the proof (that is almost the same as that of Proposition 1).

Proposition 3. *Fix some $1\le p<\infty$ and let $X_n^{(j)}$, $j\ge 1$, denote the number of faces of valency $2j$ in a random bipartite map with n edges (or the number of vertices of degree $2j$ in a random Eulerian map with n edges). Then for every functional $\boldsymbol{\ell}\in\ell^q$ the random variable $\boldsymbol{\ell}\cdot\mathbf{X}_n$ is asymptotically normally distributed with asymptotic mean*

$$\mu_{\boldsymbol{\ell}} = \sum_{i\ge1}\frac{16}{3}F_{u_i}\ell_i,$$

where $F_{u_i} = \binom{2i-1}{i}(3/16)^i$, and asymptotic variance $\sigma_{\boldsymbol{\ell}} = \boldsymbol{\ell}^T B\boldsymbol{\ell}$ with $B = (B_{ij})_{i,j\ge1}$ where

$$\begin{aligned} B_{ii} &= -\frac{2816}{81}F_{u_i}^2 + \frac{16}{3}F_{u_i} - \frac{512}{81}i^2F_{u_i}^2 + \frac{1024}{81}iF_{u_i}^2,\\ B_{ij} &= -\frac{5039}{81}F_{u_i}F_{u_j} - \frac{512}{81}ijF_{u_i}F_{u_j} + \frac{512}{81}(i+j)F_{u_i}F_{u_j} \qquad (i\ne j). \end{aligned}$$

Furthermore, the sequence $(\mathbf{X}_n - \mathbb{E}\mathbf{X}_n)/\sqrt{n}$ (considered as a sequence in ℓ^2) is tight and, thus, there exists a centered Gaussian random variable $\mathbf{X}$ with

$$\frac{\mathbf{X}_n - \mathbb{E}\mathbf{X}_n}{\sqrt{n}} \xrightarrow{w} \mathbf{X}.$$

4.3. **Subcritical Graphs.** For $k\ge 0$, a graph is k-connected if one needs to delete at least k vertices to disconnect it. Obviously, a graph G is 1-connected if and only it is connected. Furthermore, we can decompose every connected graph into 2-connected graphs by using the block structure. A block of a graph G is a maximal 2-connected induced subgraph of G. We say a vertex of G is incident to a block B of G if it belongs to B. The block structure of G yields a bipartite tree with the vertex set consisting of two types of nodes, i.e. cut-vertices and blocks of G, and the edge set describing the incidences between the cut-vertices and blocks of G. This is precisely the decomposition of connected graphs into 2-connected graphs. This decomposition leads to a unique recursive decomposition if we consider vertex-rooted graphs. The root-vertex

v of a rooted graph G is incident to a set of blocks and to each non-root vertex on these blocks is attached a rooted connected graph. In other words, a rooted connected graph rooted at v is uniquely obtained as follows: take a set of rooted 2-connected graphs and merge them at their rooted (distinguished but not labeled) vertices so that v is incident to these derived 2-connected graphs, then replace each non-root vertex w in these blocks by a rooted connected graph rooted at w (which is allowed to consist of a single vertex and in this case it has no effect).

A graph class is called *block-stable* if it contains the link-graph ℓ, which is a graph with one edge together with its two end vertices, and satisfies the property that a graph G belongs to the graph class if and only if all the blocks of G belong to this class. Block-stable classes include classes of graphs specified by a finite list of forbidden minors that are all 2-connected, for instance, planar graphs (Forbid($K_5, K_{3,3}$)), series-parallel graphs (Forbid(K_4)), and outerplanar graphs (Forbid($K_4, K_{3,2}$)). For a block-stable vertex-labeled graph class, the recursive block decomposition translates into equations for the corresponding exponential generating functions $B(x)$ for the 2-connected graphs and $C(x)$ for the connected graphs (*cf.* [18, p.10,(1.3.3)]):

$$C'(x) = e^{B'(xC'(x))}.$$

Note that $C'(x)$ is the generating function of rooted graphs, that is, one vertex is distinguished (or rooted) but not labeled.

A graph class is called subcritical if the radius of convergence of $B(x)$ is larger than η, where η is defined by the equation $\eta B''(\eta) = 1$. Several well-known graph classes are subcritical, for example series-parallel graphs or outerplanar graphs (see [9]). These graph classes are also planar graphs and have various characterizations. For example, series-parallel graphs are precisely those graphs, where the treewidth is at most 2, or there is a series-parallel extension of a tree or forest; and outerplanar graphs are planar graphs with the property that there is a non-crossing embedding into the plane, where all vertices are on the outer face. However, there are also important graph classes (like the class of planar graphs) that are not subcritical.

It has been already proved in [9] that the number of vertices of degree j in subcritical graph classes satisfies a central limit theorem. For fixed j it is sufficient to consider just a finite system of equations so that one can apply the methods of [7] to obtain the central limit theorem. However, if we want to consider all $j \ge 1$ at once then we are forced to use an infinite system.

Suppose that $B_r^\bullet(x, u_1, u_2, \ldots)$ denotes the generating function of rooted 2-connected graphs, where the root-vertex has degree r and the variables x and u_j count the number of remaining vertices and the (remaining) vertices of degree j, $j \ge 1$. Similarly we define $C_j^\bullet(x, u_1, u_2, \ldots)$ for connected graphs. Then the unique decomposition property implies that the generating functions satisfy the relations

$$C_j^\bullet(x,\mathbf{u}) = \sum_{l_1+2l_2+\cdots jl_j=j} \prod_{r=1}^{j} \frac{B_r^\bullet(x, W_1, W_2, \ldots)^{l_r}}{l_r!}, \tag{15}$$

where W_j abbreviates

$$W_j = \sum_{i\ge 0} u_{i+j} C_i^\bullet(x,\mathbf{u})$$

with the convention $C_0^\bullet = 1$ (see [9]). The generating function of interest is then

$$C^\bullet(x,\mathbf{u}) = \sum_{j\ge 0} C_j^\bullet(x,\mathbf{u}).$$

This means that we are actually in the framework of Theorems 1 and 2, however, we have to check in particular the compactness and tightness condition.

Proposition 4. *Let $\mathcal{G}$ be a subcritical class of connected vertex labeled graphs. Then the system of equations (15) satisfies all assumptions of Theorems 1 and 2 for $p, q \ge 1$. In particular, if $X_{n,j}$ denotes the number of vertices of degree j in graphs of size n and if $\mathbf{X}_n = (X_{n,1}, X_{n,2}, \ldots)$ denotes the corresponding random sequence then for every functional $\ell \in \ell^q$ the random variable*

$$\ell \cdot \frac{\mathbf{X}_n - \mathbb{E}\mathbf{X}_n}{\sqrt{n}}$$

converges weakly to a centered Gaussian random variable.

Furthermore, if $\mathcal{G}$ denotes the class of vertex labeled series-parallel or outerplanar graphs then the sequence $(\mathbf{X}_n - \mathbb{E}\mathbf{X}_n)/\sqrt{n}$ (considered as a sequence in ℓ^2) is tight and, thus, there exists a centered Gaussian random variable $\mathbf{X}$ with

$$\frac{\mathbf{X}_n - \mathbb{E}\mathbf{X}_n}{\sqrt{n}} \xrightarrow{w} \mathbf{X}.$$

Most of the conditions of Theorems 1 and 2 are easy to check. First, it is clear that for $\mathbf{u} = \mathbf{1}$ the system *collaps* to a single equation for the sum $C'(x) = \sum_{j\ge 0} C^\bullet(x,\mathbf{1})$ which is (of course) of the form $C'(x) = e^{B'(xC'(x))}$. The radius of convergence x_0 of $C'(x)$ is given by the equation $x_0C'(x_0) = \eta$. This follows from the fact that $\eta B''(\eta) = 1$ which is equivalent to $1 = e^{B'(x_0C'(x_0))}B''(x_0C'(x_0))x_0$ and which is the condition that the solution function $C'(x)$ gets singular at x_0. Furthermore, $C'(x)$ has a square root singularity. Hence the functions

$$C_j(x,\mathbf{1}) = \sum_{l_1+2l_2+\cdots jl_j=j} \prod_{r=1}^{j} \frac{B_r^\bullet(xC'(x),\mathbf{1})^{l_r}}{l_r!}$$

have the same dominant singularity (which is of square root type).

The only condition that cannot be directly checked is the compactness condition of the Jacobian. However, we can apply the following general property (that is satisfied in the present example).

Lemma 1. *Let $H(x,y,w)$ be a positive functions (as in Theorem 1 in the one dimensional setting) and suppose that $y(x)$ has a finite radius of convergence x_0 (so that $H(x,y,1)$ is analytic at (x_0,y_0)) and satisfies the functional equation $y(x) = H(x,y(x),1)$. Furthermore consider the system of equations*

$$y_j(x,\mathbf{u}) = F_j(x,\mathbf{y}(x,\mathbf{u}),\mathbf{u})$$

with positive functions that satisfy all assumptions of Theorem 1 except possibly (5). (the compactness of the Jacobian) and where F_i has the additional property that

$$F_i(x,\mathbf{y},\mathbf{1}) = [w^i]H\left(x, \sum_j y_j, w\right).$$

Then we have $y(x) = \sum_i y_i(x,\mathbf{1})$ so that all functions $y_i(x,\mathbf{1})$ have the same radius of convergence as $y(x)$ and the operator $A = \frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x,y,\mathbf{1})$ is compact.

Proof. The relation $y(x) = \sum_i y_i(x,\mathbf{1})$ is obvious. Furthermore the assumptions imply that $\frac{\partial F_i}{\partial y_j}(x,y,\mathbf{1}) = [w^i]H_y\left(x, \sum_j y_j, w\right) y_j$ and, thus, A has rank one. □

In our case we have

$$H(x,y,w) = \exp\left(\sum_{k\ge 0} B_k^\bullet(xy,\mathbf{1})w^k\right).$$

Consequently compactness of the Jacobian is granted.

Finally we have to check tightness. For the sake of shortness we will only work out the details for series-parallel graphs. Outerplanar graphs can be then handled in a similar way. The idea is to consider the variances

$$\sigma_{n,k}^2 = \mathbb{E}\left(\frac{X_{n,k} - \mathbb{E}X_{n,k}}{\sqrt{n}}\right)^2 = \frac{1}{n}\mathbb{V}\mathrm{ar}X_{n,k} = \frac{1}{n}\left(\mathbb{E}\,X_{n,k}^2 - (\mathbb{E}\,X_{n,k})^2\right)$$

and to check condition (9).

