THE NODE PROFILE OF SYMMETRIC DIGITAL SEARCH TREES

(joint with M. Drmota, H.-K. Hwang and R. Neininger)

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Node Profile of (Rooted) Trees

$B_{n,k} =$ number of external nodes at level $k$;

$I_{n,k} =$ number of internal nodes at level $k$. 
Node Profile of (Rooted) Trees

\[ B_{n,k} = \text{number of external nodes at level } k; \]
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Example:
Node Profile of (Rooted) Trees

\[ B_{n,k} = \text{number of external nodes at level } k; \]
\[ I_{n,k} = \text{number of internal nodes at level } k. \]

Example:

\[ B_{5,0} = 0, \]
\[ B_{5,1} = 0, \]
\[ B_{5,2} = 2, \]
\[ B_{5,3} = 4. \]
Node Profile of (Rooted) Trees

\( B_{n,k} = \) number of external nodes at level \( k \);

\( I_{n,k} = \) number of internal nodes at level \( k \).

Example:

\[
\begin{align*}
B_{5,0} &= 0, \quad I_{5,0} = 1; \\
B_{5,1} &= 0, \quad I_{5,1} = 2; \\
B_{5,2} &= 2, \quad I_{5,2} = 2; \\
B_{5,3} &= 4, \quad I_{5,3} = 0.
\end{align*}
\]
Relations to Other Shape Parameters

Many shape parameters can by analyzed through the profile.
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- **Depth**: $P(D_n = k) = B_{n,k}/(n + 1)$;
- **Width**: $\max\{B_{n,k} : k \geq 0\}$;
- **Total Path Length**: $\sum_k kB_{n,k}$;
- **Height**: $\max\{k : B_{n,k} > 0\}$;
- **Shortest Path**: $\min\{k : B_{n,k} > 0\}$;
- **Fill-up Level**: $\max\{k : I_{n,k} = 2^k\}$;
- **Etc.**
Profile of Random Trees

- $\sqrt{n}$-Trees:
  
  Aldous (1991); Drmota and Gittenberger (1997); Kersting (1998); Pitman (1999); etc.
Profile of Random Trees

- $\sqrt{n}$-Trees:
  Aldous (1991); Drmota and Gittenberger (1997); Kersting (1998); Pitman (1999); etc.

- $\log n$-Trees:
  - $m$-ary Search Trees: Drmota, Janson, Neininger (2008).
Tries

René de la Briandais (1959)

Name from data retrieval (suggested by Fredkin).
Tries

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Example:

```
011011
010101
101110
010000
101010
001100
```
Tries

René de la Briandais (1959)

Name from data retrieval (suggested by Fredkin).

Example:

\[
\begin{array}{c}
011011 \\
010101 \\
101110 \\
010000 \\
101010 \\
001100
\end{array}
\]
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Digital Search Trees (DSTs)


Closely related to Lempel-Ziv compression scheme.
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101010
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```
Digital Search Trees (DSTs)


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Example:

```
0 1
```

```
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101110
010000
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001100
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Digital Search Trees (DSTs)


Closely related to Lempel-Ziv compression scheme.

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```
0 1 0 1
0 1 0 1
010000
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```
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001100
```
Digital Search Trees (DSTs)


Closely related to Lempel-Ziv compression scheme.

Example:
Random Model

Bits generated by iid Bernoulli random variables with mean $p$

→ Bernoulli model
Random Model

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$\rightarrow$ Bernoulli model

Two types:

- $p = 1/2$: symmetric digital trees;
- $p \neq 1/2$: asymmetric digital trees.
Random Model

Bits generated by iid Bernoulli random variables with mean $p$

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**Question:** What can be said about the profile?
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Bits generated by iid Bernoulli random variables with mean $p$

$\rightarrow$ Bernoulli model

Two types:

- $p = 1/2$: symmetric digital trees;
- $p \neq 1/2$: asymmetric digital trees.

Question: What can be said about the profile?

In this talk, we are interested in mean, variance and limit laws of the profile for symmetric DSTs.
Profile of Digital Trees

- Tries:
  

- PATRICIA tries:
  

- Asymmetric DSTs:
  
  Mean: Drmota and Szpankowski (2011); Variance: Kazemi and Vahidi-Asl (2011); so far no limit laws.

