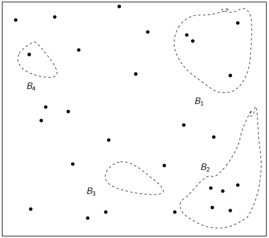
The Poisson-Voronoi cell around an isolated nucleus

Pierre Calka

October 9, 2017

Alea in Europe, TU Wien

Poisson point process



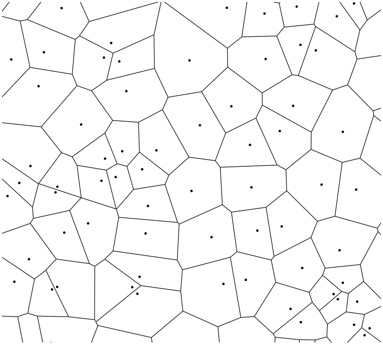
$\mathcal{P}_\lambda$ said homogeneous Poisson point process in $\mathbb{R}^2$ with intensity λ if

- $\#(\mathcal{P} \cap B_1)$ Poisson r.v. of mean $\lambda \mathcal{A}(B_1)$
- $\#(\mathcal{P} \cap B_1), \cdots, \#(\mathcal{P} \cap B_\ell)$ independent $(B_1, \cdots, B_\ell \in \mathcal{B}(\mathbb{R}^2),\ B_i \cap B_j = \emptyset,\ i \neq j)$

Two properties

- Scaling invariance: $\mu \cdot \mathcal{P}_\lambda \overset{\mathcal{D}}{=} \mathcal{P}_{\frac{\lambda}{\sqrt{\mu}}}$
- Mecke's formula: $\mathbb{E}\left(\sum_{x \in \mathcal{P}_\lambda} f(x, \mathcal{P}_\lambda) \right) = \lambda \int \mathbb{E}(f(x, \mathcal{P} \cup \{x\}))\mathrm{d}x$

Poisson-Voronoi tessellation



- $\mathcal{P}_\lambda$ homogeneous Poisson point process in $\mathbb{R}^2$ of intensity λ
- For every nucleus $x \in \mathcal{P}_\lambda$, associated cell

$$C(x|\mathcal{P}_\lambda) := \{y \in \mathbb{R}^2 : \|y - x\| \leq \|y - x'\| \ \forall x' \in \mathcal{P}_\lambda\}$$

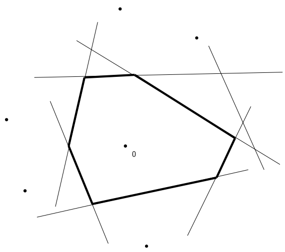
Typical Poisson-Voronoi cell

- Typical cell $\mathcal{C}$: *chosen uniformly among all cells*

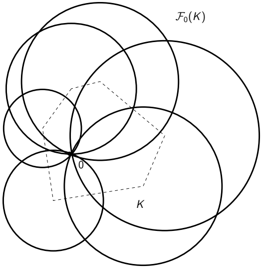
$$\mathbb{E}(f(\mathcal{C})) = \lim_{r\to\infty} \frac{1}{N_r} \sum_{\begin{cases} x\in\mathcal{P}_\lambda \\ C(x|\mathcal{P}_\lambda)\subset B_r(o)\end{cases}} f(C(x|\mathcal{P}_\lambda)) \text{ a.s.}$$

$$\mathbb{E}(f(\mathcal{C})) = \frac{1}{\lambda\mathcal{A}(B)}\mathbb{E}\Big(\sum_{x\in\mathcal{P}_\lambda\cap B} f(C(x|\mathcal{P}_\lambda) - x)\Big), \ B \in \mathcal{B}(\mathbb{R}^2)$$

- Theorem (Slivnyak): $\mathcal{C} \overset{D}{=} C(o|\mathcal{P}_\lambda \cup \{o\})$



Being in the typical cell



- K convex body containing o in its interior
- Flower of K: $\mathcal{F}_o(K) = \cup_{x \in K} B(x, \|x\|)$