In order to get access to the moments $\mathbb{E}\,X_{n,k}$ and $\mathbb{E}\,X_{n,k}^2$ we consider (again) the rooted version of our graphs and denote by $d_{n,k}$ the probability that the root-vertex has degree k in graphs of size n. Furthermore we denote by $d_{n,k,k}$ the probability that the root-vertex has degree k and a uniformly at random chosen vertex (different from the root-vertex) has degree k, too, in graphs of size n. Then we have (since we are dealing with vertex labeled graphs)

$$\mathbb{E}\,X_{n,k} = nd_{n,k} \quad \text{and} \quad \mathbb{E}\,X_{n,k}^2 = nd_{n,k} + n(n-1)d_{n,k,k}.$$

Consequently

$$\sigma_{n,k}^2 = d_{n,k} - d_{n,k,k} + n(d_{n,k,k} - d_{n,k}^2).$$

In [11] it was already proved that $d_{n,k} = O(w_0^{-k})$ and $d_{n,k,k} = O(w_0^{-k})$ uniformly in n and k, where $w_0 \approx 3.482774 > 1$. Thus we certainly obtain

$$\lim_{N\to\infty} \sup_{n\ge 1} \sum_{k\ge N} |d_{n,k} - d_{n,k,k}| = 0.$$

Furthermore we have

$$\lim_{n\to\infty} \sum_{k\ge C\log n} n|d_{n,k,k} - d_{n,k}^2| = 0$$

for a proper constant $C > 0$. Hence it suffices to show that

$$\lim_{N\to\infty} \sup_{n\ge 1} \sum_{N\le k< C\log n} n(d_{n,k,k} - d_{n,k}^2) = 0 \tag{16}$$

Actually it is known (see [11]) that, uniformly for $k \le C\log n$, $d_{n,k}$ has a limit d_k and $d_{n,k,k}$ the limit d_k^2. However, in order to prove the limiting relation (16) we have to make this relation more precise.

It was also shown in [11] that the generating function $C^\bullet(x,w)$ that takes into account the degree of the root-vertex, that is,

$$d_{n,k} = \frac{[x^{n-1}w^k]C^\bullet(x,w)}{[x^{n-1}]C^\bullet(x,1)},$$

is given by the equations

$$\begin{aligned} D(x,w) &= (1+w)\exp\left(\frac{xD(x,w)D(x,1)}{1+xD(x,1)}\right) - 1 \\ B^\bullet(x,w) &= x\left(D(x,w) - \frac{xD(x,1)}{1+xD(x,1)}D(x,w)\left(1+\frac{D(x,w)}{2}\right)\right) \\ C^\bullet(x,w) &= e^{B^\bullet(xC^\bullet(x,1),w)} \end{aligned}$$

and has a (singular) representation around $x = \rho_2 \approx 0.11.021$ and $w = w_0 \approx 3.482774$ of the form

$$C^\bullet(x,w) = G(x,X_2,w) + H(x,X_2,w)\left(1-\overline{y}(x)w\right)^{3/2}, \tag{17}$$

where the functions $G(x,v,w)$ and $H(x,v,w)$ are analytic at $(\rho_2,0,w_0)$, $X_2 = \sqrt{1-x/\rho_2}$ and

$$\overline{y}(x) = \frac{1}{w_0(xC^\bullet(x,1))},$$

with

$$w_0(x) = \left(1+\frac{1}{xD(x,1)}\right)\exp\left(-\frac{1}{1+xD(x,1)}\right) - 1.$$

In particular, $\overline{y}(x)$ has a square root singularity of the form

$$\overline{y}(x) = \overline{g}(x) - \overline{h}(x)X_2,$$

where $\overline{g}$ and $\overline{h}$ are analytic at $x = \rho_2$ which is inherited from the corresponding square root singularity of $C^\bullet(x,1)$:

$$C^\bullet(x,1) = c(x) - d(x)X_2. \tag{18}$$

Lemma 2. *Let d_k denote the limit $\lim_{n\to\infty} d_{n,k}$ and $p(w) = \sum_{k\ge1} d_k$ the generating function of d_k. Then $p(w)$ is given by*

$$p(w) = -\frac{1}{d(\rho_2)}\left(G_v(\rho_2,0,w) + H_v(\rho_2,0,w)(1-\overline{g}(\rho_2)w)^{3/2} + \frac{3}{2}H(\rho_2,0,w)\overline{h}(\rho_2)\sqrt{1-\overline{g}(\rho_2)w}\right).$$

Furthermore we have uniformly for $k \le C\log n$ (for every constant $C > 0$)

$$d_{n,k} = d_k + O\left(\frac{k^{1/2}}{n}w_0^{-k}\right).$$

Proof. First we note that the square root singularity (18) of $C^\bullet(x,1)$ leads to

$$[x^{n-1}]C^\bullet(x,1) = cn^{-3/2}\rho_2^{-n} + O\left(n^{-5/2}\rho_2^{-n}\right)$$

for $c = d(\rho_2)/(2\sqrt{\pi})$.

Next we expand $C^\bullet(x,w)$ in terms of X_2 and obtain

$$\begin{aligned}
C^\bullet(x,w) &= G(\rho_2,0,w) + G_v(\rho_2,0,w)X_2 + G_{vv}(\rho_2,0,w)X_2^2 - G_x(\rho_2,0,w)\rho_2 X_2^2 + O(X_2^3)\\
&+ \left(H(\rho_2,0,w) + H_v(\rho_2,0,w)X_2 + H_{vv}(\rho_2,0,w)X_2^2 - H_x(\rho_2,0,w)\rho_2 X_2^2 + O(X_2^3)\right)\\
&\quad\times (1-\overline{g}(\rho_2)w)^{3/2}\left(1 + \frac{3}{2}\frac{\overline{h}(\rho_2)w}{1-\overline{g}(\rho_2)w}X_2 + \frac{3}{2}\frac{\overline{g}(\rho_2)'\rho_2 w}{1-\overline{g}(\rho_2)w}X_2^2\right.\\
&\qquad\qquad\left. + \frac{3}{8}\frac{\overline{h}(\rho_2)^2w^2}{(1-\overline{g}(\rho_2)w)^2}X_2^2 + O\left(\frac{X_2^3}{(1-\overline{g}(\rho_2)w)^3}\right)\right)\\
&= G(\rho_2,0,w) + H(\rho_2,0,w)(1-\overline{g}(\rho_2)w)^{3/2} + p_1(w)X_2 + p_2(w)X_2^2 + O\left(\frac{X_2^3}{(1-\overline{g}(\rho_2)w)^{3/2}}\right),
\end{aligned}$$

where

$$p_1(w) = G_v(\rho_2,0,w) + H_v(\rho_2,0,w)(1-\overline{g}(\rho_2)w)^{3/2} + \frac{3}{2}H(\rho_2,0,w)\overline{h}(\rho_2)w\sqrt{1-\overline{g}(\rho_2)w}$$

and a proper function $p_2(w)$ (that we will not need explicitly). Note that this expansion is only valid if the ratio $X_2/(1-\overline{g}(\rho_2)w)$ is bounded. Since we assume that $k \le C\log n$ this will be certainly satisfied when we calculate the asymptotic leading part of the (double) Cauchy integral that computes the coefficient $[x^{n-1}w^k]$. In order to simplify the following calculations we will not take care of this *technical detail*, since it is really easy to correct (for more details we refer the reader to [11]).

The function $G(\rho_2,0,w)+H(\rho_2,0,w)(1-\overline{g}(\rho_2)w)^{3/2}$ depends only on w. Hence, the coefficient $[x^{n-1}w^k]$ gets zero for $n \ge 1$. Similarly, the coefficient $[x^{n-1}w^k]p_2(x)X_2 = 0$ for $n \ge 2$. The corresponding coefficient of the error term is then bounded (with the help of the transfer method by Flajolet and Odlyzko [14], where we again note that we should be slightly more precise at this point) by

$$[x^{n-1}w^k]O\left(\frac{X_2^3}{(1-\overline{g}(\rho_2)w)^{3/2}}\right) = O\left(n^{-5/2}k^{1/2}\rho_2^{-n}w_0^{-k}\right);$$

recall that $w_0 = 1/\overline{g}(\rho_2)$. Finally, the dominant term comes from

$$[x^{n-1}w^k]p_1(w)X_2 = -\frac{d(\rho_2)}{2\sqrt{\pi}}d_k n^{-3/2}\rho_2^{-n} + O\left(n^{-5/2}k^{-1/2}\rho_2^{-n}w_0^{-k}\right);$$

recall that $p_1(w) = -d(\rho_2)p(w)$. Summing up we finally obtain

$$d_{n,k} = \frac{[x^{n-1}w^k]C^\bullet(x,w)}{[x^{n-1}]C^\bullet(x,1)} = d_k + O\left(k^{1/2}n^{-1}w_0^{-k} + k^{-1/2}n^{-1}w_0^{-k}\right) = d_k + O\left(k^{1/2}n^{-1}w_0^{-k}\right),$$

which completes the proof of the lemma. □

Similarly, we can handle the probabilities $d_{n,k,k}$. For this purpose we introduce the probabilities $d_{n,k,\ell}$ (that are defined in an obvious way) and the function $C^{\bullet\bullet}(x,w,t)$ that takes into account the degree of the root-vertex (with the help of w) and the degree of a randomly chosen vertex that is different from the root-vertex (with the help of t). More precisely, we have

$$d_{n,k,\ell} = \frac{[x^{n-2}w^k t^\ell]C^{\bullet\bullet}(x,w,t)}{(n-1)[x^{n-1}]C^\bullet(x,1)}.$$

In [11] it is shown that $C^{\bullet\bullet}(x,w,t)$ has a (singular) representation locally around (ρ_2,w_0,w_0) of the form

$$\begin{aligned}
&C^{\bullet\bullet}(x,w,t)\\
&= \frac{1}{X_2}\left(\tilde{H}_1(x,X_2,w,t) + \tilde{H}_2(x,X_2,w,t)\overline{W} + \tilde{H}_3(x,X_2,w,t)\overline{T} + \tilde{H}_4(x,X_2,w,t)\overline{W}\,\overline{T}\right),
\end{aligned}$$

where $\overline{T} = \sqrt{1-\overline{y}(x)w}$ and $\tilde{H}_1$, $\tilde{H}_2$, $\tilde{H}_3$, $\tilde{H}_4$ are analytic at $(\rho_2, 0, w_0, w_0)$.

¿From this we obtain the following limiting relation.

Lemma 3. *Let $\overline{d}_{k,\ell}$ denote the limit $\lim_{n\to\infty} d_{n,k,\ell}$ and $\overline{p}(w,t) = \sum_{k,\ell\ge 1} \overline{d}_{k,\ell}$ the corresponding generating function. Then $\overline{p}(w,t)$ is given by*

$$\overline{p}(w,t) = p(w)p(t),$$

that is, $\overline{d}_{k,\ell} = d_k d_\ell$. Furthermore we have uniformly for $k,\ell \le C\log n$ (for every constant $C>0$)

$$d_{n,k,\ell} = \overline{d}_{k,\ell} + O\left(\frac{1}{n}\left(\frac{k^{1/2}}{\ell} + \frac{\ell^{1/2}}{k}\right) w_0^{-k-\ell}\right).$$

