Recurrence function of Sturmian sequences.
A probabilistic study

Pablo Rotondo
Universidad de la República, Uruguay

Ongoing work with
Valérie Berthé, Eda Cesaratto, Brigitte Vallée, and Alfredo Viola

AofA’15, 8–12 June, 2015.
Study in **combinatorics of words**.
Main aim: description of the **finite factors** of an infinite word \( u \)

- **How many** factors of length \( n \)? \( \rightarrow \) **Complexity**
- **What are the gaps** between them? \( \rightarrow \) **Recurrence**

Very easy when the word is eventually periodic!
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the “simplest” binary infinite words that are *not* eventually periodic
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Classical study : for each fixed Sturmian word,
   what are the **extreme bounds** for the recurrence function?
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Here, in a convenient **model**, we perform a **probabilistic study**:
For a “random” sturmian word, and for a given “**position**”,
    – what is the **mean value** of the recurrence?
    – what is the **limit distribution** of the recurrence?
Plan of the talk

Complexity, Recurrence, and Sturmian words
   Complexity and Recurrence
   Sturmian words
   Recurrence of Sturmian words

Our probabilistic point of view. Statement of the results
   Classical results
   Our point of view
   Our main results.

Sketch of the proof
   General description
   The dynamical system and the transfer operator
   Expressions of the main objects in terms of the transfer operator
   Asymptotic estimates.

Extensions
Complexity

\( \mathcal{L}_u(n) \) denotes the set of factors of length \( n \) in \( u \).

Definition

Complexity function of an infinite word \( u \in \mathcal{A}^\mathbb{N} \)

\[
p_u : \mathbb{N} \rightarrow \mathbb{N}, \quad p_u(n) = |\mathcal{L}_u(n)|.
\]

Two simple facts:

\[
p_u(n) \leq |\mathcal{A}|^n, \quad p_u(n) \leq p_u(n + 1).
\]

Important property

\( u \in \mathcal{A}^\mathbb{N} \) is not eventually periodic

\[
\iff p_u(n + 1) > p_u(n) \quad \implies p_u(n) \geq n + 1.
\]
Recurrence

Definition (Uniform recurrence)

A word $u \in \mathcal{A}^\mathbb{N}$ is uniformly recurrent iff each finite factor appears infinitely often and with bounded gaps.
Recurrence

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Definition (Recurrence function)

Let $u \in A^\mathbb{N}$ be uniformly recurrent. The recurrence function of $u$ is:

$$R\langle u \rangle(n) = \inf \{ m \in \mathbb{N} : \text{any } w \in \mathcal{L}_u(m) \text{ contains all the factors } v \in \mathcal{L}_u(n) \}.$$
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The recurrence function gives a notion of the cost we have to pay to ‘discover’ the factors of $u$. 

A noteworthy inequality between the two functions, the complexity function and the recurrence function

$$R_{\langle u \rangle}(n) \geq p_u(n) + n - 1.$$  

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A noteworthy inequality between the two functions, the complexity function and the recurrence function

$$R_{\langle u \rangle}(n) \geq p_u(n) + n - 1.$$
- All $P_u(n)$ factors of length $n$ appear somewhere in our window of length $R_{\omega}(n)$.
- We have $P_u(n)$ "starting points" $+ (n-1)$ characters to finish the last factor $\Rightarrow R_{\omega}(n) \geq P_u(n) + n - 1$. 
Sturmian words

These are the “simplest” words that are not eventually periodic.
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A word \( u \in \{0, 1\}^\mathbb{N} \) is Sturmian iff \( p_u(n) = n + 1 \) for each \( n \geq 0 \).
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Explicit construction
Associate with a pair $(\alpha, \beta)$ the two sequences

$$u_n = \lfloor \alpha (n + 1) + \beta \rfloor - \lfloor \alpha n + \beta \rfloor$$

$$\overline{u}_n = \lceil \alpha (n + 1) + \beta \rceil - \lceil \alpha n + \beta \rceil$$

and the two words $S(\alpha, \beta)$ and $\overline{S}(\alpha, \beta)$ produced in this way.
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\end{align*}
\]

and the two words \( \underline{S}(\alpha, \beta) \) and \( \overline{S}(\alpha, \beta) \) produced in this way.