Two properties

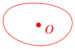
- Capacity probability: $\mathbb{P}(K \subset C(o|\mathcal{P}_\lambda \cup \{o\})) = e^{-\lambda \mathcal{A}(\mathcal{F}_o(K))}$
- Conditional distribution: $(\mathcal{P}_\lambda | K \subset C(o|\mathcal{P}_\lambda \cup \{o\})) \stackrel{D}{=} \mathcal{P}_\lambda \setminus \mathcal{F}_o(K)$

The Poisson-Voronoi cell around an isolated nucleus

- K convex body in $\mathbb{R}^2$
- An origin o chosen in $\mathrm{int}(K)$
- Point process $(\mathcal{P}_\lambda | K \subset C(o|\mathcal{P}_\lambda \cup \{o\}))$
- *Problem 1*. Asymptotics of the characteristics of the cell

$$K_\lambda = C(o|\mathcal{P}_\lambda \cup \{o\})$$

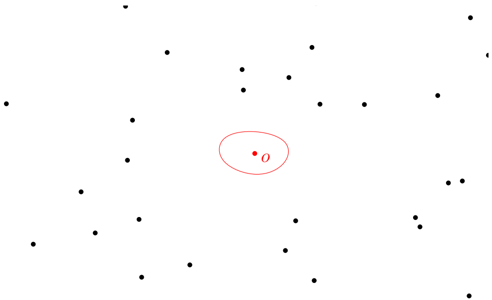
The Poisson-Voronoi cell around an isolated nucleus



- K convex body in $\mathbb{R}^2$
- An origin o chosen in $\mathrm{int}(K)$
- Point process $(\mathcal{P}_\lambda | K \subset C(o|\mathcal{P}_\lambda \cup \{o\}))$
- *Problem 1*. Asymptotics of the characteristics of the cell

$$K_\lambda = C(o|\mathcal{P}_\lambda \cup \{o\})$$

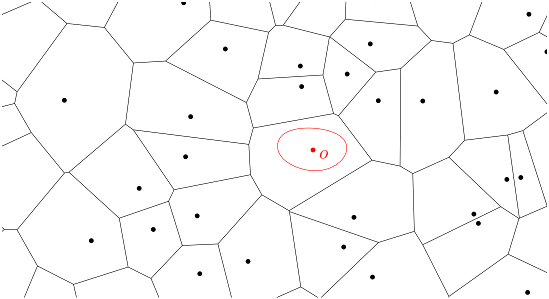
The Poisson-Voronoi cell around an isolated nucleus



- K convex body in $\mathbb{R}^2$
- An origin o chosen in $\mathrm{int}(K)$
- Point process $(\mathcal{P}_\lambda | K \subset C(o|\mathcal{P}_\lambda \cup \{o\}))$
- *Problem 1*. Asymptotics of the characteristics of the cell

$$K_\lambda = C(o|\mathcal{P}_\lambda \cup \{o\})$$

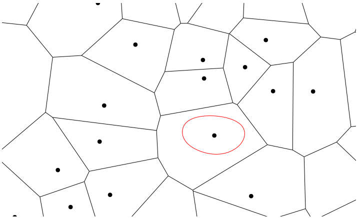
The Poisson-Voronoi cell around an isolated nucleus

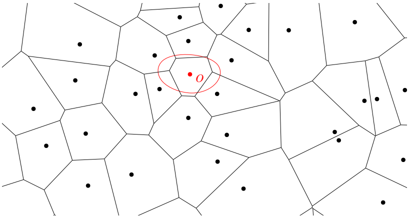


- K convex body in $\mathbb{R}^2$
- An origin o chosen in $\mathrm{int}(K)$
- Point process $(\mathcal{P}_\lambda | K \subset C(o|\mathcal{P}_\lambda \cup \{o\}))$
- *Problem 1*. Asymptotics of the characteristics of the cell

$$K_\lambda = C(o|\mathcal{P}_\lambda \cup \{o\})$$

Two associated problems





$\mathcal{D} \subset \mathbb{R}^2$, $o \in \mathrm{int}(\mathcal{D})$

Problem 2. Cell $\widehat{K}_\lambda$ containing K,
$\mathcal{P}_\lambda$ conditioned on $\{K \subset \text{ one cell}\}$