Proof. We proceed similarly to the proof of Lemma 2. ¿From the local expansion

$$\begin{aligned}
C^{\bullet\bullet}(x,w,t) &= \frac{1}{X_2}\left(\tilde{H}_1(\rho_2,0,w,t) + \tilde{H}_{1v}(\rho_2,0,w,t)X_2 + O(X_2^2)\right)\\
&+ \frac{1}{X_2}\left(\tilde{H}_2(\rho_2,0,w,t) + \tilde{H}_{2v}(\rho_2,0,w,t)X_2 + O(X_2^2)\right)\\
&\times \sqrt{1-\overline{g}(\rho_2)w}\left(1 + \frac{1}{2}\frac{\overline{h}(\rho_2)w}{1-\overline{g}(\rho_2)w}X_2 + O\left(\frac{X_2^2}{(1-\overline{g}(\rho_2)w)^2}\right)\right)\\
&+ \frac{1}{X_2}\left(\tilde{H}_3(\rho_2,0,w,t) + \tilde{H}_{3v}(\rho_2,0,w,t)X_2 + O(X_2^2)\right)\\
&\times \sqrt{1-\overline{g}(\rho_2)t}\left(1 + \frac{1}{2}\frac{\overline{h}(\rho_2)t}{1-\overline{g}(\rho_2)t}X_2 + O\left(\frac{X_2^2}{(1-\overline{g}(\rho_2)t)^2}\right)\right)\\
&+ \frac{1}{X_2}\left(\tilde{H}_3(\rho_2,0,w,t) + \tilde{H}_{3v}(\rho_2,0,w,t)X_2 + O(X_2^2)\right)\\
&\times \sqrt{1-\overline{g}(\rho_2)w}\sqrt{1-\overline{g}(\rho_2)t}\\
&\times \left(1 + \frac{1}{2}\frac{\overline{h}(\rho_2)w}{1-\overline{g}(\rho_2)w}X_2 + \frac{1}{2}\frac{\overline{h}(\rho_2)t}{1-\overline{g}(\rho_2)t}X_2 + O\left(\frac{X_2^2}{(1-\overline{g}(\rho_2)w)^2}\right) + O\left(\frac{X_2^2}{(1-\overline{g}(\rho_2)t)^2}\right)\right)\\
&= \frac{q_1(w,t)}{X_2} + q_2(w,t) + O\left(\frac{X_2}{(1-\overline{g}(\rho_2)w)^{3/2}}\right) + O\left(\frac{X_2}{(1-\overline{g}(\rho_2)t)^{3/2}}\right)
\end{aligned}$$

for proper functions $q_1(w,t)$ and $q_2(w,t)$. From this it follows that $\overline{p}(w,t) = 2q_1(w,t)/d(\rho_2)$. Furthermore, it is proved in ([10]) that $\overline{p}(w,t) = p(w)p(t)$, that is, $\overline{d}_{k,\ell} = d_k d_\ell$. Finally, by taking the error terms into account (and by doing a similar analysis as in the proof of Lemma 2) we obtain

$$d_{n,k,\ell} = \overline{d}_{k,\ell} + O\left(\frac{1}{n}\left(\frac{k^{1/2}}{\ell} + \frac{\ell^{1/2}}{k}\right) w_0^{-k-\ell}\right)$$

uniformly for $k,\ell \le C\log n$. □

By combining Lemma 2 and 3 we get

$$n(d_{n,k,k} - d_k^2) = O\left(k w_0^{-2k}\right)$$

and consequently (16). Hence, tightness follows.

4.4. **Random Walks on Groups.** Lalley [22] considered quasi-nearest neighbour walks on infinite free products of groups and proved a local limit theorem for the return probabilities by translating the problem into an infinite system of functional equations. His approach was based on what is known as Lyapunov-Schmidt reduction (see [30]) which is done by decomposing the Banach space into a direct sum of a finite dimensional section and an infinite dimensional one. In contrast, our approach is based on splitting the system into the first equation and the other equations and exploiting some spectral properties of the remaining system, followed by a simple substitution of the solution into the first one.

Lalley's problem is as follows: he considers a sequence of finite groups $\Gamma_1, \Gamma_2, \ldots$ and the free product $\Gamma = \Gamma_1 * \Gamma_2 * \cdots$ defined as the set of all words in $\left(\bigcup_i (\Gamma_i \setminus \{1\})\right)^*$ with the concatination, followed by reduction, as the group operation. Reduction is done if the last letter of the first word and the first letter of the second word are in the same group Γ_i. Then the two letters can be substituted by their product to shorten the word. This process is continued as long as possible. The neutral element of this group is the empty word ε. A random walk on Γ is defined by $S_n = \xi_1\xi_2\cdots\xi_n$ where the steps ξ_i of the random walk are i.i.d and with common distribution

$$P\{\xi_1 = \alpha\} = p_i q_\alpha \text{ if } \alpha \in \Gamma_i \setminus \{1\} \qquad \text{and } P\{\xi_1 = \varepsilon\} = p_0$$

where all p_i, $i \in \mathbb{N}$ and all q_α are strictly positive and for all $m \ge 1$ the set $\{n : P\{S_n = \varepsilon\} > 0\}$ is not contained in a proper subgroup.

Lalley proved the following theorem.

Proposition 5 (Lalley [22])**.** *Let S_n be a quasi nearest neighbor random walk on the infinite free product $\Gamma = \Gamma_1 * \Gamma_2 * \cdots$ as described above. Then there exist constants $C > 0$ and $1 < R < \infty$ such that $P\{S_n = \varepsilon\} \sim CR^{-n}n^{-3/2}$.*

The proof is based on the generating functions for the first hitting times: Set $\tau_\alpha = \min\{n \ge 0 : S_n = \alpha\}$ and

$$y_\alpha(x) = \sum_{n\ge1} P\{\tau_\alpha = n\}x^n,$$

where we write $y_{i;\alpha}(x)$ instead of $y_\alpha(x)$ if $\alpha \in \Gamma_i$. Lalley showed that these functions satisfy the following system of functional equations.

$$y_{i;\alpha}(x) = x\left(p_i q_\alpha + p_0 y_{i;\alpha}(x) + \sum_{\gamma\in\Gamma_i\setminus\{\alpha\}} p_i q_\gamma y_{i;\gamma^{-1}\alpha}(x) + \sum_{j\ne i}\sum_{\gamma\in\Gamma_i} p_j q_\gamma y_{j;\gamma^{-1}}(x) y_{i;\alpha}(x)\right).$$

Furthermore, he showed that $\mathbf{y}(x) \in \ell^1$ and that the Jacobian $\mathcal{J}(x)$ of this system is

$$\mathcal{J}(x) = \mathcal{K}(x) + \mathcal{L}(x) + \frac{G(x)-1}{xG(x)}\mathbf{I}_1, \tag{19}$$

where

$$G(x) = \sum_{n\ge1} P\{S_n = \varepsilon\}x^n = 1 \Bigg/ \left(1 - p_0 x - \sum_{\alpha\ne\varepsilon} p_\alpha x y_{\alpha^{-1}}(x)\right),$$

$\mathcal{K}(x) = (K_{(i,\alpha),(j,\beta)})$ with entries

$$K_{(i,\alpha),(j,\beta)} = p_j q_{\beta^{-1}} y_{i,\alpha}(x),$$

and $\mathcal{L}(x) = (L_{(i,\alpha),(j,\beta)})$ is a block diagonal with nonzero entries only in the $((i,\alpha),(i,\alpha))$ positions and given by

$$L_{(i,\alpha),(i,\alpha)} = \begin{cases} p_i q_{\alpha\beta^{-1}} - p_i q_{\beta^{-1}} y_{i,\alpha}(x) & \text{if } \beta \in \Gamma_i \setminus \{1,\alpha\}, \\ p_j q_{\beta^{-1}} y_{i,\alpha}(x) - p_i q_{\alpha^{-1}} y_{i,\alpha}(x) - \sum_{\gamma\in\Gamma_i} p_i q_\gamma y_{i,\gamma^{-1}}(x) & \text{if } \alpha = \beta. \end{cases}$$

It is easy to check that assumptions (1)–(4) of Theorem 1 are satisfied. The shape (19) of the Jacobian is precisely as required by assumption (5) where compactness of $\mathcal{K}(x)$ and $\mathcal{L}(x)$ is shown in [22, Lemma 4.1]. The alternative representation of the entries of $\mathcal{J}(x)$ as sum of probabilities with positive coefficients which is given in [22, Eq. (4.2)] shows irreducibility and [22, Eq. (4.3)] proves the positivity condition contained in assumption (6) of Theorem 1.

Thus Theorem 1 shows that the functions $y_{i,\alpha}(x)$ have a square-root type singularity and a transfer lemma (see [14]) finally reproves Lalley's result.

5. Proofs

5.1. **Auxiliary results.** In this section we prove some spectral properties of compact and positive operators on ℓ^p spaces and we show that the spectral radius of the Jacobian operator of $\mathbf{F}$ (under the assumptions stated in Theorem 1) is continuous.

Recall that the spectrum of a compact operator is a countable set with no accumulation point different from zero. Moreover, each nonzero element from the spectrum is an eigenvalue with finite multiplicity (see for example [19, Chapter III, § 6.7]). The following result is a generalization of the Perron-Frobenius theorem on nonnegative matrices and goes back to Kreĭn and Rutman [20] (see [30, Proposition 7.26]).

Lemma 4. *Let $T = (t_{ij})_{1\le i,j<\infty}$ be a compact positive operator on ℓ^p (where $1 \le p < \infty$) and assume that $r(T) > 0$. Then $r(T)$ is an eigenvalue of T with nonnegative eigenvector $x \in \ell^p$. Moreover, $r(T) = r(T^*)$ is an eigenvalue of T^* with nonnegative eigenvector $y \in \ell^q$.*

Lemma 5. *Let A_1 be a positive and irreducible operator on ℓ^p (where $1 \le p < \infty$) such that A_1^n is compact for some integer $n \ge 1$. Furthermore let $\alpha \ge 0$ be a real number and set $A = A_1 + \alpha\,\mathrm{I_p}$. Then we have $r(A_1) > 0$ and $r(A) = r(A_1) + \alpha$ is an eigenvalue of A with strictly positive right eigenvector $x \in \ell^p$ and strictly positive left eigenvector $y \in \ell^q$.*

Proof. First we show that $r(A_1) > 0$. Since A_1 is irreducible, there exists an integer m such that

$$d = (A_1^m)_{11} > 0.$$

Then we have $\|A_1^{mn}\| \ge d^n$ for all $n \ge 1$, where $\|\cdot\|$ denotes the operator norm that is induced by the p-norm on ℓ^p (consider $A_1^m e_1$, where $e_1 = (1, 0, 0, \ldots)$). Gelfand's formula implies $r(A) = \lim_{n\to\infty} \|A_1^n\|^{1/n} \ge d^{1/m}$. Since

$$\sigma(A_1^n) = \big(\sigma(A_1)\big)^n,$$

we have that $r := r(A_1)$ is equal to $r(A_1^n)^{1/n}$. Lemma 4 implies that r^n is an eigenvalue of A_1^n and there exist vectors $\tilde{x} \in \ell^p$ and $\tilde{y} \in \ell^q$ such that

$$A_1^n\tilde{x} = r^n\tilde{x}, \qquad \text{and} \qquad \tilde{y}A_1^n = r^n\tilde{y}.$$

Thus we have that r is also in the spectrum of A_1 and $r(A) = r(A_1) + \alpha > 0$. (Note, that $\sigma(A) = \sigma(A_1) + \alpha$.) In the following we show that

$$x := \sum_{i=0}^{n-1} r^i A_1^{n-1-i}\tilde{x}$$

is a strictly positive right eigenvector of A_1 to the eigenvalue r. It is easy to see that $A_1x = rx$. We clearly have that x is nonnegative and $x \ne 0$. Thus, there exists an index j such that $x_j > 0$. Let $k \ge 1$. Since A_1 is irreducible, there exists an integer m such that $(A_1^m)_{kj} > 0$. Since $A_1^m x = r^m x$, we obtain

$$x_k = \frac{1}{r^m}\,(A_1^m x)_k = \frac{1}{r^m}\sum_{\ell=1}^{\infty}(A_1^m)_{k\ell}\,x_\ell > \frac{1}{r^m}\,(A_1^m)_{kj}\,x_j > 0.$$

Furthermore, one can show the same way that $y := \sum_{i=0}^{n-1} r^i\tilde{y}A_1^{n-1-i}$ is a strictly positive left eigenvector of A_1 to the eigenvalue r. □