A word \( u \) is Sturmian iff there are \( \alpha, \beta \in [0, 1[ \), with \( \alpha \) irrational, such that \( u = \underline{S}(\alpha, \beta) \) or \( u = \overline{S}(\alpha, \beta) \).
Recurrence of Sturmian words

Property

Let \( u \) be a Sturmian word of the form \( S(\alpha, \beta) \) or \( \overline{S}(\alpha, \beta) \). Then

- \( u \) is uniformly recurrent
- \( R(u)(n) \) only depends on \( \alpha \), and it is written as \( R_\alpha(n) \).
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- The sequence \( (R_\alpha(n)) \) only depends on the continuants of \( \alpha \).
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Reminder:
The continuant \( q_k(\alpha) \) is the denominator of the \( k \)-th convergent of \( \alpha \).
It is obtained via the truncation at depth \( k \) of the CFE of \( \alpha \).
The sequence \( (q_k(\alpha))_k \) is strictly increasing.
Recurrence of Sturmian words

Property

Let $u$ be a Sturmian word of the form $\underline{S}(\alpha, \beta)$ or $\overline{S}(\alpha, \beta)$. Then

- $u$ is uniformly recurrent
- $R_{\langle u \rangle}(n)$ only depends on $\alpha$, and it is written as $R_{\alpha}(n)$.
- The sequence $(R_{\alpha}(n))$ only depends on the continuants of $\alpha$.

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Theorem (Morse, Hedlund, 1940)

The recurrence function is piecewise affine and satisfies

$$R_{\alpha}(n) = n - 1 + q_{k-1}(\alpha) + q_k(\alpha), \quad \text{for } n \in [q_{k-1}(\alpha), q_k(\alpha)].$$
Recurrence function for two Sturmian words

Recurrence function for \( \alpha = \varphi^2 \),
with \( \varphi = (\sqrt{5} - 1)/2 \).

Recurrence function for \( \alpha = 1/e \).
Recurrence function of Sturmian words: classical results.

Proposition

For any irrational $\alpha \in [0, 1]$, one has $\lim \inf \frac{R_{\alpha}(n)}{n} \leq 3$. 

Proof: Take the sequence $n_k = q_k - 1$.

Theorem

For almost any irrational $\alpha$, one has $\lim \sup \frac{R_{\alpha}(n)}{n \log n} = \infty$, $\lim \sup \frac{R_{\alpha}(n)}{n (\log n)^c} = 0$ for any $c > 1$.

Proof: Apply the Morse–Hedlund formula and Khinchin's Theorem.
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Our point of view

Usual studies of $R_\alpha(n)$

- consider all possible sequences of indices $n$.
- give information on extreme cases.
- give results for almost all $\alpha$. 

Here:

- we study particular sequences of indices $n$ depending on $\alpha$, defined with their position on the intervals $[q_k(\alpha) - 1, q_k(\alpha)]$.
- we then draw $\alpha$ at random.
- we perform a probabilistic study.
- we then study the role of the position in the probabilistic behaviour of the recurrence function.
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Subsequences with a fixed position

We work with particular subsequences of indices $n$

Given $\mu \in ]0, 1]$ the sequence

$$n_k^{(\mu)}(\alpha) = q_{k-1}(\alpha) + \lfloor \mu (q_k(\alpha) - q_{k-1}(\alpha)) \rfloor$$

is the subsequence of position $\mu$ of $\alpha$.

**Figure:** Sequence of indices $n$ for $\mu = 1/3$. 
We study

- the behaviour of

\[ R_\alpha(n) \frac{n}{n}, \quad n = n^{\langle \mu \rangle}_k = q_{k-1} + \lfloor \mu (q_k - q_{k-1}) \rfloor \]

when \( n \) has a fixed position \( \mu \) within \([q_{k-1}, q_k]\).

Remark that \( (n^{\langle \mu \rangle}_k)_k \) is a sequence depending on \( \alpha \in \mathcal{I} \).

- what happens when \( \alpha \) is drawn uniformly from \( \mathcal{I} = [0, 1] \).
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Remark that \((n_\mu^k)_k\) is a sequence depending on \( \alpha \in I \).

- what happens when \( \alpha \) is drawn uniformly from \( I = [0, 1] \).