Problem 3. Cell $\mathcal{C}_\lambda(\mathcal{D}) = C(o|\mathcal{P}_\lambda \cup \{o\})$,
$\mathcal{P}_\lambda$ conditioned on $\{\mathcal{P}_\lambda \cap \mathcal{D} = \emptyset\}$

Plan

Problem 1: asymptotics for $K_\lambda = C(o|\mathcal{P}_\lambda \setminus \mathcal{F}_o(K))$

Problem 2: asymptotics for $\widehat{K}_\lambda$ (no origin)

Problem 3: asymptotics for $\mathcal{C}_\lambda(\mathcal{D}) = C(o|\mathcal{P}_\lambda \setminus \mathcal{D})$

Joint work with **Yann Demichel** (Paris Nanterre) & **Nathanaël Enriquez** (Paris-Sud)

Plan

Problem 1: asymptotics for $K_\lambda = C(o|\mathcal{P}_\lambda \setminus \mathcal{F}_o(K))$

- Context
- Main results
- Support function
- Rewriting of the expectations
- Sketch of proof

Problem 2: asymptotics for $\widehat{K}_\lambda$ (no origin)

Problem 3: asymptotics for $\mathcal{C}_\lambda(\mathcal{D}) = C(o|\mathcal{P}_\lambda \setminus \mathcal{D})$

Context: large Poisson-Voronoi cells

- Large cells in a Poisson-Voronoi tessellation: *close* to the circular shape

  D. Hug, M. Reitzner & R. Schneider (2004)

- When K is the unit-disk,

  $$\mathbb{E}(\mathcal{A}(K_\lambda)) - \mathcal{A}(K) \underset{\lambda\to\infty}{\sim} \lambda^{-\frac{2}{3}} 2^{-1} 3^{-\frac{1}{3}} \pi \Gamma\left(\frac{2}{3}\right), \ \mathbb{E}(\mathcal{N}(K_\lambda)) \underset{\lambda\to\infty}{\sim} \lambda^{\frac{1}{3}} 2^2 3^{-\frac{4}{3}} \Gamma\left(\frac{2}{3}\right)$$

  PC & T. Schreiber (2005)

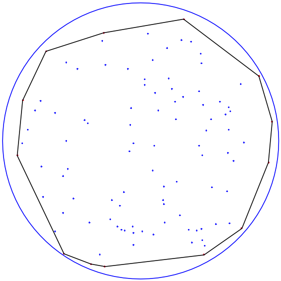
- Estimate of the Hausdorff distance between K_λ and K for a Poisson line tessellation

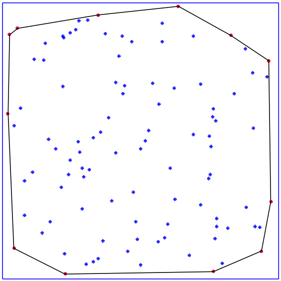
  D. Hug & R. Schneider (2014), **R. Schneider** (1988)

Context: approximation of a convex body from the inside

$$K^\lambda = \mathrm{Conv}(\mathcal{P}_\lambda \cap K)$$

Efron's relation (1965): $\mathbb{E}(\mathcal{N}(K^\lambda)) = \lambda(\mathcal{A}(K) - \mathbb{E}(\mathcal{A}(K^\lambda)))$





K with a smooth boundary

$$\mathcal{A}(K) - \mathbb{E}(\mathcal{A}(K^\lambda)) \underset{\lambda\to\infty}{\sim} \lambda^{-\frac{2}{3}} 2^{\frac{4}{3}} 3^{-\frac{4}{3}} \Gamma\left(\frac{2}{3}\right) \int_{\partial K} r_s^{-\frac{1}{3}} \mathrm{d}s$$