Proposition 6. *Let $1 \le p < \infty$ and $A = A_1 + \alpha\,\mathrm{I_p}$, $C = C_1 + \gamma\,\mathrm{I_p}$ be operators on ℓ^p with $\alpha \in \mathbb{R}^+$, $\gamma \in \mathbb{C}$ and such that there exists an integer n such that A_1^n and C_1^n are compact. Furthermore let A_1 be positive and irreducible such that $|C_1| \le A_1$ and $|\gamma| \le \alpha$ but $|C_1| + |\gamma\,\mathrm{I_p}| \ne A$. Then we have*

$$r(C) < r(A).$$

Proof. Lemma 5 implies that $r(A) \ge r(A_1) > 0$. If $r(C_1) = 0$, we have $r(C) = |\gamma|$ and

$$r(A) = r(A_1) + \alpha > \alpha \ge |\gamma| = r(C).$$

Assume now that $r(C) > 0$. Since C_1^n is compact, there exists an eigenvector $z \in \ell^p$ to some eigenvalue s with $|s| = r(C_1)$. Since $r(C) \le r(C_1) + |\gamma|$, we get

$$r(C)|z| \le (r(C_1) + |\gamma|)|z| = |C_1z| + |\gamma z| \le (|C_1| + |\gamma\,\mathrm{I_p}|)|z| \le A|z|.$$

If we assume that $r(A) \le r(C)$, then we have

$$r(A)|z| \le A|z|. \tag{20}$$

Next we show that this inequality can only hold true if $|z| = 0$ or if $|z|$ is strictly positive and a right eigenvector of A to the eigenvalue $r(A)$ (cf. [28, Lemma 5.2]): If $|z| = 0$, then (20) holds trivially true. Hence we assume that $|z| \ne 0$. Lemma 5 implies that there exists a strictly positive left eigenvector $y \in \ell^q$ associated to the operator A. Hölders inequality and the fact that $|z| \in \ell^p$ imply

$$\frac{1}{r(A)} yA|z| = y \cdot |z| = \sum_{n=1}^{\infty} x_n \alpha_n < \infty.$$

Thus we have $y \cdot (A|z| - r(A)|z|) = 0$ and since y is strictly positive this can only hold true if $|z|$ is an eigenvector of A to the eigenvalue $r(A)$. The same way as in the proof of Lemma 5 one can now show that the irreducibility of A_1 implies the strict positivity of the eigenvector $|z|$.

It remains to show that $r(A) \le r(C)$ yields a contradiction. Since z is an eigenvector (of C_1^n) we clearly have $|z| \neq 0$. Hence, let us assume that $|z|$ is a strictly positive eigenvector of A. We obtain

$$A|z| = r(A)|z| \le r(C)|z| \le (|C_1| + |\gamma \, \mathrm{I_p}\,|)|z| \le A|z|.$$

Thus, we have $(A - (|C_1| + |\gamma \, \mathrm{I_p}\,|))|z| = 0$. But since $|z|$ is strictly positive and $A \ge |C_1| + |\gamma \, \mathrm{I_p}\,|$ but $A \neq |C_1| + |\gamma \, \mathrm{I_p}\,|$, this is impossible. □

Remark 5. Let A be given as in Proposition 6. Furthermore, let B be obtained through eliminating the first row and first column of A, that is $B = B_1 + \alpha \, \mathrm{I_p}$, where $B_1 = ((B_1)_{ij})_{1 \le i,j < \infty}$ is defined by $(B_1)_{ij} = (A_1)_{i+1\, j+1}$. Then we have

$$r(B) < r(A).$$

In order to see this, note that B is also compact, $r(A) = r(A_1) + \alpha$ and $r(B) = r(B_1) + \alpha$. It is easy to show that Proposition 6 (with $\alpha = \gamma = 0$) implies $r(B_1) < r(A_1)$, which shows the desired result.

Lemma 6. *Let the function* $\mathbf{F}$ *satisfy the assumptions of Theorem 1. Then we have that the map*

$$(x, \mathbf{y}) \mapsto r\left(\frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x, \mathbf{y}, \mathbf{0})\right)$$

is continuous for all positive $(x, \mathbf{y}) \in B \times U$. *Furthermore, if there exists an arbitrary point* $(\tilde{x}, \tilde{\mathbf{y}}, \tilde{\mathbf{v}}) \in B \times U \times V$ *such that*

$$r\left(\frac{\partial \mathbf{F}}{\partial \mathbf{y}}(\tilde{x}, \tilde{\mathbf{y}}, \tilde{\mathbf{v}})\right) < 1,$$

then the same holds true in a neighborhood of $(\tilde{x}, \tilde{\mathbf{y}}, \tilde{\mathbf{v}})$.

Proof. First note, that $(x, \mathbf{y}) \mapsto \frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x, \mathbf{y}, \mathbf{0}) = A(x, \mathbf{y}) + \alpha(x, \mathbf{y})$ is continuous. Let us fix some positive $(x, \mathbf{y}) \in B \times U$ (in the following, we suppress x and $\mathbf{y}$ for brevity). The positivity properties of $\mathbf{F}$ and Lemma 5 imply that $r(\partial \mathbf{F}/\partial \mathbf{y}) = r(A) + \alpha$. (Note, that we have $\sigma(\partial \mathbf{F}/\partial \mathbf{y}) = \sigma(A) + \alpha$.) Furthermore, we have (compare with the proof of Lemma 5)

$$r(A)^n = r(A^n).$$

Thus it remains to show that $r(A^n)$ is continuous for positive $(x, \mathbf{y})$. Let $r(A^n) > 0$. Since A^n is compact and isolated eigenvalues with finite multiplicity must vary continuously (see [19, Chapter IV, § 3.5]), we obtain the desired result. If $r(A^n) = 0$, then the continuity follows from the upper semi-continuity of the spectrum of closed operators (see [19, Chapter IV, § 3.1, Theorem 3.1]).

Now suppose that there exists a point $(\tilde{x}, \tilde{\mathbf{y}}, \tilde{\mathbf{v}}) \in B \times U \times V$ such that

$$r := r\left(\frac{\partial \mathbf{F}}{\partial \mathbf{y}}(\tilde{x}, \tilde{\mathbf{y}}, \tilde{\mathbf{v}})\right) < 1.$$

This means, that the spectrum of $(\partial \mathbf{F}/\partial \mathbf{y})(\tilde{x}, \tilde{\mathbf{y}}, \tilde{\mathbf{v}})$ is contained in a ball with radius r. We can use again [19, Chapter IV, § 3.1, Theorem 3.1] (the upper semi-continuity of the spectrum of closed operators) in order to deduce that there exists a neighborhood D of $(\tilde{x}, \tilde{\mathbf{y}}, \tilde{\mathbf{v}})$ such that for all $(x, \mathbf{y}, \mathbf{v}) \in D$ the spectrum of $(\partial \mathbf{F}/\partial \mathbf{y})(x, \mathbf{y}, \mathbf{v})$ is contained in a ball with radius $1 - (1 - r)/2$. In particular, it follows that

$$r\left(\frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x, \mathbf{y}, \mathbf{v})\right) \le 1 - (1 - r)/2 < 1.$$

This proves the second assertion of Lemma 6. □

5.2. **Proof of Theorem 1 and Corollary 1.**

Proof of Theorem 1. First, we fix the vector $\mathbf{v} = \mathbf{0}$. The implicit function theorem for Banach spaces (see for example [6, Theorem 15.3]) implies that there exists a unique analytic solution $\mathbf{y} = \mathbf{y}(x, \mathbf{0})$ of the functional equation (4) in a neighborhood of $(0, \mathbf{0})$. It also follows from the Banach fixed-point theorem that the sequence $\mathbf{y}^{(0)} \equiv \mathbf{0}$ and

$$\mathbf{y}^{(n+1)}(x, \mathbf{0}) = \mathbf{F}(x, \mathbf{y}^{(n)}(x, \mathbf{0}), \mathbf{0}), \quad n \geq 1,$$

converges uniformly to the unique solution $\mathbf{y}(x, \mathbf{0})$ of (4). Since $\mathbf{F}$ is positive for $\mathbf{v} = \mathbf{0}$, we get that $\mathbf{y}(x, \mathbf{0})$ is positive. Next we show that

$$\begin{aligned} \mathbf{y}_0 &= \mathbf{F}(x_0, \mathbf{y}_0, \mathbf{0}), \\ r\left(\frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x_0, \mathbf{y}_0, \mathbf{0})\right) &= 1, \end{aligned} \tag{21}$$

holds true. The first equation follows from analyticity. Since $\mathbf{F}$ is positive, we obtain that the Jacobian operator (evaluated at x, $\mathbf{y}(x, \mathbf{0})$ and $\mathbf{0}$) is positive. Lemma 6 and Proposition 6 imply that the function

$$x \mapsto r\left(\frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x, \mathbf{y}(x, \mathbf{0}), \mathbf{0})\right)$$

is continuous and strictly monotonically increasing. We get for each $x < x_0$ that

$$r\left(\frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x, \mathbf{y}(x, \mathbf{0}), \mathbf{0})\right) < 1.$$

In order to see this note that implicit differentiation yields

$$\left(I - \frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x, \mathbf{y}(x, \mathbf{0}), \mathbf{0})\right) \frac{\partial \mathbf{y}}{\partial x}(x, \mathbf{0}) = \frac{\partial \mathbf{F}}{\partial x}(x, \mathbf{y}(x, \mathbf{0}), \mathbf{0}). \tag{22}$$

Suppose that the spectral radius of the positive and irreducible Jacobian operator at $(x, \mathbf{y}(x, \mathbf{0}), \mathbf{0})$ for some $x < x_0$ is equal to 1. Lemma 5 implies that there exists a strictly positive left eigenvector to the eigenvalue 1. Multiplying this vector to equation (22) from the left yields a contradiction since $\frac{\partial \mathbf{F}}{\partial x}(x, \mathbf{y}(x, \mathbf{0}), \mathbf{0}) \neq \mathbf{0}$ (note that $\mathbf{F}(x, \mathbf{0}, \mathbf{0}) \not\equiv \mathbf{0}$ and that $\mathbf{F}$ is positive). Since $\mathbf{y}$ cannot be analytically continued at the point x_0 and since $(x_0, \mathbf{y}(x_0)) = (x_0, \mathbf{y}_0)$ lies in the domain of analyticity of $\mathbf{F}$, we obtain that (21) holds true. Indeed, otherwise the implicit function theorem would imply that there exists an analytic continuation.