We consider the sequence of random variables

\[ S_\mu^k = \frac{R_\alpha(n) + 1}{n} = 1 + \frac{q_k - 1 + q_k}{n}, \quad n = n_\mu^k. \]

For any fixed \( \mu \in [0, 1] \), we perform an asymptotic study

- for expected values: \( \lim_{k \to \infty} \mathbb{E}[S_\mu^k] \)

- for distributions: \( \lim_{k \to \infty} \Pr[S_\mu^k \in J] \)
First result: Expectations

For each $\mu \in ]0, 1]$, the sequence of random variables $S_k^{(\mu)}$ satisfies

$$\mathbb{E}[S_k^{(\mu)}] = 1 + \frac{1}{\log 2} \frac{|\log \mu|}{1 - \mu} + O \left( \frac{\varphi^{2k}}{\mu} \right) + O \left( \varphi^k \frac{|\log \mu|}{1 - \mu} \right),$$

(for $k \to \infty$). Here, $\varphi = (\sqrt{5} - 1)/2 \approx 0.6180339 \ldots$ and the constants of the $O$-terms are uniform in $\mu$ and $k$.

Remark: The result only holds for $\mu > 0$.

Limit of the expected value as a function of $\mu$. 
Second result : Distributions

For each $\mu \in [0, 1]$ with $\mu \neq 1/2$, the sequence of random variables $S^{(\mu)}_k$ has a limit density

$$s_{\mu}(x) = \frac{1}{\log 2(x - 1) \left| 2 - \mu - x(1 - \mu) \right|} 1_{I_\mu}(x).$$

Here, $I_\mu$ is the interval with endpoints 3 and $1 + 1/\mu$. 
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For all $b \geq \min\{3, 1 + \frac{1}{\mu}\}$

$$\Pr\left[ S_k^{(\mu)} \leq b \right] = \int_0^b s_\mu(x)dx + \frac{1}{b} O\left( \varphi^k \right).$$

where the constant of the $O$-term is uniform in $b$ and $k$.

When $|\mu - 1/2| \geq \epsilon$ for a fixed $\epsilon > 0$, it is also uniform in $\mu$. 
Limit density $s_\mu$

\[ s_\mu(x) \]

- $\mu = \frac{2}{3}$
- $\mu = \frac{7}{10}$
- $\mu = \frac{3}{4}$
- $\mu = \frac{4}{5}$

$x$-axis: $2.3$ to $3.0$

$y$-axis: $1.2$ to $2.4$
Limit density for $\mu = 1/4$

<table>
<thead>
<tr>
<th>Interval</th>
<th>Empirical Pr</th>
<th>Asymptotic Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>[3.0, 3.0]</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>[3.0, 3.5]</td>
<td>0.485237</td>
<td>0.4854…</td>
</tr>
<tr>
<td>[3.0, 4.0]</td>
<td>0.737139</td>
<td>0.7369…</td>
</tr>
<tr>
<td>[3.0, 4.5]</td>
<td>0.893511</td>
<td>0.8931…</td>
</tr>
<tr>
<td>[3.0, 5.0]</td>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

In **blue**, the scaled histogram for $k = 25$, bin-width $\delta = 1/10$, obtained with $10^6$ samples.

In **red**, the graph of the limit distribution $s_{1/4}(x) = \frac{1}{\log 2} \frac{4}{(x - 1)(3x - 7)}$. 
Four steps in the proof

i) We drop the integer part in $S_k^{\langle \mu \rangle}$ getting

$$\tilde{S}_k^{\langle \mu \rangle} = 1 + \frac{q_k + q_{k-1}}{q_{k-1} + \mu (q_k - q_{k-1})},$$

which depends only on $\frac{q_{k-1}}{q_k}$. Indeed

$$\tilde{S}_k^{\langle \mu \rangle} = f_\mu \left( \frac{q_{k-1}}{q_k} \right), \quad \text{with} \quad f_\mu(x) = 1 + \frac{1 + x}{x + \mu (1 - x)}.$$
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ii) The expected value and the distribution of $\tilde{S}_k^{(\mu)}$ are expressed with the $k$–th iterate of the Perron-Frobenius operator $H$. 
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iii) The asymptotics for $k \to \infty$ is obtained by using the spectral properties of $H$, when acting on the space of functions of bounded variation.
Four steps in the proof

i) We drop the integer part in $S_k^{\langle \mu \rangle}$ getting

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iii) The asymptotics for $k \to \infty$ is obtained by using the spectral properties of $H$, when acting on the space of functions of bounded variation.