K polygon

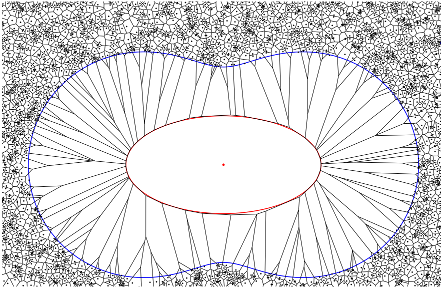
$$\mathcal{A}(K) - \mathbb{E}(\mathcal{A}(K^\lambda)) \underset{\lambda\to\infty}{\sim} (\lambda^{-1} \log \lambda) \cdot 2 \cdot 3^{-1} n_K$$

A. Rényi & R. Sulanke (1963, 1964)

Main results: smooth case

$\mathcal{A}(\cdot)$: area, $\mathcal{U}(\cdot)$: perimeter, $\mathcal{N}(\cdot)$: number of vertices

r_s: radius of curvature, n_s: outer unit normal vector at $s \in \partial K$



$$\mathbb{E}(\mathcal{A}(K_\lambda)) - \mathcal{A}(K) \underset{\lambda\to\infty}{\sim} \lambda^{-\frac{2}{3}} 2^{-2} 3^{-\frac{1}{3}} \Gamma\left(\frac{2}{3}\right) \int_{\partial K} r_s^{\frac{1}{3}} \langle s, n_s \rangle^{-\frac{2}{3}} \, \mathrm{d}s$$

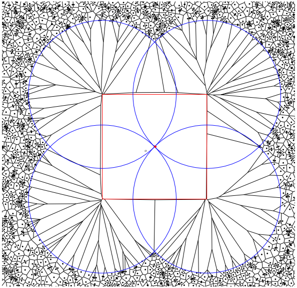
$$\mathbb{E}(\mathcal{U}(K_\lambda)) - \mathcal{U}(K) \underset{\lambda\to\infty}{\sim} \lambda^{-\frac{2}{3}} 3^{-\frac{4}{3}} \Gamma\left(\frac{2}{3}\right) \int_{\partial K} r_s^{-\frac{2}{3}} \langle s, n_s \rangle^{-\frac{2}{3}} \mathrm{d}s$$

$$\mathbb{E}(\mathcal{N}(K_\lambda)) \underset{\lambda\to\infty}{\sim} \lambda^{\frac{1}{3}} 2^{2} 3^{-\frac{4}{3}} \Gamma\left(\frac{2}{3}\right) \int_{\partial K} r_s^{-\frac{2}{3}} \langle s, n_s \rangle^{\frac{1}{3}} \, \mathrm{d}s$$

Main results: polygonal case

$\mathcal{A}(\cdot)$: area, $\mathcal{U}(\cdot)$: perimeter, $\mathcal{N}(\cdot)$: number of vertices

n_K: number of vertices of K, $\{a_i\}$: vertices of K, o_i: projection of o onto (a_i, a_{i+1})

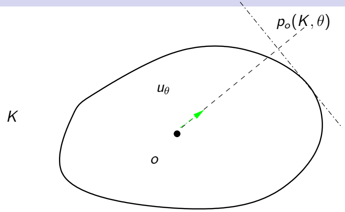


$$\mathbb{E}(\mathcal{A}(K_\lambda)) - \mathcal{A}(K) \underset{\lambda\to\infty}{\sim} \lambda^{-\frac{1}{2}} 2^{-\frac{9}{2}} \pi^{\frac{3}{2}} \sum_{i=1}^{n_K} \|o_i\|^{-\frac{1}{2}} \|a_{i+1} - a_i\|^{\frac{3}{2}}$$

$$\mathbb{E}(\mathcal{U}(K_\lambda)) - \mathcal{U}(K) \underset{\lambda\to\infty}{\sim} (\lambda^{-1}\log\lambda) \cdot 2^{-1} 3^{-1} \sum_{i=1}^{n_K} \|o_i\|^{-1}$$

$$\mathbb{E}(\mathcal{N}(K_\lambda)) \underset{\lambda\to\infty}{\sim} (\log\lambda) \cdot 2 \cdot 3^{-1} n_K.$$