Next, we divide equation (4) up into two equations (we project equation (4) onto the subspace spanned by the first standard vector and onto its complement):

$$y_1 = \mathbf{F}_1(x, y_1, \overline{\mathbf{y}}, \mathbf{0}), \tag{23}$$

$$\overline{\mathbf{y}} = \overline{\mathbf{F}}(x, y_1, \overline{\mathbf{y}}, \mathbf{0}), \tag{24}$$

where $\overline{\mathbf{y}} = S_\ell \, \mathbf{y}$, $\overline{\mathbf{F}} = S_\ell \, \mathbf{F}$ and S_ℓ denotes the left shift defined by $S_\ell(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots)$. Observe, that the Jacobian operator of $\overline{\mathbf{F}}$ (with respect to $\overline{\mathbf{y}}$) can be obtained by deleting the first row and column of the matrix of the Jacobian operator of $\mathbf{F}$. The tuple $(x_0, (\mathbf{y}_0)_1, \overline{\mathbf{y}_0})$ is a solution of (23) and (24). We can employ the implicit function theorem and obtain that there exists a unique positive analytic solution $\overline{\mathbf{y}} = \overline{\mathbf{y}}(x, y_1, \mathbf{0})$ of (24) with $\overline{\mathbf{y}}(0, 0, \mathbf{0}) = \overline{\mathbf{0}}$. For simplicity, we use the abbreviation $y_{01} = (\mathbf{y}_0)_1$ and $\overline{\mathbf{y}_0} = S_\ell \mathbf{y}_0$. Set

$$A = \frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x_0, \mathbf{y}_0, \mathbf{0}) \quad \text{and} \quad B = \frac{\partial \overline{\mathbf{F}}}{\partial \overline{\mathbf{y}}}(x_0, y_{01}, \overline{\mathbf{y}_0}, \mathbf{0}).$$

Proposition 6 and Remark 5 implies that $r(B) < r(A) = 1$. Thus, we can employ the implicit function theorem another time (at the point $(x_0, y_{01}, \overline{\mathbf{y}_0}, \mathbf{0})$) and obtain that $\overline{\mathbf{y}}(x, y_1, \mathbf{0})$ is also analytic in a neighborhood of $(x_0, y_{01}, \mathbf{0})$. Furthermore, we have $\overline{\mathbf{y}}(x_0, y_{01}, \mathbf{0}) = \overline{\mathbf{y}_0}$. If we insert this function into equation (23), we get a single equation

$$y_1 = \mathbf{F}_1(x, y_1, \overline{\mathbf{y}}(x, y_1, \mathbf{0}), \mathbf{0})$$

for $y_1 = y_1(x, \mathbf{0})$. The function $G(x, y_1) = \mathbf{F}_1(x, y_1, \overline{\mathbf{y}}(x, y_1, \mathbf{0}), \mathbf{0})$ is an analytic function around $(0, 0)$ with $G(0, y_1) = 0$ and such that all Taylor coefficients of G are real and non-negative (this follows from the positivity of $\mathbf{F}$ and $\mathbf{y}(x, y_1, \mathbf{0})$). Furthermore, the tuple $(x_0, y_{01}, \mathbf{0})$ belongs to the region of convergence of $G(x, y)$. In what follows, we show that $(x_0, y_{01}, \mathbf{0})$ is a positive solution of the system of equations

$$\begin{aligned} y_1 &= G(x, y_1), \\ 1 &= G_{y_1}(x, y_1), \end{aligned}$$

with $G_x(x_0, y_{01}) \neq 0$ and $G_{y_1 y_1}(x_0, y_{01}) \neq 0$.

In order to see that $G_{y_1}(x_0, y_{01})$ is indeed equal to 1, note that the classical implicit function theorem otherwise implies that there exists an analytic solution of $y_1 = G(x, y_1)$ locally around x_0. Inserting this function into equation (24), we obtain that there also exists an analytic solution $\mathbf{y}(x, \mathbf{0})$ of (4) in a neighborhood of x_0. As in (22), implicit differentiation yields a contradiction since the spectral radius of the (positive and irreducible) Jacobian operator of $\mathbf{F}$ at $(x_0, \mathbf{y}_0, \mathbf{0})$ is equal to 1.

Next suppose that $G_x(x_0, y_{01}) = 0$. The positivity implies that the unique solution of $y_1 = G(x, y_1)$ is given by $y_1(x, \mathbf{0}) \equiv 0$. Consider the solution $\mathbf{y}(x, \mathbf{0})$ of (4) for some real $x > 0$ in the vicinity of 0. Since the spectral radius of the Jacobian operator is smaller than 1 (for x small), we can express the resolvent with the aid of the Neumann series, i.e., we have (cf. (22))

$$\begin{aligned} \frac{\partial \mathbf{y}}{\partial x}(x, \mathbf{0}) &= \left(I - \frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x, \mathbf{y}(x), \mathbf{0}) \right)^{-1} \frac{\partial \mathbf{F}}{\partial x}(x, \mathbf{y}(x), \mathbf{0}) \\ &= \sum_{n \geq 0} \left(\frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x, \mathbf{y}(x), \mathbf{0}) \right)^{n} \frac{\partial \mathbf{F}}{\partial x}(x, \mathbf{y}(x), \mathbf{0}). \end{aligned}$$

Since $\partial \mathbf{F}/\partial \mathbf{y}$ is irreducible and $\partial \mathbf{F}/\partial x \neq 0$ we obtain that no component of the solution $\mathbf{y}(x, \mathbf{0})$ is a constant function. In particular, $y_1(x, \mathbf{0})$ cannot be constant.

Finally, if $G_{y_1 y_1}(x_0, y_{01}) = 0$, it follows from the positivity of G that G is a linear function in y_1. But then the conditions $G(x_0, y_{01}) = y_{01}$ and $G_{y_1}(x_0, y_{01}) = 1$ imply

$$y_{01} = G(x, y_{01}) = G(x, 0) + G_{y_1}(x, y_{01}) \cdot y_{01} = G(x, 0) + y_{01}.$$

Thus we have in this case that $G(x, 0) = 0$. But then (since $G_{y_1}(x, y_1) = G(x, 1)$ and $G(0, y_1) = 0$), the only solution of

$$y_1 = G(x, y_1) = G_{y_1}(x, y_{01}) \cdot y_1$$

is $y_1(x, \mathbf{0}) \equiv 0$. As we have seen before, this is impossible.

It follows from [8, Theorem 2.19] that there exists a unique solution $\tilde{y}(x_1, \mathbf{0})$ of the equation $y_1 = G(x, y_1)$ with $\tilde{y}_1(0, \mathbf{0}) = 0$. It is analytic for $|x| < x_0$ and there exist functions $g_1(x, \mathbf{0})$ and $h_1(x, \mathbf{0})$ that are analytic around x_0 such that $\tilde{y}_1(x, \mathbf{0})$ has a representation of the form

$$\tilde{y}_1(x, \mathbf{0}) = g_1(x, \mathbf{0}) - h_1(x, \mathbf{0}) \sqrt{1 - \frac{x}{x_0}} \tag{25}$$

locally around x_0 with $h_1(x_0, \mathbf{0}) > 0$ and $g_1(x_0, \mathbf{0}) = y_{01}$. Due to the uniqueness of the solution $y(x, \mathbf{0})$ of the functional equation (4), we have that the first component of $y(x, \mathbf{0})$ coincide with $\tilde{y}_1(x, \mathbf{0})$, i.e., $y_1(x, \mathbf{0}) = \tilde{y}_1(x, \mathbf{0})$. Moreover, we get $\overline{\mathbf{y}}(x, y_1(x, \mathbf{0}), \mathbf{0}) = (y_2(x, \mathbf{0}), y_3(x, \mathbf{0}), \ldots)$. More precisely, the analyticity of $\overline{\mathbf{y}}$ implies that there exist an $s > 0$ and vectors $a_n(x) := a_n(x, g_1(x, \mathbf{0}), \mathbf{0}) \in \ell^p$ such that $\sum_n \|a_n(x)\| \, s^n < \infty$ and

$$\overline{\mathbf{y}}(x, y_1, \mathbf{0}) = \overline{\mathbf{y}}(x, g_1(x, \mathbf{0}), \mathbf{0}) + \sum_{n \geq 1} \frac{a_n(x)}{n!} \Big((y_1 - g_1(x, \mathbf{0}))^n \Big), \tag{26}$$

and we obtain

$$\begin{aligned}\overline{\mathbf{y}}(x, y_1(x,\mathbf{0}),\mathbf{0}) &= \overline{\mathbf{y}}(x, g_1(x,\mathbf{0}),\mathbf{0}) + \sum_{n\ge 1} \frac{\left(1-\frac{x}{x_0}\right)^n}{(2n)!} a_{2n}(x)\Big(h_1(x,\mathbf{0})^{2n}\Big) \\ &\quad - \sqrt{1-\frac{x}{x_0}} \sum_{n\ge 0} \frac{\left(1-\frac{x}{x_0}\right)^n}{(2n+1)!} a_{2n+1}(x)\Big(h_1(x,\mathbf{0})^{2n+1}\Big) \\ &= \overline{\mathbf{g}}(x,\mathbf{0}) - \overline{\mathbf{h}}(x,\mathbf{0})\sqrt{1-\frac{x}{x_0}}.\end{aligned}$$

In particular, we get the desired representation

$$\mathbf{y}(x,\mathbf{0}) = \mathbf{g}(x,\mathbf{0}) - \mathbf{h}(x,\mathbf{0})\sqrt{1-\frac{x}{x_0}}$$

with $\mathbf{g}(x,\mathbf{0}) = (g_1(x,\mathbf{0}),\overline{\mathbf{g}}(x,\mathbf{0}))$ and $\mathbf{h}(x,\mathbf{0}) = (h_1(x,\mathbf{0}),\overline{\mathbf{h}}(x,\mathbf{0}))$. Furthermore, we have the property $\mathbf{h}_1(x,\mathbf{0}) > 0$. Since the same result can be obtained when equation (4) is projected onto the subspace spanned by the *i-th* standard vector and onto its complement, we obtain that $\mathbf{h}_i(x,\mathbf{0}) > 0$, either. (Note, that the reasoning of Remark 5 also works when the *i-th* row and column of the Jacobian matrix is deleted.)

Until now, we have shown that the statement of Theorem 1 is true for $\mathbf{v} = \mathbf{0}$. Next, we prove that the solution $\mathbf{y}(x,\mathbf{v})$ is also analytic in $\mathbf{v}$. We have seen before, that

$$r\left(\frac{\partial \overline{F}}{\partial \overline{\mathbf{y}}}(x_0,(\mathbf{y}_0)_1,\overline{\mathbf{y}_0},\mathbf{0})\right) < 1.$$

It follows, that there exists a unique solution $\overline{\mathbf{y}}(x,y_1,\mathbf{v})$ of the function equation

$$\overline{\mathbf{y}} = \overline{\mathbf{F}}(x,y_1,\overline{\mathbf{y}},\mathbf{v})$$

for all $(x,\mathbf{y}_1,\mathbf{v})$ in a neighborhood of $(x_0,(\mathbf{y}_0)_1,\mathbf{0})$. Inserting this solution into (23) (but this time with the additional variable $\mathbf{v}$), we have already seen that the functional equations

$$\begin{aligned} y_1 &= G(x,y_1,\mathbf{v}),\\ 1 &= G_{y_1}(x,y_1,\mathbf{v}),\end{aligned}$$

with $G(x,y_1,\mathbf{v}) = \mathbf{F}_1(x,y_1,\overline{\mathbf{y}}(x,y_1,\mathbf{v}),\mathbf{v})$ have a positive solution $(x_0,(\mathbf{y}_0)_1,\mathbf{0})$. Furthermore, note that $G_x(x_0,(\mathbf{y}_0)_1,\mathbf{0}) \neq 0$ and $G_{y_1y_1}(x_0,(\mathbf{y}_0)_1,\mathbf{0}) \neq 0$. Since we have (evaluated at $(x_0,(\mathbf{y}_0)_1,\mathbf{0})$) that

$$\det\begin{pmatrix} -G_x & 1-G_{y_1} \\ -G_{y_1,x} & -G_{y_1,y_1}\end{pmatrix} = G_x\cdot G_{y_1y_1} \neq 0,$$

the implicit function theorem implies that there exist unique analytic functions $x_0(\mathbf{v})$ and $y_1(\mathbf{v})$ in a neighborhood of $\mathbf{0}$, such that we have $y_1(\mathbf{v}) = G(x_0(\mathbf{v}),y_1(\mathbf{v}),\mathbf{v})$ and $G_{y_1}(x_0(\mathbf{v}),y_1(\mathbf{v}),\mathbf{v}) = 1$. In particular, we have $x_0(\mathbf{0}) = x_0$ and $y_1(\mathbf{0}) = (\mathbf{y}_0)_1$. ¿From continuity it follows that for any $\mathbf{v}$ in a neighborhood of $\mathbf{0}$ we have $G_x(x_0(\mathbf{v}),y_1(\mathbf{v}),\mathbf{v}) \neq 0$ and $G_{y_1y_1}(x_0(\mathbf{v}),y_1(\mathbf{v}),\mathbf{v}) \neq 0$. Thus, the Weierstrass preparation theorem implies that there exist analytic functions $g_1(x,\mathbf{v})$ and $h_1(x,\mathbf{v})$ such that

$$y_1(x,\mathbf{v}) = g_1(x,\mathbf{v}) - h_1(x,\mathbf{v})\sqrt{1-\frac{x}{x_0(\mathbf{v})}} \tag{27}$$

(see for example the proof of [8, Theorem 2.19]). Inserting this solution into $\overline{\mathbf{y}}(x,y_1,\mathbf{v})$ (cf. 26), this finally proves (5).