- Symmetric DSTs:
  
Profile of Digital Trees

- **Tries:**
  

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  Mean: Magner, Knessler, Szpankowski (2014); Variance & limit laws: Szpankowski & Magner (→ Thursday).

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  Variance & limit laws: Drmota, F., Hwang, Neininger (→ this talk).
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- **Symmetric DSTs:**
  Variance & limit laws: Drmota, F., Hwang, Neininger (→ this talk).
Profile of Tries

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Abstract. The profile of a trie, the most popular data structures on words, is a parameter that represents the number of nodes (either internal or external) with the same distance to the root. Several, if not
Hwang, Nicodéme, Park, Szpankowski (2009):
Symmetric Tries: Mean

We have,

\[ \mu_{n,k} := \mathbb{E}(B_{n,k}) \sim \begin{cases} 
  n(1 - 2^{-k})^{n-1}, & \text{if } 2^{-k}n \to \infty; \\
  \tilde{M}_{k,1}(n), & \text{if } 4^{-k}n \to 0,
\end{cases} \]

where

\[ \tilde{M}_{k,1}(z) = z(e^{-z/2^k} - e^{-z/2^{k-1}}). \]
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where

\[ \tilde{M}_{k,1}(z) = z(e^{-z/2^k} - e^{-z/2^{k-1}}). \]

In particular,

\[ \tilde{M}_{k,1}(n) \sim \begin{cases} ne^{-n/2^k}, & \text{if } 2^{-k}n \to \infty; \\ \Theta(n), & \text{if } 2^{-k}n = \Theta(1); \\ 2^{-k}n^2, & \text{if } 2^{-k}n \to 0. \end{cases} \]
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  2^{-k}n^2, & \text{if } 2^{-k}n \to 0.
\end{cases} \]

Thus, the profile has maximum of order \( n \) (asymmetric tries: \( n/\sqrt{\log n} \))
Symmetric Tries: Variance

We have,

\[ \sigma_{n,k}^2 := \text{Var}(B_{n,k}) \sim \begin{cases} n(1 - 2^{-k})^{n-1}, & \text{if } 2^{-k}n \to \infty; \\ \tilde{V}_k(n), & \text{if } 4^{-k}n \to 0, \end{cases} \]

where

\[ \tilde{V}_k(z) = z(e^{-z/2^k} - e^{-z/2^{k-1}}) + 2^{-k}z^2e^{-z/2^{k-1}} - 2^{1-k}z^2(e^{-z/2^k} - e^{-z/2^{k-1}})^2. \]
Symmetric Tries: Variance

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\quad - 2^{1-k} z^2 (e^{-z/2^k} - e^{-z/2^{k-1}})^2.
\]

In particular,

\[
\tilde{V}_k(n) \sim \begin{cases} 
  n e^{-n/2^k} \sim \tilde{M}_k(n), & \text{if } 2^{-k}n \to \infty; \\
  \Theta(n), & \text{if } 2^{-k}n = \Theta(1); \\
  2^{1-k} n^2 \sim 2\tilde{M}_k(n), & \text{if } 2^{-k}n \to 0.
\end{cases}
\]
Poissonization and Depoissonization

**Poisson Model:** Build digital tree from Poisson-distributed number of records.

\[ \tilde{M}_{k,\ell}(z) = E(B_{\ell \text{Pois}(z)}, k) = e^{-z} \sum_{n \geq 0} E(B_{\ell n}, k) \frac{z^n}{n!} . \]

**Poisson Heuristic:** \( \tilde{M}_{k,\ell}(z) \) sufficiently smooth \( \Rightarrow \) 
\[ E(B_{\ell n}, k) \approx \tilde{M}_{k,\ell}(n) . \]

Poisson heuristic made precise by the Theory of Analytic Depoissonization (Jacquet & Szpankowski; 1998).
Poissonization and Depoissonization

**Poisson Model:** Build digital tree from Poisson-distributed number of records.

Poisson moments:

\[
\tilde{M}_{k,\ell}(z) = \mathbb{E}(B_{\text{Pois}(z),k}) = e^{-z} \sum_{n \geq 0} \mathbb{E}(B_{n,k}^\ell) \frac{z^n}{n!}.
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Poisson Variance