Support function



Support function

$$p_o(K,\theta) = p_o(K,u_\theta) = \sup_{x\in K} \langle x, u_\theta\rangle$$

with $u_\theta = (\cos(\theta), \sin(\theta))$

Two properties

- Link with the flower: $\sup\{r>0 : ru_\theta \in \mathcal{F}_o(K)\} = 2p_o(K,\theta)$
- Cauchy-Crofton formula: $\mathcal{U}(K) = \int_0^{2\pi} p_o(K,\theta)\mathrm{d}\theta$

Rewriting of the expectations

- Mean defect area

$$\mathbb{E}(\mathcal{A}(K_\lambda)) - \mathcal{A}(K) = \int_{\mathbb{R}^2\setminus K} \mathbb{P}(x \in K_\lambda)\mathrm{d}x$$
$$= \int_{\mathbb{R}^2\setminus K} e^{-\lambda(\mathcal{A}(\mathcal{F}_o(K\cup\{x\}))-\mathcal{A}(\mathcal{F}_o(K)))}\mathrm{d}x$$

where $\mathcal{A}(\mathcal{F}_o(K)) = 2\int_0^{2\pi} p_o(K,\theta)^2\mathrm{d}\theta$

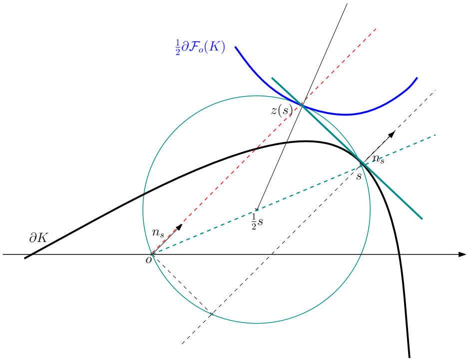
- Mean defect perimeter

$$\mathbb{E}(\mathcal{U}(K_\lambda)) - \mathcal{U}(K) = \int_0^{2\pi} \mathbb{E}(p_o(K_\lambda,\theta) - p_o(K,\theta))\mathrm{d}\theta$$

- Mean number of vertices: Efron-type relation

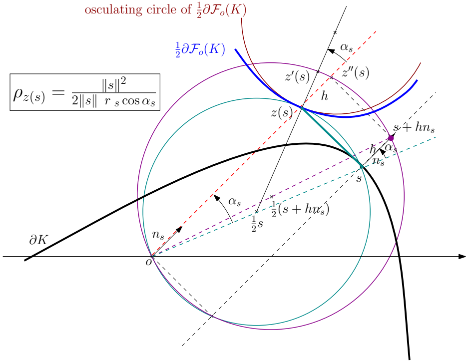
$$\mathbb{E}(\mathcal{N}(K_\lambda)) = \lambda(\mathbb{E}(\mathcal{A}(\mathcal{F}_o(K_\lambda)) - \mathcal{A}(\mathcal{F}_o(K)))$$
$$\underset{\lambda\to\infty}{\sim} 4\lambda\int_0^{2\pi} p_o(K,\theta)\mathbb{E}(p_o(K_\lambda,\theta) - p_o(K,\theta))\mathrm{d}\theta$$

Sketch of proof, smooth case



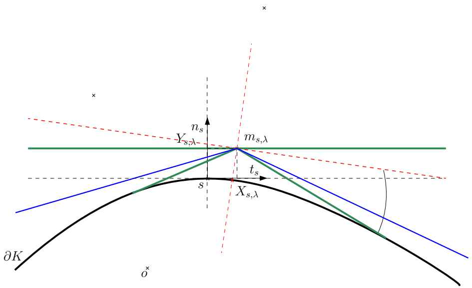
$$\mathcal{A}(\mathcal{F}_o(K \cup \{s + hn_s\})) - \mathcal{A}(\mathcal{F}_o(K)) \underset{h\to 0}{\sim} h^{\frac{3}{2}} 2^{\frac{9}{2}} 3^{-1} r_s^{-\frac{1}{2}} \langle s, n_s \rangle$$