In what follows, we show that $x_0(\mathbf{v})$ is the only singularity of $\mathbf{y}(x,\mathbf{v})$ on the circle $|x| = x_0(\mathbf{v})$. Recall that by assumption, there exist two integers n_1 and n_2 that are relatively prime such that $[x^{n_1}]\mathbf{y}_1(x,\mathbf{0}) > 0$ and $[x^{n_2}]\mathbf{y}_1(x,\mathbf{0}) > 0$. In order to show the desired result, it suffices to show that

$$G_{y_1}(x,\mathbf{y}_1(x,\mathbf{v}),\mathbf{v}) \neq 1 \tag{28}$$

for $|x| = x_0(\mathbf{v})$ but $x \neq x_0(\mathbf{v})$ (compare with the proof of [8, Theorem 2.19]). Let us first study the case $\mathbf{v} = \mathbf{0}$. Since $\mathbf{y}_1(x, \mathbf{0})$ is positive, we clearly have $|\mathbf{y}_1(x, \mathbf{0})| \leq \mathbf{y}_1(|x|, \mathbf{0})$. If equality occurs, then

$$x^{n_1} = |x|^{n_1} = x_0^{n_1} \quad \text{and} \quad x^{n_2} = |x|^{n_2} = x_0^{n_2}.$$

Since n_1 and n_2 are relatively prime we obtain $x = x_0$, which is impossible. Thus, we actually have $|\mathbf{y}_1(x, \mathbf{0})| < \mathbf{y}_1(|x|, \mathbf{0})$. The positivity of G implies

$$\begin{aligned}|G_{y_1}(x, \mathbf{y}_1(x, \mathbf{0}), \mathbf{0})| &\leq G_{y_1}(|x|, |\mathbf{y}_1(x, \mathbf{0})|, \mathbf{0}) \\ &< G_{y_1}(|x|, \mathbf{y}_1(|x|, \mathbf{0}), \mathbf{0}) = G_{y_1}(x_0, (\mathbf{y}_0)_1, \mathbf{0}) = 1.\end{aligned}$$

¿From continuity we obtain that $|G_{y_1}(x, \mathbf{y}_1(x, \mathbf{v}), \mathbf{v})| < 1$ and (28) follows. Thus, there exists an analytic continuation of $\mathbf{y}_1(x, \mathbf{v})$ locally around x. ¿From positivity, it follows that

$$\left| \frac{\partial \overline{\mathbf{F}}}{\partial \overline{\mathbf{y}}}(x, (\mathbf{y}(x, \mathbf{0}))_1, \overline{\mathbf{y}}(x, \mathbf{0}), \mathbf{0}) \right| \leq \left| \frac{\partial \overline{\mathbf{F}}}{\partial \overline{\mathbf{y}}}(x_0, y_{01}, \overline{\mathbf{y}}_0, \mathbf{0}) \right|.$$

Employing Proposition 6 yields

$$r\left(\frac{\partial \overline{\mathbf{F}}}{\partial \overline{\mathbf{y}}}(x', \mathbf{y}(x', \mathbf{v}), \mathbf{v}) \right) < 1$$

for $x' = x$ and $\mathbf{v} = \mathbf{0}$. Lemma 6 implies that the same holds true for all $(x', \mathbf{v})$ in a neighborhood of $(x, \mathbf{0})$. The implicit function theorem implies that we can insert the function $\mathbf{y}_1(x, \mathbf{v})$ into the solution of equation (24). We obtain that $x_0(\mathbf{v})$ is the only singularity of $\mathbf{y}(x, \mathbf{v})$ on the circle $|x| = x_0(\mathbf{v})$ and there exist constants δ and η such that $\mathbf{y}(x, \mathbf{v})$ is analytic in $\{x : |x| < x_0(\mathbf{v}) + \eta, |\arg(x/x_0(\mathbf{v}) - 1)| > \delta\}$ (note, that locally around $x_0(\mathbf{v})$ the representation (27) yields an analytic continuation). □

Proof of Corollary 1. The first part of the proof is similar to (26). Since $G(x, \mathbf{y}, \mathbf{v})$ is analytic in $(x_0(\mathbf{0}), \mathbf{y}(x_0(\mathbf{0}), \mathbf{0}), \mathbf{0})$ there exist an $s > 0$ and continuous symmetric n-linear forms $A_n(x, \mathbf{v}) := A_n(x, \mathbf{g}(x, \mathbf{v}), \mathbf{v})$ (defined on the the right space) such that

$$\sum_{n \geq 1} \|A_n(x, \mathbf{v})\| \, s^n < \infty$$

and

$$G(x, \mathbf{y}, \mathbf{v}) = G(x, \mathbf{g}(x, \mathbf{v}), \mathbf{v}) + \sum_{n \geq 1} \frac{A_n(x, \mathbf{v})}{n!} \Big((\mathbf{y} - \mathbf{g}(x, \mathbf{v}))^n \Big).$$

Note, that

$$A_1(x, \mathbf{v})(\mathbf{y}) = \frac{\partial G}{\partial \mathbf{y}}(x, \mathbf{g}(x, \mathbf{v}), \mathbf{v}) \in \ell^q$$

and

$$A_1(x_0(\mathbf{0}), \mathbf{0}) = \frac{\partial G}{\partial \mathbf{y}}(x_0(\mathbf{0}), \mathbf{y}(x_0(\mathbf{0}), \mathbf{0}), \mathbf{0}) \neq \mathbf{0}$$

by assumption. We can write

$$\begin{aligned}G(x, \mathbf{y}(x, \mathbf{v}), \mathbf{v}) &= G(x, \mathbf{g}(x, \mathbf{v}), \mathbf{v}) + \sum_{n \geq 1} \frac{\left(1 - \frac{x}{x_0(\mathbf{v})}\right)^n}{(2n)!} A_{2n}(x, \mathbf{v}) \Big(\mathbf{h}(x, \mathbf{v})^{2n} \Big) \\ &\quad - \sqrt{1 - \frac{x}{x_0(\mathbf{v})}} \sum_{n \geq 0} \frac{\left(1 - \frac{x}{x_0(\mathbf{v})}\right)^n}{(2n+1)!} A_{2n+1}(x, \mathbf{v}) \Big(\mathbf{h}(x, \mathbf{v})^{2n+1} \Big) \\ &= \bar{g}(x, \mathbf{v}) - \bar{h}(x, \mathbf{v}) \sqrt{1 - \frac{x}{x_0(\mathbf{v})}}.\end{aligned}$$

Moreover, we have that $\bar{g}$ and $\bar{h}$ are analytic and

$$\bar{h}(x_0(\mathbf{0}), \mathbf{0}) = A_1(x_0(\mathbf{0}), \mathbf{0}) \cdot \mathbf{h}(x_0(\mathbf{0}), \mathbf{0}) \neq 0.$$

(Recall that $\mathbf{h} \in \ell^p$ and $\mathbf{h}_i(x_0(\mathbf{0}), \mathbf{0}) > 0$ for all $i \geq 1$, see Theorem 1). The analyticity of $x_0(\mathbf{v})$ and $G(x, \mathbf{y}(x,\mathbf{v}), \mathbf{v})$ follows from Theorem 1. Using the transfer lemma of [14] (the region of analyticity Δ from Theorem 1 is uniform in $\mathbf{v}$) we finally obtain that

$$[x^n]G(x, \mathbf{y}(x,\mathbf{v}), \mathbf{v}) = \frac{\bar{h}(x_0(\mathbf{v}), \mathbf{v})}{2\sqrt{\pi}} x_0(\mathbf{v})^{-n} n^{-3/2} \left(1 + O\left(\frac{1}{n}\right)\right)$$

uniformly for $\mathbf{v}$ in a neighborhood of $\mathbf{0}$. (Note, that the part coming from $g(x,\mathbf{v})$ is exponentially smaller than the stated term.) □

5.3. **Proof of Theorem 2 and Corollary 2.** Recall that $G(x, \mathbf{y}, \mathbf{v})$ is the generating function of some combinatorial object of the form

$$G(x, \mathbf{y}(x,\mathbf{v}), \mathbf{v}) = \sum_{n=0}^{\infty} c_n(\mathbf{v}) x^n,$$

where $\mathbf{y}(x,\mathbf{v})$ satisfies a functional equation

$$\mathbf{y} = \mathbf{F}(x, \mathbf{y}, \mathbf{v})$$

with $\mathbf{y}(0,\mathbf{v}) = \mathbf{0}$ such that the assumptions of Theorem 1 are satisfied. Moreover, $\mathbf{X}_n$ denotes an ℓ^p-valued $(1 \leq p < \infty)$ random variable defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\mathbb{E}\left[e^{it\boldsymbol{\ell}\cdot\mathbf{X}_n}\right] = \frac{c_n(it\boldsymbol{\ell})}{c_n(\mathbf{0})}$$

for all $\boldsymbol{\ell} \in \ell^q$.