Correct choice is crucial!
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- Asymmetric Digital Trees:

\[
\tilde{V}_k(z) = \tilde{M}_{k,2}(z) - \tilde{M}_{k,1}(z)^2.
\]
Poisson Variance

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- **Asymmetric Digital Trees:**
  \[ \tilde{V}_k(z) = \tilde{M}_{k,2}(z) - \tilde{M}_{k,1}(z)^2. \]

- **Symmetric Digital Trees:**
  \[ \tilde{V}_k(z) = \tilde{M}_{k,2}(z) - \tilde{M}_{k,1}(z)^2 - z\tilde{M}_{k,1}'(z)^2. \]
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  \]

With this choice:
\[
\text{Var}(B_{n,k}) \sim \tilde{V}_k(n)
\]
when \(4^{-k}n \to 0\).
Symmetric DSTs: Mean

Let

\[ Q(z) = \prod_{\ell=1}^{\infty} \left(1 - z2^{-\ell}\right), \quad Q_n = \prod_{\ell=1}^{n} \left(1 - 2^{-\ell}\right) = \frac{Q(2^{-n})}{Q(1)}. \]
Symmetric DSTs: Mean

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\[ Q(z) = \prod_{\ell=1}^{\infty} \left( 1 - z 2^{-\ell} \right), \quad Q_n = \prod_{\ell=1}^{n} \left( 1 - 2^{-\ell} \right) = \frac{Q(2^{-n})}{Q(1)}. \]

**Theorem**

We have,

\[ \mu_{n,k} \begin{cases} \sim \frac{2^k}{Q_k} \left( 1 - 2^{-k} \right)^n, & \text{if } 2^{-k} n \to \infty; \\ = 2^k F(n/2^k) + O(1), & \text{if } 4^{-k} n \to 0, \end{cases} \]

where \( F(x) \) is the positive function

\[ F(x) = \sum_{j \geq 0} \frac{(-1)^j 2^{-\binom{j}{2}}}{Q_j Q(1)} e^{-2jx}. \]
As $x \to \infty$, 

$$F(x) = \frac{e^{-x}}{Q(1)} + \mathcal{O}(e^{-2x})$$
As \( x \to \infty \),

\[
F(x) = \frac{e^{-x}}{Q(1)} + O(e^{-2x})
\]

and as \( x \to 0 \),

\[
F(x) \sim \frac{X^{1/\log 2}}{\sqrt{2\pi x}} \exp \left( -\frac{\log X \log X}{\log 2} - \sum_{j \in \mathbb{Z}} c_j (X \log X)^{-\chi_j} \right),
\]

where \( X = 1/(x \log 2) \), \( \chi_j = \frac{2j\pi i}{\log 2} \),

\[
c_0 = \frac{\log 2}{12} + \frac{\pi^2}{6 \log 2}
\]

and

\[
c_j = \frac{1}{2j \sinh(2j\pi/\log 2)}, \quad (j \neq 0).
\]
$F(x)$ (ii)
Some Details of the Proof (i)

We have,

\[ \tilde{M}_{k,1}(z) + \tilde{M}'_{k,1}(z) = 2\tilde{M}_{k-1,1}(z/2). \]
Some Details of the Proof (i)

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\[ \tilde{M}_{k,1}(z) + \tilde{M}'_{k,1}(z) = 2\tilde{M}_{k-1,1}(z/2). \]

By Laplace transform and its inverse,

\[ \tilde{M}_{k,1}(z) = 2^k \sum_{0 \leq j \leq k} \frac{(-1)^j 2^{-(j/2)}}{Q_j Q_{k-j}} e^{-z/2^{k-j}}. \]
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From this,
\[ \mu_{n,k} = 2^k \sum_{0 \leq j \leq k} \frac{(-1)^j 2^{-j^2}}{Q_j Q_{k-j}} \left(1 - 2^{j-k}\right)^n. \]

This formula was first derived by Louchard (1987).
Some Details of the Proof (i)

We have,
\[ \tilde{M}_{k,1}(z) + \tilde{M}_{k,1}'(z) = 2\tilde{M}_{k-1,1}(z/2). \]

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This formula was first derived by Louchard (1987).