iv) Finally we return from $\tilde{S}_k^{\langle \mu \rangle}$ to $S_k^{\langle \mu \rangle}$.
The Euclidean dynamical system

The Gauss map $T : [0, 1] \rightarrow [0, 1]$

$$T(x) = \left\{ \frac{1}{x} \right\} = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor .$$

The inverse branches of $T$ are:

$$\mathcal{H} = \left\{ h_m : x \mapsto \frac{1}{m + x} : m \geq 1 \right\} .$$
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The inverse branches of $T^k$ are:

$$\mathcal{H}^k = \left\{ h_{m_1,m_2,...,m_k} = h_{m_1} \circ h_{m_2} \circ \ldots \circ h_{m_k} : m_1, \ldots, m_k \geq 1 \right\}.$$
The LFT \( h_{m_1,\ldots,m_k} \in \mathcal{H}^k \) is expressed with continuants

\[
h_{m_1,\ldots,m_k}(x) = \frac{1}{m_1 + \frac{1}{\ldots + \frac{1}{m_k + x}}} = \frac{p_{k-1}x + p_k}{q_{k-1}x + q_k},
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and satisfies the mirror property

$$h_{m_k,\ldots,m_1}(x) = \frac{1}{m_k + \frac{1}{\ldots + \frac{1}{m_1 + x}}} = \frac{p_{k-1} x + q_{k-1}}{p_k x + q_k}.$$
The Perron-Frobenius operator \( H \)

If \( g \in C^0(I) \) is the density of \( \alpha \), what is the density of \( T(\alpha) \)?
The Perron-Frobenius operator $H$

If $g \in C^0(\mathcal{I})$ is the density of $\alpha$, what is the density of $T(\alpha)$?
The Perron-Frobenius operator $\mathbf{H}$

If $g \in C^0(\mathcal{I})$ is the density of $\alpha$, what is the density of $T(\alpha)$?

**Answer:** The density is

$$H[g](x) = \sum_{h \in \mathcal{H}} \left| h'(x) \right| g(h(x))$$

$$= \sum_{m=1}^{\infty} \frac{1}{(m + x)^2} g\left(\frac{1}{m + x}\right).$$
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= \sum_{m=1}^{\infty} \frac{1}{(m+x)^2} \ g \left( \frac{1}{m+x} \right).
$$

For $k \geq 1$, the density of $T^k(\alpha)$ is given by the $k$-th iterate of $\mathbf{H}$

$$
\mathbf{H}^k[g](x) = \sum_{h \in \mathcal{H}^k} |h'(x)| \ g(h(x))
$$

$\mathbf{H}$ is called the Perron-Frobenius operator (or the density transform).
Evaluating at $x = 0$

\[ H^k[g](0) = \sum_{m_1,\ldots,m_k \geq 1} \frac{1}{q_k^2} g \left( \frac{p_k}{q_k} \right). \]
Evaluating at $x = 0$

\[ \mathcal{H}^k[g](0) = \sum_{m_1, \ldots, m_k \geq 1} \frac{1}{q_k^2} g \left( \frac{p_k}{q_k} \right). \]

As the sum is over all $k$-tuples, we apply the mirror property, and

\[ \mathcal{H}^k[g](0) = \sum_{m_1, \ldots, m_k \geq 1} \frac{1}{q_k^2} g \left( \frac{q_k - 1}{q_k} \right). \]
Expressions in terms of the operator $\textbf{H}$.

Three main facts:

- The intervals $h(I)$ for $h \in \mathcal{H}^k$ form a partition of $(0, 1)$
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- The intervals $h(I)$ for $h \in H^k$ form a partition of $(0, 1)$
- $\tilde{S}^\langle\mu\rangle_k$ is a step function, constant on each $h_{m_1, \ldots, m_k}(I)$,

$$\tilde{S}^\langle\mu\rangle_k = f_\mu \left( \frac{q_k - 1}{q_k} \right)$$
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$$\tilde{S}_k^{(\mu)} = f_\mu \left( \frac{q_{k-1}}{q_k} \right)$$

- The length of the interval $h_{m_1,\ldots,m_k}(I)$ is

$$|h(0) - h(1)| = \frac{1}{q_k (q_k + q_{k-1})} = \frac{1}{q_k^2} \cdot \frac{1}{1 + \frac{q_{k-1}}{q_k}}$$
Expressions in terms of the operator $\mathbf{H}$.

