Generating asymptotics for factorially divergent sequences

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Introduction

- Singularity analysis is a great tool to obtain asymptotic expansions of combinatorial classes.
- Caveat: Only applicable if the generating function has a non-zero, finite radius of convergence.
- Topic of this talk: Power series with vanishing radius of convergence and factorial growth.

- Consider the class of power series $\mathbb{R}[[x]]^\alpha_\beta \subset \mathbb{R}[[x]]$ which admit an asymptotic expansion of the form,

$$
\begin{aligned}
f_n &= \alpha^{n+\beta}\Gamma(n+\beta)\left(c_0 + \frac{c_1}{\alpha(n+\beta)} + \frac{c_2}{\alpha^2(n+\beta)(n+\beta-1)} + \ldots\right) \\
&= \sum_{k=0}^{R-1} c_k \alpha^{n+\beta-k}\Gamma(n+\beta-k) + \mathcal{O}\left(\alpha^{n+\beta-R}\Gamma(n+\beta-R)\right)
\end{aligned}
$$

- $\mathbb{R}[[x]]^\alpha_\beta$ a linear subspace of $\mathbb{R}[[x]]$.
- Includes power series with non-vanishing radius of convergence: In this case all $c_k = 0$.
- These power series appear in
  - Graph counting
  - Permutations
  - Perturbation expansions in physics

- Consider a power series $f(x) \in \mathbb{R}[[x]]^\alpha_\beta$:

$$f_n = \sum_{k=0}^{R-1} c_k \alpha^{n+\beta-k}\Gamma(n+\beta-k) + \mathcal{O}\left(\alpha^{n+\beta-R}\Gamma(n+\beta-R)\right)$$

- Interpret the coefficients c_k of the **asymptotic** expansion as a new power series.

Definition

$\mathcal{A}$ maps a power series to its asymptotic expansion:

$$\begin{array}{cccccc} \mathcal{A} & : & \mathbb{R}[[x]]^\alpha_\beta & \to & \mathbb{R}[[x]] \\ & & f(x) & \mapsto & \gamma(x) = \sum_{k=0}^{\infty} c_k x^k \end{array}$$

Theorem

$\mathcal{A}$ is a derivation on $\mathbb{R}[[x]]^\alpha_\beta$:

$$(\mathcal{A}f \cdot g)(x) = f(x)(\mathcal{A}g)(x) + (\mathcal{A}f)(x)g(x)$$

$\Rightarrow$ $\mathbb{R}[[x]]^\alpha_\beta$ is a subring of $\mathbb{R}[[x]]$.

Proof sketch

With $h(x) = f(x)g(x)$,

$$h_n = \underbrace{\sum_{k=0}^{R-1} f_{n-k} g_k + \sum_{k=0}^{R-1} f_k g_{n-k}}_{\text{High order times low order}} + \underbrace{\sum_{k=R}^{n-R} f_k g_{n-k}}_{\mathcal{O}(\alpha^n \Gamma(n+\beta-R))}$$

.

- $\sum_{k=R}^{n-R} f_k g_{n-k} \in \mathcal{O}(\alpha^n \Gamma(n+\beta-R))$ follows from the *log-convexity* of the Γ function.

Example

- Set $F(x) = \sum_{n=1}^{\infty} n! x^n = \sum_{n=1}^{\infty} 1^{n+1} \Gamma(n+1) x^n$,
- By definition: $F \in \mathbb{R}[[x]]_1^1$ and $(\mathcal{A}F)(x) = 1$
- Because $\mathbb{R}[[x]]_1^1$ is a ring: $F(x)^2 \in \mathbb{R}[[x]]_1^1$
- Because of the product rule for $\mathcal{A}$:

$$(\mathcal{A}F(x)^2)(x) = F(x)(\mathcal{A}F)(x) + (\mathcal{A}F)(x)F(x) = 2F(x)$$

- Asymptotic expansion of $F(x)^2$ is given by $2F(x)$:

$$[x^n]F(x)^2 = \sum_{k=0}^{R-1} c_k (n-k)! + \mathcal{O}\left((n-R)!\right) \qquad \forall R \in \mathbb{N}_0$$

where $c_k = [x^k]2F(x)$.

- What happens for **composition** of power series $\in \mathbb{R}[[x]]^\alpha_\beta$?