This is useful if \( n 2^{-k} \to \infty. \)
Some Details of the Proof (ii)

If $4^{-k} n \to 0$, Poisson heuristic holds.
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**Lemma**

We have,

$$\tilde{M}_{k,1}(z) = 2^k \sum_{r \geq 0} \frac{2^{-(r+1)/2} - kr}{Q_r} F(r) \left( \frac{z}{2^k} \right).$$
If $4^{-k}n \to 0$, Poisson heuristic holds.

**Lemma**

We have,

$$\tilde{M}_{k,1}(z) = 2^k \sum_{r \geq 0} \frac{2^{-\left(\frac{r+1}{2}\right) - kr}}{Q_r} F(r) \left( \frac{z}{2^k} \right).$$

This gives,

$$\tilde{M}_{k,1}(z) = 2^k F \left( \frac{z}{2^k} \right) + O(1).$$

Result follows from depoissonization.
Symmetric DSTs: Variance

Theorem (Drmota, F., Hwang, Neininger)

We have,

\[
\sigma_{n,k}^2 \begin{cases} 
\sim \frac{2^k}{Q_k} \left(1 - 2^{-k}\right)^n, & \text{if } 2^{-k}n \to \infty; \\
= 2^k H\left(n/2^k\right) + O(1), & \text{if } 4^{-k}n \to 0,
\end{cases}
\]

where \(H(x)\) is a function with

\[
H(x) = \frac{e^{-x}}{Q(1)} + O(x e^{-2x}), \quad (x \to \infty)
\]

and

\[
H(x) \sim 2F(x), \quad (x \to 0).
\]
We have,

\[ H(x) = \sum_{j,r=0}^{\infty} \sum_{0 \leq h, \ell \leq j} \frac{2^{-j}(-1)^{r+h+\ell}2^{-r}(\binom{r}{2})-\binom{h}{2}-\binom{\ell}{2}+2h+2\ell}{Q_r Q(1) Q_h Q_j - h Q_\ell Q_j - \ell} \varphi(2^{r+j}, 2^h + 2^\ell; x), \]

where

\[ \varphi(u, v; x) = \begin{cases} 
    \frac{e^{-ux} - ((v - u)x + 1)e^{-vx}}{(v - u)^2}, & \text{if } u \neq v; \\
    x^2e^{-ux}/2, & \text{if } u = v.
\]
We have,

\[ H(x) = \sum_{j,r=0}^{\infty} \sum_{0 \leq h, \ell \leq j} 2^{-j} (-1)^{r+h+\ell} 2^{-\binom{r}{2}} - \left( \binom{h}{2} + \binom{\ell}{2} \right) + 2h + 2\ell \]

\[ \frac{Q_r Q(1) Q_h Q_j - h Q_\ell Q_j - \ell}{(Q_\ell Q_1 Q_0)^2} \]

\[ \varphi(2^{r+j}, 2^h + 2^{\ell}; x), \]

where

\[ \varphi(u, v; x) = \begin{cases} 
    e^{-ux} - ((v - u)x + 1)e^{-vx} \\
    \frac{(v - u)^2}{x^2 e^{-ux}} \
\end{cases}, \quad \text{if } u \neq v; \]

\[ x^2 e^{-ux}/2, \quad \text{if } u = v. \]

**Proposition (Drmota, F., Hwang, Neininger)**

\( H(x) \) is a positive function on \((0, \infty)\).
$H(x)$ (ii)
Some Details of the Proof (i)

We have,

$$\tilde{V}_k(z) + \tilde{V}'_k(z) = 2\tilde{V}_{k-1}(z/2) + z\tilde{M}''_{k,2}(z)^2.$$
Some Details of the Proof (i)

We have,

\[ \tilde{V}_k(z) + \tilde{V}_k'(z) = 2\tilde{V}_{k-1}(z/2) + z\tilde{M}_{k,2}''(z)^2. \]