Proof of Theorem 2. We have $\boldsymbol{\ell} \in \ell^q$. Corollary 1 implies that uniformly in t (for small values of t) that

$$c_n(it\boldsymbol{\ell}) = \frac{\bar{h}\big(x_0(it\boldsymbol{\ell}), it\boldsymbol{\ell}\big)}{2\sqrt{\pi}} x_0(it\boldsymbol{\ell})^{-n} n^{-3/2} \left(1 + O\left(\frac{1}{n}\right)\right),$$

where $\bar{h}$ and x_0 are analytic functions . Thus we get

$$\mathbb{E}\left[e^{it\boldsymbol{\ell}\cdot\mathbf{X}_n}\right] = \frac{c_n(it\boldsymbol{\ell})}{c_n(\mathbf{0})} = \frac{\bar{h}\big(x_0(it\boldsymbol{\ell}), it\boldsymbol{\ell}\big)}{\bar{h}(x_0(\mathbf{0}), \mathbf{0})} \left(\frac{x_0(\mathbf{0})}{x_0(it\boldsymbol{\ell})}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right).$$

Set

$$\Phi_{\boldsymbol{\ell}}(s) := x_0(s\boldsymbol{\ell}), \qquad f_{\boldsymbol{\ell}}(s) = \log\left(\frac{\Phi_{\boldsymbol{\ell}}(0)}{\Phi_{\boldsymbol{\ell}}(s)}\right), \qquad \text{and} \qquad g_{\boldsymbol{\ell}}(s) = \log\left(\frac{\bar{h}(\Phi_{\boldsymbol{\ell}}(s), s\boldsymbol{\ell})}{\bar{h}(\Phi_{\boldsymbol{\ell}}(0), \mathbf{0})}\right).$$

These functions are analytic in a neighborhood of $s = 0$ and we have $f_{\boldsymbol{\ell}}(0) = g_{\boldsymbol{\ell}}(0) = 0$ and $\Phi_{\boldsymbol{\ell}}(0) = x_0(\mathbf{0})$. We obtain

$$\begin{aligned}\mathbb{E}\left[e^{it\boldsymbol{\ell}\cdot\mathbf{X}_n}\right] &= e^{f_{\boldsymbol{\ell}}(it)\cdot n + g_{\boldsymbol{\ell}}(it)} \left(1 + O\left(\frac{1}{n}\right)\right) \\ &= e^{it\mu_{\boldsymbol{\ell}} n - \sigma_{\boldsymbol{\ell}}^2 n \frac{t^2}{2} + O(nt^3) + O(t)} \left(1 + O\left(\frac{1}{n}\right)\right),\end{aligned}$$

where $\mu_{\boldsymbol{\ell}} = f'_\ell(0)$ and $\sigma_{\boldsymbol{\ell}}^2 = f''_\ell(0)$. Replacing t by $t/\sqrt{n}$ we get

$$\mathbb{E}\left[e^{it\boldsymbol{\ell}\cdot\mathbf{X}_n/\sqrt{n}}\right] = e^{it\mu_{\boldsymbol{\ell}}\sqrt{n} - \sigma_{\boldsymbol{\ell}}^2 \frac{t^2}{2} + O(t^3/\sqrt{n})} \left(1 + O\left(\frac{1}{n}\right)\right). \tag{29}$$

By definition, $\boldsymbol{\ell} \cdot \mathbb{E}\mathbf{X}_n = \mathbb{E}[\boldsymbol{\ell} \cdot \mathbf{X}_n]$. If we set $\chi_n(t) = \mathbb{E}e^{it\boldsymbol{\ell}\cdot\mathbf{X}_n}$, then $\mathbb{E}[\boldsymbol{\ell} \cdot \mathbf{X}_n] = -i \cdot \chi'_n(0)$. By Cauchy's formula, we have

$$-i \cdot \chi'_n(0) = -\frac{1}{2\pi} \int_{|u|=\rho} \frac{\chi_n(u)}{u^2} \mathrm{d}u.$$

Setting $\rho = 1/n$, we get

$$\begin{aligned}\mathbb{E}[\boldsymbol{\ell}\cdot\mathbf{X}_n] &= -\frac{1}{2\pi}\int_{|u|=1/n}\frac{1+iu\mu_{\boldsymbol{\ell}}n+iug_{\boldsymbol{\ell}}'(0)+O(u^2)}{u^2}\left(1+O\left(\frac{1}{n}\right)\right)\mathrm{d}u\\ &= \frac{1}{2\pi i}\int_{|u|=1/n}\frac{\mu_{\boldsymbol{\ell}}n}{u}\mathrm{d}u+O(1).\end{aligned}$$

Hence, $\boldsymbol{\ell}\cdot\mathbb{E}\mathbf{X}_n = \mu_{\boldsymbol{\ell}}n+O(1)$. Setting

$$\mathbf{Y}_n := \frac{\mathbf{X}_n-\mathbb{E}\mathbf{X}_n}{\sqrt{n}},$$

we finally obtain (see (29))

$$\lim_{n\to\infty}\mathbb{E}\left[e^{it\boldsymbol{\ell}\cdot\mathbf{Y}_n}\right] = e^{-\frac{\sigma_{\boldsymbol{\ell}}^2t^2}{2}}.$$

In particular, this implies that $\mathbf{Y}_n$ weakly converges to a centered Gaussian random variable with variance $\sigma_{\boldsymbol{\ell}}^2$. It remains to calculate $\mu_{\boldsymbol{\ell}}$ and $\sigma_{\boldsymbol{\ell}}^2$: Since $x_0:\ell^q\to\mathbb{C}$ is an analytic function, it follows that there exists a vector $\partial x_0/\partial\mathbf{v}:\ell^q\to\ell^p\approx(\ell^q)^*$ (the first derivative) and an operator $\partial^2x_0/\partial\mathbf{v}^2:\ell^q\to L(\ell^q,\ell^p)$ (the second derivative) such that

$$x_0(\mathbf{h}) = x_0(\mathbf{0})+\frac{\partial x_0}{\partial\mathbf{v}}(\mathbf{0})\cdot\mathbf{h}+\frac{1}{2}\left(\frac{\partial^2x_0}{\partial\mathbf{v}^2}(\mathbf{0})\right)(\mathbf{h})\cdot\mathbf{h}+o(\|\mathbf{h}\|^2)$$

in a neighborhood of $\mathbf{0}$. Note, that the second derivative can be associated with the infinite dimensional Hessian matrix $A=(a_{ij})_{1\le i,j<\infty}$ via

$$\left(\frac{\partial^2x_0}{\partial\mathbf{v}^2}(\mathbf{0})\right)(\mathbf{h})\cdot\mathbf{h} = \mathbf{h}^HA\mathbf{h},$$

where

$$a_{ij} = \frac{\partial^2x_0}{\partial\mathbf{v}_i\mathbf{v}_j}(\mathbf{0}).$$

We obtain

$$\mu_{\boldsymbol{\ell}} = -\frac{1}{\Phi_{\boldsymbol{\ell}}(0)}\Phi_{\boldsymbol{\ell}}'(0) = -\frac{1}{\Phi_{\boldsymbol{\ell}}(0)}\cdot\frac{\partial x_0}{\partial\mathbf{v}}(\mathbf{0})\cdot\boldsymbol{\ell},$$

and

$$\sigma_{\boldsymbol{\ell}}^2 = \frac{\Phi_{\boldsymbol{\ell}}'(0)^2}{\Phi_{\boldsymbol{\ell}}(0)^2}-\frac{\Phi_{\boldsymbol{\ell}}''(0)}{\Phi_{\boldsymbol{\ell}}(0)} = \mu_{\boldsymbol{\ell}}^2-\frac{\Phi_{\boldsymbol{\ell}}''(0)}{\Phi_{\boldsymbol{\ell}}(0)} = \mu_{\boldsymbol{\ell}}^2-\frac{1}{\Phi_{\boldsymbol{\ell}}(0)}\left(\frac{\partial^2x_0}{\partial\mathbf{v}^2}(\mathbf{0})\right)(\boldsymbol{\ell})\cdot\boldsymbol{\ell}.$$

If we define $B\in L(\ell^q,\ell^p)$ by

$$\frac{1}{\Phi_{\boldsymbol{\ell}}(0)^2}\left(\frac{\partial x_0}{\partial\mathbf{v}}(0)\cdot\frac{\partial x_0}{\partial\mathbf{v}}(0)^T\right)-\frac{1}{\Phi_{\boldsymbol{\ell}}(0)}A,$$

then we have

$$\sigma_{\boldsymbol{\ell}}^2 = \boldsymbol{\ell}^HB\boldsymbol{\ell}.$$

This finally proves Theorem 2. □

Proof of Corollary 2. Set $\mathbf{Y}_n=(\mathbf{X}_n-\mathbb{E}\mathbf{X}_n)/\sqrt{n}$. Since the set of laws of $(\mathbf{Y}_n)_{n\ge1}$ is tight, we know from Prohorov's theorem (see [3, Chapter I, Section 5]) that the set of laws of $(\mathbf{Y}_n)_{n\ge1}$ is a relatively compact set. In particular, it follows that there exists a subsequence $(\mathbf{Y}_{n_k})_{k\ge1}$ that weakly converges to some random variable $\mathbf{X}$. Let $\chi_{\boldsymbol{\ell}}(t)$ be the characteristic function of $\boldsymbol{\ell}\cdot\mathbf{X}$, that is, $\chi_{\boldsymbol{\ell}}(t)=\mathbb{E}e^{it\boldsymbol{\ell}\cdot\mathbf{X}}$. ¿From weak convergence of $\mathbf{Y}_{n_k}$, we obtain on the one hand that

$$\lim_{k\to\infty}\mathbb{E}\left[e^{it\boldsymbol{\ell}\cdot\mathbf{Y}_{n_k}}\right] = \chi_{\boldsymbol{\ell}}(t)$$

for all $\boldsymbol{\ell}\in\ell^q$. One the other hand, Theorem 2 implies that there exist constants $\sigma_{\boldsymbol{\ell}}^2$ such that

$$\lim_{n\to\infty}\mathbb{E}\left[e^{it\boldsymbol{\ell}\cdot\mathbf{Y}_n}\right] = e^{-\frac{\sigma_{\boldsymbol{\ell}}^2t^2}{2}}$$

for all $\boldsymbol{\ell} \in \ell^q$. Hence we see that we actually have

$$\chi_{\boldsymbol{\ell}}(t) = e^{-\frac{\sigma_{\boldsymbol{\ell}}^2 t^2}{2}},$$

and thus, $\mathbf{X}$ is a centered Gaussian random variable. Moreover, we obtain that not only $(\mathbf{Y}_{n_k})_{k\ge 1}$ but $(\mathbf{Y}_n)_{n\ge 1}$ weakly converges to $\mathbf{X}$. Since the distribution of $\mathbf{X}$ is uniquely determined by the distributions of $\boldsymbol{\ell}\cdot\mathbf{X}$, $\boldsymbol{\ell} \in \ell^q$, we obtain that $\mathbf{X}$ is uniquely determined by the operator B stated in Theorem 2. □

5.4. **Proof of Theorem 3.** For the sake of brevity we just give an outline of the proof. Corollary 2 implies that we only have to show the tightness condition. Theorem 6.2.3 of [17] states that tightness follows from the property (9). First of all we know by assumption that $X_n^{(j)} = 0$ if $j > cn$. Hence the condition (9) reduces to

$$\lim_{N\to\infty} \sup_{n\ge 1} \sum_{N<j\le cn} \sigma_{n,j}^2 = 0 \tag{30}$$

where $\sigma_{n,j}^2$ denote the variance of the normalized random variable $(X_n^{(j)} - \mathbb{E}X_n^{(j)})/\sqrt{n}$. Now assume that we know that $\sigma_{n,j}^2$ is asymptotically given by

$$\sigma_{n,j}^2 = \sigma_j^2 + \frac{\tau_j}{n} + O(n^{-2}), \tag{31}$$

where

$$\sum_{j\ge 0} \sigma_j^2 < \infty, \quad \text{and} \quad \tau_j = o(1) \quad (j\to\infty),$$

and the error term is uniform for all $j \ge 0$. Then it is clear that (31) implies (30) and, hence, tightness follows. By Theorem 2.23 of [8] we already know that we have an expansion of the form (31) and that σ_j^2 is given by

$$\sigma_j^2 = \frac{F_{v_j}}{x_0 F_x} + \left(\frac{F_{v_j}}{x_0 F_x}\right)^2 + \frac{1}{x_0 F_x^3 F_{yy}}\Big(F_x^2(F_{yy}F_{v_jv_j} - F_{yv_j}^2) - 2F_xF_{v_j}(F_{yy}F_{xv_j} - F_{yx}F_{yv_j}) + F_{v_j}^2(F_{yy}F_{xx} - F_{yx}^2)\Big).$$

By assumption it is then clear that the sum $\sum \sigma_j^2$ is convergent. In a similar (but slightly more involved) way it is also possible to calculate τ_j explicitly, from which we easily deduce the convergence $\tau_j \to 0$.