Sketch of proof, smooth case



$$\mathcal{A}(\mathcal{F}_o(K \cup \{s + hn_s\})) - \mathcal{A}(\mathcal{F}_o(K)) \underset{h\to 0}{\sim} h^{\frac{3}{2}} 2^{\frac{9}{2}} 3^{-1} r_s^{-\frac{1}{2}} \langle s, n_s \rangle$$

Sketch of proof, smooth case



- $m_{s,\lambda}$: support point of K_λ in direction n_s
- $(X_{s,\lambda}, Y_{s,\lambda})$: coordinates of $m_{s,\lambda}$ in the Frenet frame at s

$$(\lambda^{\frac{1}{3}} X_{s,\lambda}, \lambda^{\frac{2}{3}} Y_{s,\lambda}) \xrightarrow{D} (X, Y) \text{ with explicit density.}$$

$$\mathbb{E}(p_o(K_\lambda, n_s) - p_o(K, n_s)) = \mathbb{E}(Y_{s,\lambda}) \underset{\lambda \to \infty}{\sim} \lambda^{-\frac{2}{3}} 3^{-\frac{4}{3}} \Gamma\left(\frac{2}{3}\right) r_s^{\frac{1}{3}} \langle s, n_s \rangle^{-\frac{2}{3}}$$

Problem 2: asymptotics for $\widehat{K}_\lambda$ (no origin)

- Steiner point

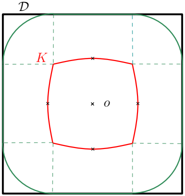
$$\mathrm{st}(K) = \frac{1}{\pi}\int_0^{2\pi} p_o(K,\theta)u_\theta \mathrm{d}\theta = \mathrm{argmin}_x \mathcal{A}(\mathcal{F}_x(K))$$

- K: convex body with Steiner point at o
- $\widehat{K}_\lambda$: cell containing K, conditional on $\mathscr{S}_\lambda = \{K \subset \text{ one cell}\}$
- Z_λ: nucleus of $\widehat{K}_\lambda$

Conditional on $\mathscr{S}_\lambda$, $\lambda^{\frac{1}{2}} Z_\lambda \xrightarrow{D} \mathcal{N}(o, (4\pi)^{-1} I_2)$.

The expectation asymptotics of $\widehat{K}_\lambda$ coincide with those of K_λ when $o = \mathrm{st}(K)$.

Problem 3: asymptotics for $\mathcal{C}_\lambda(\mathcal{D}) = C(o|\mathcal{P}_\lambda \setminus \mathcal{D})$



- $\mathcal{D}$ closed domain, $o \in \mathrm{int}(\mathcal{D})$
- $\mathcal{C}_\lambda(\mathcal{D}) = C(o|\mathcal{P}_\lambda \setminus \mathcal{D})$
- K: convex body such that $\mathcal{F}_o(K)$ is the largest flower in $\mathcal{D}$
- $\mathcal{D}^*$: maximal starlike set in $\mathcal{D}$, with piecewise C^3 equation $d(\cdot)$

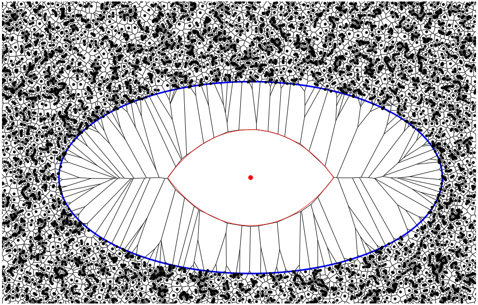
$$\mathcal{C}_\lambda(\mathcal{D}) \xrightarrow{\mathbb{P}} K \text{ in the Hausdorff metric}$$