By Laplace transform and its inverse,

\[ \tilde{V}_k(z) = \sum_{(j, r, h, \ell) \in \mathcal{V}} \frac{2^{k-j}(-1)^{r+h+\ell}2^{-(r^2) - (h^2) - (\ell^2) + 2h + 2\ell}}{Q_r Q_{k-j-r} Q_h Q_{j-h} Q_\ell Q_{j-\ell}} \varphi \left( 2^{r+j}, 2^h + 2^\ell, \frac{z}{2^k} \right) \]

with

\[ \mathcal{V} = \{(j, r, h, \ell) : 0 \leq j \leq k, 0 \leq r \leq k - j, 0 \leq h, \ell \leq j\} \]

and

\[ \varphi(u, v; x) = \begin{cases} 
    e^{-ux} - ((v - u)x + 1)e^{-vx} \\
    (v - u)^2 \\
    x^2e^{-ux}/2, \end{cases} \]

if \( u \neq v; \)

\[ e^{-ux}/2, \quad \text{if } u = v. \]
Some Details of the Proof (ii)

Lemma

We have,

\[ \tilde{V}_k(z) = 2^k \sum_{m \geq 0} \frac{2^{-(m+1)} - km}{Q_m} H^{(m)} \left( \frac{z}{2^k} \right). \]
Lemma

We have,

\[ \tilde{V}_k(z) = 2^k \sum_{m \geq 0} \frac{2^{-(m+1)} - km}{Q_m} H^{(m)} \left( \frac{z}{2^k} \right). \]

The Laplace transform of \( H(z) \):

\[ \mathcal{L}[H(z); s] = \sum_{j \geq 0} 4^{-j} \tilde{g}_j^*(2^{-j} s) \frac{Q(-2^{1-j} s)}{Q(-2^1 s)} \]

where

\[ \tilde{g}_j^*(s) = \sum_{0 \leq k, \ell \leq j} \frac{(-1)^{h+\ell} 2^{-\left(\frac{h}{2}\right)-\left(\frac{\ell}{2}\right)+2h+2\ell}}{Q_k Q_{j-k} Q_\ell Q_{j-\ell}} \frac{1}{(2^j s + 2^h + 2^\ell)^2}. \]
Lemma

We have, as $s \to \infty$,

\[
\frac{\tilde{g}_0^*(s)}{Q(-2s)} \sim \frac{1}{s^2 Q(-2s)},
\]

\[
4^{-1} \frac{\tilde{g}_1^*(2^{-1}s)}{Q(-s)} \sim \frac{9}{sQ(-2s)}
\]

and, for $j \geq 2$,

\[
4^{-j} \frac{\tilde{g}_j^*(2^{-j}s)}{Q(-2^{1-j}s)} \sim \frac{(2j - 3)!}{((j - 2)!)^2} \frac{2^{(j_2)}}{s^{j-2}Q(-2s)}.
\]

Thus,

\[
\mathcal{L}[H(z); s] \sim \frac{2}{Q(-2s)}
\]

and hence, $H(x) \sim 2F(x)$ as $x \to 0$. 

Corollary (Drmota, F., Hwang, Neininger)

We have,

$$\mu_{n,k} \to \infty \quad \text{iff} \quad \sigma_{n,k}^2 \to \infty.$$  

Theorem (Drmota, F., Hwang, Neininger)

Assume that $\mu_{n,k} \to \infty$. Then,

$$\frac{B_{n,k} - \mu_{n,k}}{\sigma_{n,k}} \overset{d}{\to} N(0, 1),$$

where $N(0, 1)$ denotes a standard normal distribution.
Application to the Height

\[ H_n = \text{height of a symmetric DST of size } n. \]
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$H_n =$height of a symmetric DST of size $n$.

**Theorem** (Drmota, F., Hwang, Neininger)

Set

$$k_n = \min\{ k \geq \log_2 n : 2^k F(n/2^k) \leq 1 \}.$$ 

Then,

$$k_n = \log_2 n + \sqrt{2 \log_2 n} - \log_2 \left(\sqrt{\log_2 n}\right) + O(1).$$

Moreover,

$$P(H_n = k_n - 2 \text{ or } H_n = k_n - 1) \rightarrow 1.$$
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This solves an open problem of Aldous & Shields.
Summary of Results for Symmetric DSTs

- Mean profile tends to infinity when $k$ is roughly in the range
  \[ \log_2 n - \log_2 \log n \leq k \leq \log_2 n + \sqrt{2 \log_2 n}; \]
  otherwise it is bounded.

Maximum of mean profile is of linear order. Variance has same order as the mean. Thus, it tends to infinity iff mean tends to infinity. If mean tends to infinity, a central limit theorem holds. Our results have many applications, e.g., they allow us to solve a problem of Aldous & Shields.
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