Theorem Bender [1975]

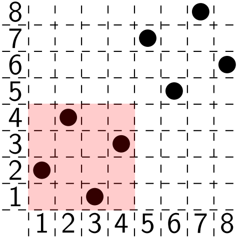
If $|f_n| \leq C^n$ then, for $g \in \mathbb{R}[[x]]^\alpha_\beta$ with $g_0 = 0$:

$$f \circ g \in \mathbb{R}[[x]]^\alpha_\beta$$
$$(\mathcal{A} f \circ g)(x) = f'(g(x))(\mathcal{A} g)(x).$$

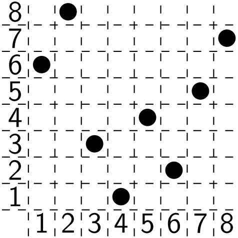
- Bender considered much more general power series, but this is a direct corollary of his theorem in 1975.

Example

A reducible permutation:



An irreducible permutation:



- A permutation π of $[n] = \{1, \ldots, n\}$ is called irreducible if there is **no** $m < n$ such that $\pi([m]) = [m]$.
- Set $F(x) = \sum_{n=1}^{\infty} n! x^n$ - the OGF of all permutations.
- The OGF of irreducible permutations I fulfills

$$I(x) = 1 - \frac{1}{1 + F(x)}.$$

$$I(x) = 1 - \frac{1}{1+F(x)} \qquad F(x) = \sum_{n=1}^{\infty} n! x^n.$$

- By definition: $F \in \mathbb{R}[[x]]_1^1$ and $(\mathcal{A}F)(x) = 1$.
- $\frac{1}{1+x}$ is analytic at the origin, therefore by the chain rule

$$(\mathcal{A}I)(x) = \left(\mathcal{A}\left(1 - \frac{1}{1+F(x)}\right)\right)(x) = \frac{1}{(1+F(x))^2}$$

Theorem Comtet [1972]

Therefore the asymptotic expansion of the coefficients of $I(x)$ is

$$[x^n]I(x) = \sum_{k=0}^{R-1} c_k (n-k)! + \mathcal{O}((n-R)!) \qquad \forall R \in \mathbb{N}_0,$$

where $c_k = [x^k]\frac{1}{(1+F(x))^2}$.

This chain rule can easily be generalized to multivalued analytic functions:

Theorem MB [2016]

More general: For $f \in \mathbb{R}\{y_1, \ldots, y_L\}$ and $g^1, \ldots, g^L \in x\mathbb{R}[[x]]^\alpha_\beta$:

$$(\mathcal{A}(f(g^1, \ldots, g^L))(x) = \sum_{l=1}^{L} \frac{\partial f}{\partial g^l}(g^1, \ldots, g^L)(\mathcal{A}^\alpha_\beta g^l)(x).$$

- What happens if f is not an analytic function?
- $\mathcal{A}$ fulfills a general 'chain rule':

Theorem MB [2016]

If $f, g \in \mathbb{R}[[x]]^\alpha_\beta$ with $g_0 = 0$ and $g_1 = 1$, then $f \circ g \in \mathbb{R}[[x]]^\alpha_\beta$ and

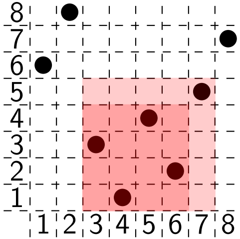
$$(\mathcal{A} f \circ g)(x) = f'(g(x))(\mathcal{A} g)(x) + \left(\frac{x}{g(x)}\right)^\beta e^{\frac{g(x)-x}{\alpha x g(x)}} (\mathcal{A} f)(g(x))$$

$\Rightarrow$ $\mathbb{R}[[x]]^\alpha_\beta$ is closed under composition and inversion.

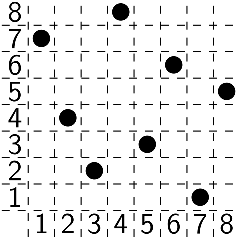
$\Rightarrow$ We can solve for asymptotics of implicitly defined power series.