5.5. **Proof of Theorem 4.** In this section we prove Theorem 4. Let us recall that $\mathbf{F}$ can be written as

$$\mathbf{F}(x,\mathbf{y},\mathbf{v}) = \mathbf{L}(x,\mathbf{v})\mathbf{y} + \mathbf{b}(x,\mathbf{v}),$$

where $\mathbf{L}(x,\mathbf{0}) = A(x) + \alpha(x)\,\mathrm{I_p}$, α is an analytic function, and there exists an integer n such that A^n is compact. Furthermore, $A(x)$ is irreducible and strictly positive for $x > 0$ and α and $\mathbf{b}(x,\mathbf{0})$ has positive Taylor coefficients. Moreover, $\mathbf{F}(0,\mathbf{y},\mathbf{v}) = \mathbf{0}$ for all $\mathbf{y}$ and $\mathbf{v}$ in a neighborhood of $\mathbf{0}$.

In order to show the desired result we proceed as in the proof of Theorem 1. Let us first assume that $\mathbf{v} = \mathbf{0}$ (in the following, we suppress the variable $\mathbf{v} = \mathbf{0}$ in order to make text more readable). The implicit function theorem (and the Banach fixed-point theorem) implies that there exists a unique analytic and positive solution $\mathbf{y}(x)$ of the function equation $\mathbf{y} = \mathbf{F}(x,\mathbf{y})$, $\mathbf{y}(0) = \mathbf{0}$. Since $\mathbf{F}$ is linear, this can be also deduced from the following reasoning: Since $r(\mathbf{L}(0)) = 0$ (note, that $\mathbf{F}(0,\mathbf{y}) = \mathbf{0}$ for all $\mathbf{y}$ in a neighborhood of $\mathbf{0}$), we see that the solution $\mathbf{y}(x)$ is given by

$$\mathbf{y}(x) = (\mathrm{I_p} - \mathbf{L}(x))^{-1}\mathbf{b}(x) = \sum_{k=0}^{\infty} \mathbf{L}(x)^k \mathbf{b}(x).$$

Here we also used that the inverse of $(\mathrm{I_p} - \mathbf{L}(x))$ can be represented by the corresponding Neumann series as long as $r(\mathbf{L}(x)) < 1$. Since $\mathbf{L}$ and $\mathbf{b}$ is positive, the solution $\mathbf{y}$ is also positive. Note, that the solution exists for all $x < x_0$ and that there is a singularity at x_0.

In what follows we split the functional equation $\mathbf{y} = \mathbf{F}(x, \mathbf{y})$ up into two equations (cf. (23) and (24)). Since $\mathbf{F}$ is linear, this gives

$$y_1 = \mathbf{l}_{11}(x) \cdot \mathbf{y}_1 + \overline{\mathbf{l}_r}(x) \cdot \overline{\mathbf{y}} + \mathbf{b}_1(x) \tag{32}$$

$$\overline{\mathbf{y}} = \mathbf{y}_1 \overline{\mathbf{l}_c}(x) + \overline{\mathbf{L}}(x)\overline{\mathbf{y}} + \overline{\mathbf{b}}(x), \tag{33}$$

where we denote by $\mathbf{l}_r$ the vector in ℓ^p associated to the first row of the infinite matrix $\mathbf{L}$ (that is, $\mathbf{l}_r^T = e_1^T \mathbf{L}$), by $\mathbf{l}_c$ the first column of $\mathbf{L}$, and by $\mathbf{l}_{11}$ the element $e_1^T \mathbf{L} e_1$. The operator $\overline{\mathbf{L}}$ is defined as the operator $S_\ell \mathbf{L} S_r$, where S_ℓ is the left shift- and S_r is the right shift operator; moreover we set $\overline{\mathbf{a}} = S_\ell \mathbf{a}$. Note that the matrix representation of $\overline{\mathbf{L}}$ is equal to the matrix representation of $\mathbf{L}$ without the first row and column.

Since $r(\mathbf{L}(x_0)) = 1$, we obtain (cf. Remark 5) that $r(\mathbf{L}(x_0)) < 1$. Thus, the solution of (33) is given by

$$\overline{\mathbf{y}}(x, \mathbf{y}_1) = (\mathrm{I_p} - \overline{\mathbf{L}}(x))^{-1} \Big(\mathbf{y}_1 \overline{\mathbf{l}_c}(x) + \overline{\mathbf{b}}(x) \Big).$$

Inserting this solution into Equation (32) gives

$$\mathbf{y}_1(x) = \frac{\overline{\mathbf{l}_r}(x)(\mathrm{I_p} - \overline{\mathbf{L}}(x))^{-1}\overline{\mathbf{b}}(x) + \mathbf{b}_1(x)}{1 - \mathbf{l}_{11}(x) - \overline{\mathbf{l}_r}(x)(\mathrm{I_p} - \overline{\mathbf{L}}(x))^{-1}\overline{\mathbf{l}_c}(x)}.$$

We finally obtain

$$\overline{\mathbf{y}}(x) = (\mathrm{I_p} - \overline{\mathbf{L}}(x))^{-1} \left(\frac{\overline{\mathbf{l}_r}(x)(\mathrm{I_p} - \overline{\mathbf{L}}(x))^{-1}\overline{\mathbf{b}}(x) + \mathbf{b}_1(x)}{1 - \mathbf{l}_{11}(x) - \overline{\mathbf{l}_r}(x)(\mathrm{I_p} - \overline{\mathbf{L}}(x))^{-1}\overline{\mathbf{l}_c}(x)} \cdot \overline{\mathbf{l}_c}(x) + \overline{\mathbf{b}}(x) \right).$$

Set $\gamma(x) = \mathbf{l}_{11}(x) + \overline{\mathbf{l}_r}(x)(\mathrm{I_p} - \overline{\mathbf{L}}(x))^{-1}\overline{\mathbf{l}_c}(x)$ and define $\mathbf{k}(x)$ by

$$\mathbf{k}_1(x) = \overline{\mathbf{l}_r}(x)(\mathrm{I_p} - \overline{\mathbf{L}}(x))^{-1}\overline{\mathbf{b}}(x) + \mathbf{b}_1(x),$$

and

$$\overline{\mathbf{k}}(x) = (\mathrm{I_p} - \overline{\mathbf{L}}(x))^{-1} \Big(\big(\overline{\mathbf{l}_r}(x)(\mathrm{I_p} - \overline{\mathbf{L}}(x))^{-1}\overline{\mathbf{b}}(x) + \mathbf{b}_1(x)\big) \cdot \overline{\mathbf{l}_c}(x) + (1 - \gamma(x)) \cdot \overline{\mathbf{b}}(x) \Big).$$

Then we have

$$\mathbf{y}(x) = \frac{\mathbf{k}(x)}{1 - \gamma(x)}.$$

Note, that $\mathbf{k}(x)$ is analytic for x in a neighborhood of x_0. Note furthermore, that $\gamma(x)$ is also analytic for x in a neighborhood of x_0 and that it is a positive function, and thus, a monotonically increasing function (again, this can be shown with the help of the Neumann series). We also know that $\gamma(x_0) = 1$ since otherwise $(\mathrm{I_p} - \mathbf{L}(x_0))$ would be an invertible operator (contrary to $r(\mathbf{L}(x_0)) = 1$). Finally we set (for $x \neq x_0$)

$$\mathbf{f}(x) = \frac{x_0 - x}{1 - \gamma(x)} \cdot \frac{\mathbf{k}(x)}{x_0}.$$

We obtain that

$$\mathbf{y}(x) = \frac{1}{1 - \frac{x}{x_0}} \mathbf{f}(x).$$

In order to finish the proof of Theorem 4 for $\mathbf{v} = \mathbf{0}$ it suffices to show that $\mathbf{f}(x)$ can be analytically continued to x_0 and that $\mathbf{f}(x_0)_j \neq 0$ for all j. First note that

$$\lim_{x \to x_0} \mathbf{f}(x) = \lim_{x \to x_0} \frac{x_0 - x}{\gamma(x_0) - \gamma(x)} \cdot \frac{\mathbf{k}(x)}{x_0} = \frac{1}{\gamma'(x_0)} \cdot \frac{\mathbf{k}(x_0)}{x_0}. \tag{34}$$

This implies that for every $\boldsymbol{\ell} \in \ell^q$ the limit

$$\lim_{x \to x_0} \boldsymbol{\ell} \cdot \mathbf{f}(x)$$

exists. Riemann's theorem on removable singularities now implies that $\boldsymbol{\ell} \cdot \mathbf{f}(x)$ can be continued analytically to x_0 for all $\boldsymbol{\ell} \in \ell^q$ which finally implies that $\mathbf{f}(x)$ can be analytically continued to x_0. Since $\gamma'(x_0) \neq 0$ (γ is positive) and $\mathbf{k}(x_0)_j \neq 0$ for all j (this follows from irreducibility with the help of the Neumann series), we have also proved that $\mathbf{f}(x_0)_j \neq 0$ for all j.

In the second part of the proof we show that the result holds also true for $\mathbf{v}$ in a neighborhood of $\mathbf{0}$. First we see that Equation (33) can also be solved with the additional parameter $\mathbf{v}$. Indeed, the analyticity of $\mathbf{F}$ and Lemma 6 imply that $r(\mathbf{L}(x_0, \mathbf{v})) < 1$ for all $\mathbf{v}$ in a neighborhood of $\mathbf{0}$. Inserting this solution into Equation (32), we obtain in the same way as above that

$$\mathbf{y}(x, \mathbf{v}) = \frac{\mathbf{k}(x, \mathbf{v})}{1 - \gamma(x, \mathbf{v})}.$$

for some analytic function $\mathbf{k}(x, \mathbf{v})$ and for $\gamma(x, \mathbf{v}) = \mathbf{l}_{11}(x, \mathbf{v}) + \overline{\mathbf{l}_r}(x, \mathbf{v})(\mathrm{I_p} - \overline{\mathbf{L}}(x, \mathbf{v}))^{-1}\overline{\mathbf{l}_c}(x, \mathbf{v})$. Since $\gamma'(x_0, \mathbf{0}) > 0$, the implicit function theorem implies that there exists an analytic function $x_0(\mathbf{v})$ in a neighborhood of $\mathbf{0}$ such that

$$\gamma(x_0(\mathbf{v}), \mathbf{v}) = 1.$$

Thus we obtain with

$$\mathbf{f}(x) = \frac{x_0(\mathbf{v}) - x}{1 - \gamma(x, \mathbf{v})} \cdot \frac{\mathbf{k}(x, \mathbf{v})}{x_0(\mathbf{v})}, \qquad x \neq x_0(\mathbf{v}),$$

that

$$\mathbf{y}(x, \mathbf{v}) = \frac{1}{1 - \frac{x}{x_0(\mathbf{v})}} \mathbf{f}(x, \mathbf{v}).$$

As before, we see that $\mathbf{f}$ can be continued analytically to $x_0(\mathbf{v})$. This finally proves Theorem 4.

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