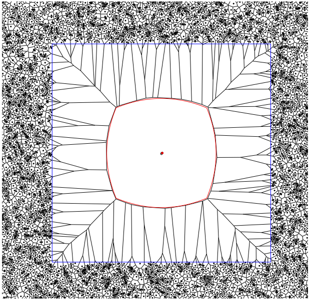
$$\mathbb{E}(\mathcal{A}(\mathcal{C}_\lambda(\mathcal{D}))) - \mathcal{A}(K) \underset{\lambda\to\infty}{\sim} \lambda^{-\frac{2}{3}} 2^{-\frac{8}{3}} 3^{-\frac{1}{3}} \Gamma\left(\frac{2}{3}\right) \int (d(\theta) + d''(\theta))^{\frac{4}{3}} d(\theta)^{-\frac{2}{3}} \mathrm{d}\theta$$

$$\mathbb{E}(\mathcal{U}(\mathcal{C}_\lambda(\mathcal{D}))) - \mathcal{U}(K) \underset{\lambda\to\infty}{\sim} \lambda^{-\frac{2}{3}} 2^{-\frac{2}{3}} 3^{-\frac{4}{3}} \Gamma\left(\frac{2}{3}\right) \int (d(\theta) + d''(\theta))^{\frac{1}{3}} d(\theta)^{-\frac{2}{3}} \mathrm{d}\theta$$

$$\mathbb{E}(\mathcal{N}(\mathcal{C}_\lambda(\mathcal{D}))) \underset{\lambda\to\infty}{\sim} \lambda^{\frac{1}{3}} 2^{-\frac{8}{3}} 3^{-\frac{4}{3}} \Gamma\left(\frac{2}{3}\right) \int (d(\theta) + d''(\theta))^{\frac{1}{3}} d(\theta)^{\frac{1}{3}} \mathrm{d}\theta$$

Problem 3: asymptotics for $\mathcal{C}_\lambda(\mathcal{D}) = C(o|\mathcal{P}_\lambda \setminus \mathcal{D})$

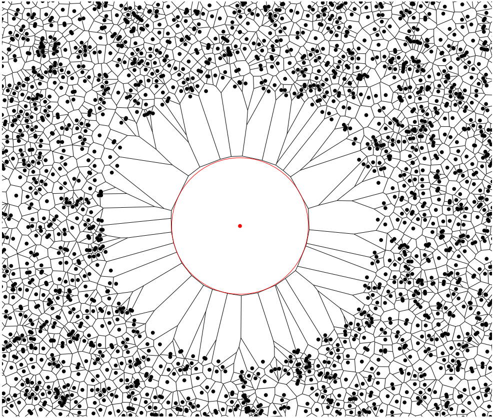




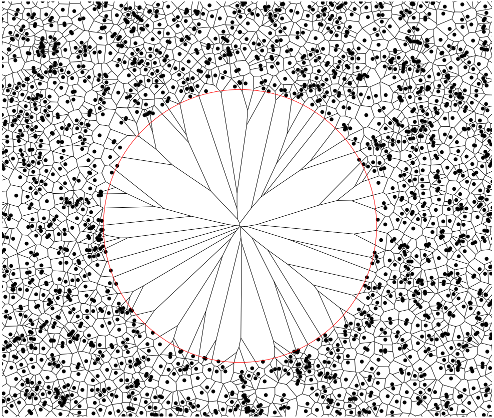
Concluding remarks

- Higher dimension
- Variances
- Similar results for the zero-cell of a Poisson line tessellation
- Inlets

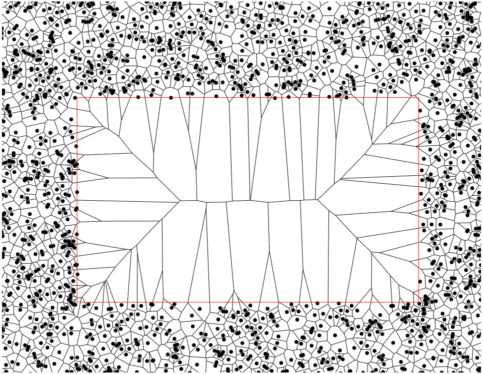
Concluding remarks



Concluding remarks



Concluding remarks



Thank you for your attention!