Example: Simple permutations

A non-simple permutation:



A simple permutation:



- A permutation π of $[n] = \{1, \ldots, n\}$ is called simple if there is **no** (non-trivial) interval $[i, j] = \{i, \ldots, j\}$ such that $\pi([i, j])$ is another interval.
- The OGF $S(x)$ of simple permutations fulfills

$$\frac{F(x) - F(x)^2}{1 + F(x)} = x + S(F(x)),$$

with $F(x) = \sum_{n=1}^{\infty} n! x^n$ [Albert, Klazar, and Atkinson, 2003].

$$\frac{F(x) - F(x)^2}{1 + F(x)} = x + S(F(x)).$$

- By definition: $F \in \mathbb{R}[[x]]^1_1$ and $(\mathcal{A}F)(x) = 1$.
- Extract asymptotics by applying the $\mathcal{A}$-derivative:

$$\mathcal{A}\left(\frac{F(x) - F(x)^2}{1 + F(x)}\right) = \mathcal{A}\left(x + S(F(x))\right).$$

- Apply chain rule on both sides

$$\begin{aligned}\frac{1 - 2F(x) - F(x)^2}{(1 + F(x))^2}(\mathcal{A}F)(x) &= S'(F(x))(\mathcal{A}F)(x) \\ &\quad + \left(\frac{x}{F(x)}\right)^1 e^{\frac{F(x)-x}{xF(x)}}(\mathcal{A}S)(F(x)),\end{aligned}$$

which can be solved for $(\mathcal{A}S)(x)$.

- After simplifications:

$$(\mathcal{A}S)(x) = \frac{1}{1+x} \frac{1 - x - (1+x)\frac{S(x)}{x}}{1 + (1+x)\frac{S(x)}{x^2}} e^{-\frac{2+(1+x)\frac{S(x)}{x^2}}{1-x-(1+x)\frac{S(x)}{x}}}$$

- We get the full asymptotic expansion for S:

$$[x^n]S(x) = \sum_{k=0}^{R-1} c_k (n-k)! + \mathcal{O}((n-R)!) \qquad \forall R \in \mathbb{N}_0$$

where $c_k = [x^k](\mathcal{A}S)(x)$.

$$[x^n]S(x) = e^{-2} n! \left(1 - \frac{4}{n} + \frac{2}{n(n-1)} - \frac{40}{3n(n-1)(n-2)} + \ldots\right),$$

the first three coefficients have been obtained by Albert, Klazar, and Atkinson [2003].

Meta asymptotics

$$(\mathcal{A}S)(x) = \frac{1}{1+x} \frac{1 - x - (1+x)\frac{S(x)}{x}}{1 + (1+x)\frac{S(x)}{x^2}} e^{-\frac{2+(1+x)\frac{S(x)}{x^2}}{1-x-(1+x)\frac{S(x)}{x}}} := g(x, S(x))$$

- $g(x, S(x))$ is an analytic function in $S(x)$:
- Because of the chain rule for analytic functions,

$$(\mathcal{A}(\mathcal{A}S))(x) = \frac{\partial g(x,S)}{\partial S}(\mathcal{A}S)(x),$$

we obtain the **asymptotics of the asymptotic expansion**.

$$g(x,S) = \frac{1}{1+x} \frac{1 - x - (1+x)\frac{S}{x}}{1 + (1+x)\frac{S}{x^2}} e^{-\frac{2+(1+x)\frac{S}{x^2}}{1-x-(1+x)\frac{S}{x}}}$$

- This way we can obtain the GF for **meta asymptotics**:

$$f(t,x) = \sum_{k=0}^{\infty} t^k \frac{(\mathcal{A}^k S)(x)}{k!} = q^{-1}(t + q(S(x))),$$

  where $q(S) = \int_0^S \frac{dS'}{g(x,S')}$ and $q^{-1}(q(S)) = S$.
- $[t^k]f(t,x)$ is the GF of the k-th order asymptotics of S.
- Using this information to resum such a series leads to the theory of resurgence.

Conclusions

- $\mathbb{R}[[x]]^{\alpha}_{\beta}$ forms a subring of $\mathbb{R}[[x]]$ **closed under mutliplication, composition and inversion**.
- $\mathcal{A}$ is a **derivation** on $\mathbb{R}[[x]]^{\alpha}_{\beta}$ which can be used to obtain asymptotic expansions of **implicitly defined power series**.
- Closure properties under asymptotic derivative $\mathcal{A}$.

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Edward A Bender. An asymptotic expansion for the coefficients of some formal power series. *Journal of the London Mathematical Society*, 2(3):451–458, 1975.

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