Three main facts:

- The intervals $h(I)$ for $h \in \mathcal{H}^k$ form a partition of $(0, 1)$
- $\tilde{S}^{(\mu)}_k$ is a step function, constant on each $h_{m_1, \ldots, m_k}(I)$,
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Then: \[ \mathbb{E} \left[ \tilde{S}^{(\mu)}_k \right] = \sum_{m_1, \ldots, m_k \geq 1} \frac{1}{q_k^2} \frac{f_\mu(q_k-1/q_k)}{1 + (q_k-1/q_k)} = \mathbf{H}^k \left[ \frac{f_\mu(x)}{1 + x} \right] (0), \]
Expressions in terms of the operator $H$.

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And

$$
\Pr \left[ \tilde{S}_k^{(\mu)} \in J \right] = \mathbb{E} \left[ 1_J \circ \tilde{S}_k^{(\mu)} \right] = H^k \left[ \frac{1_J \circ f_\mu(x)}{1 + x} \right](0).
$$
Analytic properties of $H$

The operator $H$ acts on the Banach space $BV(I)$ of functions of bounded variation,

$$\| f \|_{BV} = V_0^1(f) + \| f \|_1.$$
Analytic properties of $\mathbf{H}$

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The following dominant spectral properties are well-known

- Dominant eigenvalue (simple) : $\lambda = 1$
- Dominant eigenfunction: $\psi(x) = \frac{1}{\log 2} \frac{1}{1 + x}$.
- Dominant eigenmeasure for the adjoint: Lebesgue measure
- Subdominant spectral radius: $\varphi^2$ with $\varphi = (\sqrt{5} - 1)/2$. 
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Then, for any $g \in BV(I)$, the asymptotic estimate holds:

$$H^k[g](x) = \frac{1}{\log 2} \frac{1}{1 + x} \int_0^1 g(x)dx + O\left(\varphi^{2k} \|g\|_{BV}\right).$$
Going back to the expectations and distributions.

With the expressions for the expectations and distributions,

\[ \mathbb{E} \left[ \tilde{S}_k^{\langle \mu \rangle} \right] = H^k \left[ \frac{f_\mu(x)}{1 + x} \right](0), \quad \Pr \left[ \tilde{S}_k^{\langle \mu \rangle} \in J \right] = H^k \left[ \frac{1 J \circ f_\mu(x)}{1 + x} \right](0) \]

We apply the previous result to the “red” functions:
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We apply the previous result to the “red” functions:

- The first function belongs to $BV(\mathcal{I})$ only for $\mu \neq 0$, with a $BV$-norm $O(1/\mu)$.  

- The second function always belongs to $BV(\mathcal{I})$, even for $\mu = 0$ with a bounded $BV$-norm with respect to $\mu$.  

The limit distribution

\[
\lim_{k \to \infty} \text{Pr} \left[ \tilde{S}_k^{\langle \mu \rangle} \in J \right] = \frac{1}{\log 2} \int_1^0 1_J \circ f_\mu(x) \, dx,
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is expressed with the inverse of $f_\mu$ in the interval $\mathcal{I}_\mu$. Therefore, the asymptotics are obtained for $\tilde{S}_k^{\langle \mu \rangle}$. We then return to $S^{\langle \mu \rangle}$.  

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Possible extensions: variable $\mu$
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As our estimates are uniform wrt position $\mu$, and index $k$, thus making it possible to deal with a position depending on $k$.

We then let $\mu_k \to 0$ as $k \to \infty$. 

Theorem

For each $\tau \in [\varphi^2, 1]$, considering $\mu_k = \tau k$ we have

$$E_{\alpha}[R_{\alpha}(n) n - \frac{\log \tau}{\pi^2 \log n}] = O(1),$$

as $k \to \infty$, where the constant depends on $\tau$.

Theorem

If $b \in (0, 1)$ and for each $k$ we pick $\mu_k \in [0, 1]$ uniformly, then

$$\lim_{k \to \infty} E_{\alpha, \mu_k}[S_{\langle \mu_k \rangle} k \bigg| \mu_k \geq b k] = 1 + \frac{\pi^2}{6}.$$
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