ALEA in Europe Workshop
TU Vienna

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Stochastic fixed-points and periodicities in combinatorial structures

Exercise 1. Consider a sequence of random variables $(X_n)_{n\geq 0}$ with $X_0 = 0$ and

$$X_n \stackrel{d}{=} X_{I_n} + n, \quad n \geq 1,$$

where I_n has the binomial $B(n, p)$ distribution with $0 < p < 1$ and I_n is independent of $X_0, \ldots, X_n$.

(a) Find a scaling $Y_n := \frac{X_n}{\sigma_n}$ with a suitable sequence $(\sigma_n)_{n\geq 0}$ such that Y_n leads to a limit equation. Guess its (unique) solution.

(b) Part (a) suggests that $\mathbb{E}[X_n] = (1-p)^{-1}n + o(n)$ as $n \to \infty$. Find a refined scaling (i.e., with $\sigma_n = o(n)$) of the form $Y_n := \frac{X_n - \mu_n}{\sigma_n}$ with suitable sequences $(\mu_n)_{n\geq 0}$, $(\sigma_n)_{n\geq 0}$, leading to a limit equation.

(c) Guess the (unique) solution of the limit equation in (b). What does this suggest for the asymptotic behavior of $\mathbb{E}[X_n]$ and $\mathrm{Var}(X_n)$?

(d) Use the general theorems presented in the course to prove these claims.

Exercise 2. The number of recursive calls R_n of `Quickselect` when selecting the minimum within a uniformly permuted set of n data satisfies $R_0 = R_1 = 0$ and

$$R_n \stackrel{d}{=} R_{I_n} + 1, \quad n \geq 2,$$

where I_n is independent of $R_0, \ldots, R_{n-1}$ and uniformly distributed over $\{0, \ldots, n-1\}$. Show (or believe) that as $n \to \infty$ we have:

$$\mathbb{E}[R_n] = \log n + O(1), \qquad \mathrm{Var}(R_n) = \log n + O(1).$$

Derive the limit equation for the standardized sequence.

Exercise 3. Let $0 < p < 1$, U a unif$[0,1]$ distributed random variable and $\mu, \nu \in \mathcal{M}_p$ with quantile functions F_μ^{-1} and F_ν^{-1}. Show that in general $(F_\mu^{-1}(U), F_\nu^{-1}(U))$ is not an optimal ℓ_p-coupling of μ and ν.

Hint: You may construct a counter example where μ and ν are supported by sets of two elements each.

Exercise 4. The number of red balls drawn from an urn with r red and b blue balls when drawing k times without replacement has the hypergeometric $\mathrm{Hyp}(k, r, r+b)$ distribution, where $r, b \in \mathbb{N}_0$ and $k \leq r+b$. Let U be uniformly on $[0,1]$ distributed and let Z_n conditional $U = u$ have the $\mathrm{Hyp}(\lfloor nu \rfloor, \lfloor n(1-u) \rfloor, n-1)$ distribution for all $u \in [0,1)$ and $n \geq 2$. Show that

$$\frac{Z_n}{n} \xrightarrow{L_2} U(1-U) \quad (n \to \infty).$$

Hint: You may use that for $\mathrm{Hyp}(k, r, r+b)$ distributed G we have

$$\mathbb{E}[G] = \frac{kr}{r+b}, \quad \mathrm{Var}(G) = \frac{krb(r+b-k)}{(r+b)^2(r+b-1)}.$$

Exercise 5. Let Y_n denote the number of key exchanges of `Quickselect` while selecting the smallest element within a uniformly permuted list of n numbers. (To partition the list two pointers are used to scan the list from left and right respectively). Deduce a recurrence of the form $Y_n =_d Y_{I_n} + b_n$ with suitable (I_n, b_n). Use a normalization such that the normalized Y_n converge towards a limit. You may use the general theorems presented in the course and exercise 4.

Exercise 6. Let $(Y_n)_{n\geq 0}$ be a sequence of random variables with $Y_0 = 0$, $Y_1 = 1$ and

$$Y_n \stackrel{d}{=} Y^{(1)}_{I_n} + Y^{(2)}_{I_n}, \quad n \geq 2,$$

where $(Y^{(1)}_n)_{n\geq 0}$, $(Y^{(2)}_n)_{n\geq 0}$ and I_n are independent, $(Y^{(r)}_n)_{n\geq 0}$ is distributed as $(Y_n)_{n\geq 0}$ for $r = 1, 2$ and I_n is uniformly distributed over $\{0, \dots, n-1\}$. Find a normalization of the Y_n leading to a non-degenerate limit law. Identify this limit distribution.

Hint: $\mathbb{E}[Y_n]$ can be computed elementary. The fixed point equation allows an interpretation via a homogeneous Poisson process on $\mathbb{R}^+_0$.

Exercise 7. The map $S : \mathcal{M} \to \mathcal{M}$ is given by

$$\mu \mapsto \mathcal{L}\left(\bigvee_{r=1}^{K} (A_r Z_r + b_r)\right),$$

where $Z_1, \dots, Z_K, (A_1, b_1, \dots, A_K, b_K)$ are independent and $\mathcal{L}(Z_r) = \mu$ for $r = 1, \dots, K$. Show: If A_r and b_r are L_p-integrable for a $p \geq 1$ and

$$\sum_{r=1}^{K} \mathbb{E}\left[|A_r|^p\right] < 1,$$

then the restriction of S to $\mathcal{M}_p$ has a unique fixed-point.

Hinweis: Use the ℓ_p-metric. You may also use that $|a \vee b - c \vee d|^p \leq |a-c|^p + |b-d|^p$ for all $a, b, c, d \in \mathbb{R}$.

Exercise 8. Let $(A_1, \dots, A_K, b)$ be a vector of L_2-integrable random variables in $\mathbb{C}$ and

$$T : \mathcal{M}^{\mathbb{C}} \to \mathcal{M}^{\mathbb{C}}, \qquad \mu \mapsto \mathcal{L}\left(\sum_{r=1}^{K} A_r Z^{(r)} + b\right)$$

where $(A_1, \dots, A_K, b)$, $Z^{(1)}, \dots, Z^{(K)}$ are independent and $Z^{(r)}$ has distribution μ for $r = 1, \dots, K$. Show, that for a suitable $\iota \in \mathbb{C}$ the restriction of T to $\mathcal{M}^{\mathbb{C}}_2(\iota)$ has a unique fixed-point, if $\mathbb{E}\sum_{r=1}^{K} |A_r|^2 < 1$.

What do we obtain from this about the solutions of

$$X \stackrel{d}{=} \sum_{r=1}^{b} V_r^{\gamma} X^{(r)}$$

where $(V_1, \dots, V_b)$, $X^{(1)}, \dots, X^{(b)}$ are independent, the complex valued random variables $X^{(r)}$ are distributed as X, moreover $\gamma \in \mathbb{C}$ and $(V_1, \dots, V_b)$ are random probabilities, i.e., random variables with $0 \leq V_r \leq 1$ and $\sum_{r=1}^{b} V_r = 1$ almost surely?