The southeast Corner of a Young Tableau

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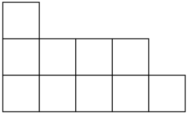
Filling at random a Young diagram

Consider $\mathcal{F}$ a Young (or Ferrers) diagram of size N : $\mathcal{F}$ is the shape of a Young tableau.



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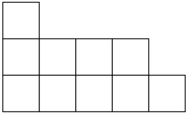

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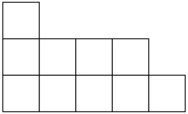

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What is the entry of a given cell ?

In which cell does one find a given entry ?

The rectangular case : scaling limit

Take a rectangular tableau of size (m, n).
Associated surface : function $f : [0,1] \times [0,1] \to [0,1]$
If the cell (i, j) has entry k, put $f(i/m, j/n) = k/mn$.

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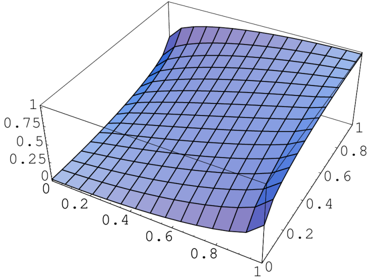
Pittel-Romik (2007) : if $m, n \to \infty$, $m/n \to \ell$, existence of a deterministic limit function f, expressed as the solution of a variational problem.



The rectangular case : fluctuations along the edge

Two asymptotic regimes (M., 2016)

In the corner

Let $X_{m,n}$ be the entry in the southeast corner.

$$\frac{\sqrt{2}(1+\ell)\left(X_{m,n} - \mathbb{E}X_{m,n}\right)}{n^{3/2}} \xrightarrow{law} Gaussian$$

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Along the edge

Suppose the tableau is an (n,n) square. Let $Y_{i,n}$ be the entry in the cell $(1,i)$. Fix $0 < t < 1$. Then for large n,

$$\frac{r(t)(Y_{1,\lfloor tn \rfloor} - \mathbb{E}Y_{1,\lfloor tn \rfloor})}{n^{4/3}} \overset{law}{\to} Tracy - Widom$$

The southeast corner

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$$\mathbb{P}(X_n = k) = \frac{\binom{k-1}{m-1}\binom{mn-k}{n-1}}{\binom{mn}{m+n-1}}$$

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This is the same as the distribution of an entry in a hook tableau.
A hook tableau is also a tree.

Linear extension of a tree

If T is a tree of size $N+1$, a linear extension is a function $f : T \to \{0, 1 \ldots N\}$ such that $f(child) > f(parent)$.

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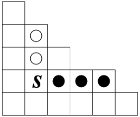
Analogue of the hook length formula for the number of standard fillings of a diagram $\mathcal{F}$:

$$\frac{N!}{\prod_{e \in \mathcal{F}} h(e)}$$

where $h(e)$ is the hook length of the cell e.



The tree associated with a diagram

If $\mathcal{F}$ is a Young diagram, associate a planar rooted tree T with a distinguished vertex v :

- The hook lengths along the first row of $\mathcal{F}$ are the same as the hook lengths along the branch of T from the root to v.
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Enlarge T to obtain $\overline{T}$ by adding a father R to the root of T and adding children to R so that the size of $\overline{T}$ is $N+1$.

The main result

Theorem

Let $\mathcal{F}$ be a Young diagram to which one associates a tree $\overline{T}$ with a distinguished vertex v.
Let X be the entry in the southeast corner of $\mathcal{F}$ when one picks a random, uniform standard filling of $\mathcal{F}$.
Let $Y = \ell(v)$ where ℓ is a random, uniform linear extension of $\overline{T}$.
Then X and Y have the same law.

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This enables to recover the law of the corner for a rectangular Young tableau.

Triangular tableaux and periodic trees

Consider a staircase tableau. The associated tree T is a comb. More generally, if $\mathcal{F}$ is a discretized triangle, then along the branch of T between the root and v, we have a periodic pattern.

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More generally, if $\mathcal{F}$ is a discretized triangle, then along the branch of T between the root and v, we have a periodic pattern.
If $\mathcal{F}$ is large, the entry in the southeast corner is $N - o(N)$. Say that this entry is

$$N + 1 - Z$$

Z is the number of cell having a greater entry than the southeast corner. This corresponds to the number of vertices w of the tree having $\ell(w) \geq \ell(v)$.

Trees and urns

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Consider the set

$$E_n = \{\ell(v), \ell(u_1), \ell(w_1) \ldots \ell(u_{n-1}), \ell(w_{n-1}), \ell(w_n)\}$$

Let r be the rank of $\ell(w_n)$ in E_n and k be the rank of $\ell(v)$ in E_n.
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$\ell(w_n) > \ell(v)$ iff $r > k$.
Fact : r is uniform in $\{1, 2 \dots 2n\}$. Therefore

$$\mathbb{P}(\ell(w_n) > \ell(v)) = \mathbb{P}(r > k) = \frac{2n - k}{2n}$$

Note that $2n - k$ is the number of elements a in $E_n - \{\ell(w_n)\}$ having $\ell(a) \geq \ell(v)$: this is the number of white balls.
Thus we get an urn model.